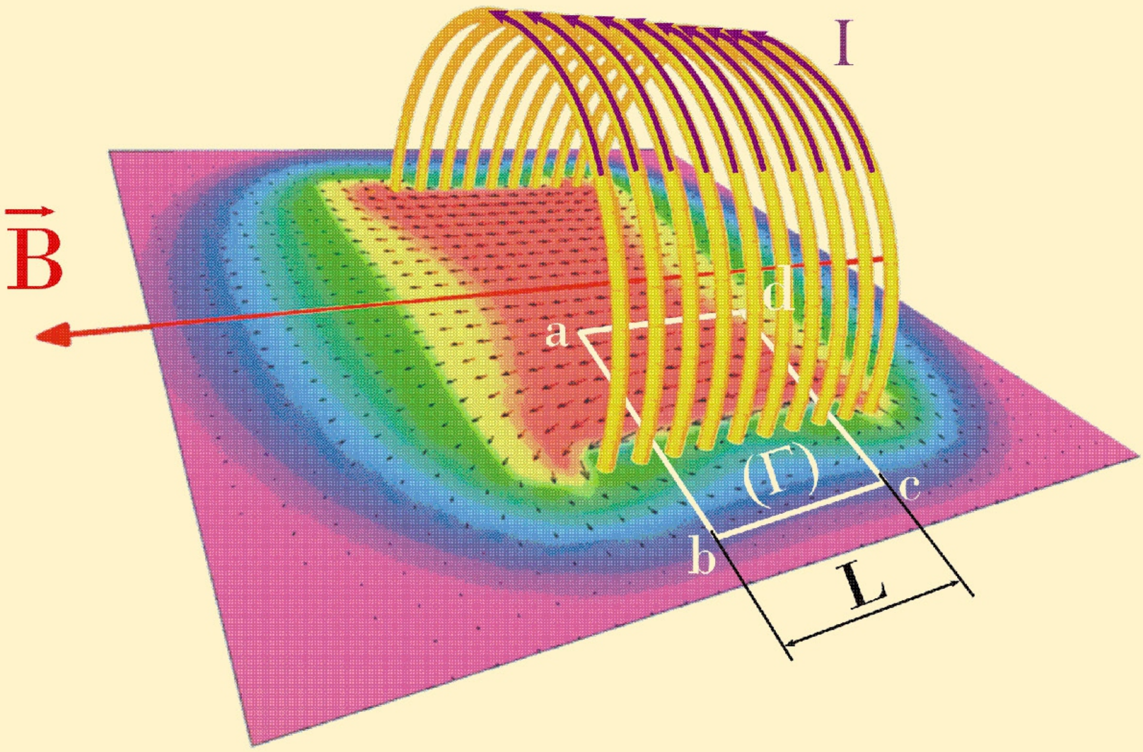
Simona C. Nicoara Manuela C. Badea Radu Fechete

Problems and Applications of PHYSICS for Students of Engineering



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CONTENTS

INTRODUCTION

This collection of problems and applications meant to be studied at the Physics seminars covers a broad range of Physics chapters related to the main subjects usually taught to the students of engineering in university courses. The problems are grouped by themes, although it is difficult to actually completely separate the physical processes, and often phenomena studied in one chapter may very well relate to another chapter, such as for example, the photoelectric effect that can illustrate an application of electromagnetic waves, but also of quantum physics.

Each problem is followed by its suggested solution, but in certain cases, other approaches may also be possible to solve the problem. Most of the problems settings are illustrated by a corresponding figure, aiming to clarify the conditions and the symbols considered in the text.

Wherever it was possible, the figures are also provided with a link to their dynamic representations, in order to help the student to better understand and imagine the physical process described in the problem. The figures numbers and the links to their corresponding numerical simulations (☘) are listed at the end of the book.

The authors encourage the students to review the theoretical aspects of the Physics courses and they hope that by solving these problems, the students become more familiar with calculating unknown quantities in hypothesized practical examples. This will support them to better clarify and correlate the principles of various chapters of physics, it will stimulate their learning curiosity to get a closer grip to the joy of understanding the physical phenomena and their engineering applications that affect our lives in so many aspects.

1. PHYSICAL QUANTITIES AND UNITS

The phenomena and processes that we experience or witness everyday are driven by the objective laws of science. These laws operate with equations that interconnect the physical quantities involved in the respective processes. A physical quantity is a measurable property of a physical system (e.g. a car, a train, a building, a grain of sand, the amount of gas from a hot air balloon, an amount of liquid, a molecule, an atom, an electron, etc.), and has its own unit of measurement. The physical quantities can be classified based on different criteria, two of which will be considered below.

1.1 Vectors and scalars

Many quantities in physics can be assigned with a magnitude, origin (point of application) and direction. They are called *vectors*. The others only have magnitude and are called *scalars*.

A scalar is expressed by (i) magnitude (a number) and (ii) its unit of measurement.

Examples of scalars:

> length, width, temperature, pressure, density, mass, volume, energy, work, power, light intensity, amount of substance, etc. .

A vector is expressed by (i) magnitude, (ii) unit of measurement, (iii) origin (point of application) and (iv) a direction.

Examples of vectors:

> velocity ($\vec{\mathrm{v}}$), acceleration ($\vec{\mathrm{a}}$), force ($\vec{\mathrm{F}}$), linear momentum ($\vec{\mathrm{p}}$), angular momentum ($\vec{\mathrm{L}}$), momentum of a force ($\vec{\mathrm{M}}$), electric field intensity ($\vec{\mathrm{E}}$), electric induction ($\vec{\mathrm{D}}$), magnetic field intensity ($\vec{\mathrm{H}}$), magnetic field induction ($\vec{\mathrm{B}}$), etc. .

A vector (Figure 1) has the following characteristics: origin (A), arrow (B), magnitude (V), and a versor or a unit vector ($\vec{\mathrm{v}}$). In Figure 1, the vector has a magnitude of 10 units:

$$\overrightarrow{\mathrm{AB}} = \vec{V} = V \cdot \vec{\mathrm{v}} = 10\,\vec{\mathrm{v}}.$$

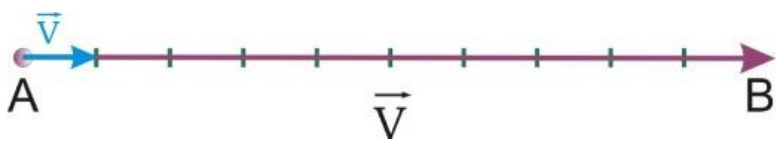


Fig. 1. A vector quantity: $\vec{V}$ with magnitude (V), unit vector ($\vec{v}$) and direction from A (origin) to B (arrow head). The unit vector is also called a versor, of magnitude equal to 1 unit.

1.2 Fundamental quantities and units

There are 7 physical quantities (having their own units of measurement) that have been selected by the scientific community as *fundamental* or *base* physical quantities (i.e. they are not defined by mathematical equations). These fundamental quantities cover all the chapters of science and of phenomena. All the other are called *derived physical quantities* (and their units), they can be defined and deduced from the base quantities, based on mathematical equations. The 7 fundamental quantities are presented in Table 1. The international system of physical quantities and units, abbreviated SI - from the French: Système International (d' Unitées) - is also called the 'LMT' (length, mass, time) or 'MKS' (metre, kilogram, second). By international convention, the main SI units of fundamental quantities are defined as:

Table 1. Fundamental physical quantities and their coherent units in the International System (SI).

Fundamental or base quantity	Quantity symbols	Coherent unit	Unit symbol
Length	L, ℓ	meter	m
Mass	M, m	kilogram	kg
Time	T, t	second	s
Absolute temperature	T_a	kelvin	K
Amount of substance, number of moles	ν	mole	Mole
Intensity of electric current	I	ampere	A
Light intensity	I_{cd}	candela	cd

The SI unit of length: $[L]_{SI} = 1\ \text{metre}$

- $1\ \text{m} = \frac{1}{10^7}$ of the distance between the North pole to Equator.
- $1\ \text{m} = 1\,650\,763.73\ \lambda_{\text{Kr}}$, where $\lambda_{\text{Kr}} =$ wavelength of the orange-red light emitted by krypton $^{86}_{36}\text{Kr}$ gas at low pressure, excited under high voltage.
- $1\ \text{m} = \text{c}\frac{1}{299\,792\,458}\ \text{s}$ = is the distance that light travels in approximately $\frac{1}{3} \times 10^{-8}$ seconds.

The SI unit of time: $[\text{t}]_{\text{SI}} = 1\ \text{second}$

- $1\ \text{s} = \frac{1}{86\,400}$ of a mean solar day, the average duration of the day over one year.
- $1\ \text{s} = 9\,192\,631\,770\ \nu_{\text{Cs}} = 9.2 \times 10^9 \nu_{\text{Cs}}$, where ν_{Cs} is the frequency of vibration of the $^{133}_{55}\text{Cs}$ atom.

The SI unit of mass: $[\mathrm{m}]_{\mathrm{SI}} = 1$ kilogram.

One kilogram ($1\ \mathrm{kg}$) is the mass of the *International Prototype Kilogram* (IPK), a cylinder of 3.9 cm high × 3.9 cm diameter, made of a platinum (90 %) - iridium (10 %) alloy preserved under special conditions of temperature and pressure, isolated under vacuumed glass jars (see Figure 2).

A more accurate definition of the fundamental unit of mass has recently (2017 - 2018) been proposed by the scientific community. The new definition eliminates the need of the kilogram artefact IPK, as it is based on assigning a fixed value to the Planck's constant **h**, given that its SI unit is: $[\mathrm{h}]_{\mathrm{SI}} = 1\ \mathrm{J{\cdot}s} = 1\ \mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}}$. Defining the *kilogram* based on the natural constant **h** requires the use of a Kibble balance[1], named after the British physicist *Bryan Kibble*. This balance is a complex equipment that allows precise mass weighing through the use of electrical measurements. The idea that led to the 'electronic' or 'electric' kilogram is that, if a Kibble balance could use a defined mass to measure the unknown value of **h**, then the process could also be reversed: by setting an exact fixed value of Plank's constant $\mathrm{h} = 6.626070150 \times 10^{-34}\ \mathrm{kg}\frac{m^2}{\mathrm{s}}$, previously determined by other accurate experiments, the Kibble equipment could be used to measure exactly an unknown mass.

In the metric system of units SI, the physical quantities have defined coherent (or principal, main) units, as well as their *multiples* and *submultiples*, named with prefixes and defined using the powers of ten, as shown in Table 2. The only exception among the base quantities and units is mass, with its coherent unit considered the kilogram (i.e. with a prefix), a choice based on historic reasons.

Table 2. Multiples and submultiples of the coherent (or principal) units of physical quantities.

Multiples			Submultiples		
Prefix	*Symbol*	*Power of ten*	*Prefix*	*Symbol*	*Power of ten*
deca	da	10^{1}	deci	d	10^{-1}
hecto	h	10^{2}	centi	c	10^{-2}
kilo	k	10^{3}	milli	m	10^{-3}
mega	M	10^{6}	micro	µ	10^{-6}
giga	G	10^{9}	nano	n	10^{-9}
terra	T	10^{12}	pico	p	10^{-12}
peta	P	10^{15}	femto	f	10^{-15}
exa	E	10^{18}	atto	a	10^{-18}

[1] https://en.wikipedia.org/wiki/Kibble_balance#/media/File:NIST-4_Kibble_balance.jpg

Fig. 2. Image of the IPK (credit: https://www.nist.gov/si-redefinition/kilogram-past).

In the followings, we give some examples of physical quantities that illustrate the sizes of objects with various orders of magnitude:

Length

- 1 km = 1000 m;
- 1 LY = 1 light year = the distance travelled by light (at speed v ≅ 3 x 10^8 m/s) during a time interval of one year;
- 1 AU (astronomical unit) is the average distance Earth-Sun;
- 1 cm = approximate diameter of a human finger;
- 1 mm = diameter of a pencil lead, or of a ball-point pen;
- 1 µm = 1/7 of the diameter of a human red blood cell;
- 1 nm = diameter of an atom.

Mass

- 60 - 100 kg = typical mass range of human body;
- 1 kg = average mass of one medium size pineapple fruit;
- 100 g = half of a package of butter;
- 1 g = mass of a 5 cents coin;
- 1 mg = mass of an average raindrop;
- 1 µg = mass of a small grain of sand.

Time

- 1 s = average time between two pulses for a person when relaxing, at rest;
- 1 ms = 1/14 of the time interval between the human heart beats (the heart beats do not have the same pace as the pulse detected at one's wrist);
- 1 µs = the time interval required for a bullet fired from a gun, to travel along 1 mm distance;
- 1 ns = the time interval for light to travel along 30 cm (at the speed of light $\mathrm{v} \cong 3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}$).

Other systems of units and conversion factors

The metric system has been adopted by the wide scientific community worldwide. Though, in different technical fields or different countries, other units are traditionally in official and popular use, and some examples of conversion factors between SI units and units of other systems, are shown in the following:

Length

- $1\ \text{Å}\ (1\ \text{Ångstrom}) = 10^{-10}\ \text{m}$;
- $1\ \text{inch} = 1\ \text{in} \cong 2.54\ \text{cm}$;
- $1\ \text{foot} = 1\ \text{ft} \cong 0.3\ \text{m}$;
- $1\ \text{mile (on land)} = 1\ \text{mi} = 5280\ \text{ft} \cong 1.609\ \text{km}$;
- $1\ \text{mile (nautical)} = 1\ \text{mi} \cong 1852\ \text{m}\ \left(\text{at sea, where: 1 knot} = 1\ \frac{\text{nautical mile}}{\text{hour}}\right)$;
- $1\ \text{light year} = 9.461\times10^{15}\ \text{m}$ = distance travelled during one year at the speed of light;
- $1\ \text{AU (astronomical unit)} = 1.496\times10^{11}\ \text{m} \cong 150\times10^{6}\ \text{km}$ = average distance from the Earth to the Sun;
- $1\ \text{pc (parsec) is } 3.26\ \text{light years} = 3.086\times10^{13}\ \text{km} \cong 206\ 264.81\ \text{AU}$.

Volume

- $1\ \text{litre} = 1\ \ell = 10^{-3}\ \text{m}^3 = 1\ \text{dm}^3 = 1000\ \text{cm}^3 = 1000\ \text{m}\ell$;
- $1\ \text{ft}^3 = 0.0283\ \text{m}^3$;
- $1\ \text{fluid ounce} = 1\ \text{fl oz} = \begin{cases} \frac{1}{20}\ \text{of a UK (imperial) pint} \cong 0.028\ \ell \\ \frac{1}{16}\ \text{of a US pint} \cong 0.03\ \ell \end{cases}$;
- $1\ \text{pint} = \begin{cases} \frac{1}{8}\ \text{of an Imperial gallon} \cong 528.3\ \text{ml} \\ \frac{1}{8}\ \text{of a US gallon} \cong 473.2\ \text{ml} \end{cases}$;
- $1\ \text{gallon} = 1\ \text{gal} \cong \begin{cases} 4.546\ \ell\ \text{(Imperial)} \\ 3.785\ \ell\ \text{(US)}. \end{cases}$;

Time

- $1\ \text{min} = 60\ s$;
- $1\ \text{h (hour)} = 60\ \text{min} = 3600\ \text{s}$;
- $1\ \text{day} = 24\ \text{h} = 1440\ \text{min} = 86.4\times10^{6}\ \text{s} = 8.64\times10^{7}\ \text{s}$.

Mass

- $1\ \text{amu (atomic mass unit)} \cong 1.66\cdot10^{-27}\ \text{kg}$;
- $1\ \text{oz (ounce)} \cong 28.3\ \text{g (commonly)}$;
- $1\ \text{lb (pound)} \cong 0.454\ \text{kg}$;
- $1\ \text{stone} = 14\ \text{lbs} \cong 6.035\ \text{kg}$;
- $1\ \text{slug} \cong 32.1740\ \text{lb} \cong 14.593\ \text{kg}$.

Speed

- $1\,\frac{\text{km}}{\text{h}} = \frac{1000\text{ m}}{3600\text{ s}} = 0.277(7)\,\frac{\text{m}}{\text{s}} \cong 27.8\,\frac{\text{cm}}{\text{s}}$;
- $1\,\frac{\text{mi}}{\text{h}} = \frac{1609\text{ m}}{3600\text{ s}} = 0.44694(4)\,\frac{\text{m}}{\text{s}} \cong 44.7\,\frac{\text{cm}}{\text{s}}$;
- $1\text{ knot} = \frac{1\text{ nautical mile}}{\text{h}} = \frac{1852\text{ m}}{3600\text{ s}} = 0.514(4)\,\frac{\text{m}}{\text{s}} \cong 51.4\,\frac{\text{cm}}{\text{s}}$.

Power

- $1\text{ W} = 1\,\frac{\text{J}}{\text{s}}$.
- $1\text{ horse power} = 1\text{ HP} \cong 745.7\text{ W}$.

Pressure

- $1\text{ Pa} = 1\,\frac{\text{N}}{\text{m}^2}$;
- $1\text{ bar} = 10^5\text{ Pa} = 10^5\,\frac{\text{N}}{\text{m}^2}$;
- $1\text{ pound force per square inch} = 1\text{ psi} = 1\,\frac{\text{lbf}}{\text{in}^2} \cong 6900\text{ Pa}$;
- $1\text{ torr} = 1\text{ mm Hg} = \rho_{\text{Hg}}\cdot 1\text{ mm}\cdot\text{g} \cong 133.3\,\frac{\text{N}}{\text{m}^2}$; where $\text{g} \cong 9.81\,\frac{\text{m}}{\text{s}^2}$;
- $1\text{ atm} = 760\text{ mm} \cong 1.013\times 10^5\,\frac{\text{N}}{\text{m}^2} \cong 1013\text{ bar} \cong 11\text{ psi}$;
- $1\text{ mbar} = 1\text{ milli bar} = 10^{-3}\text{ bar} = 100\,\frac{\text{N}}{\text{m}^2}$.

Force

- $1\text{ kg force} = 1\text{ kg f} = \text{weight of a 1 kg mass} \cong 9.81\text{ N}$, where g is considered $9.81\,\frac{\text{m}}{\text{s}^2}$;
- $1\text{ pound force} = 1\text{ lbf} = \text{weight of a 1 lb mass} \cong 4.448\text{ N}$;
- $1\text{ dyne} = 10^{-5}\text{ N} = 10\ \mu\text{N} =$ a unit of force that, acting on a mass of one gram, increases its velocity by one centimetre per second, every second along the direction.
- $1\text{ N} = 10^5\text{ dynes} \cong 0.2248\text{ lbf}$.

Work and energy

- $1\text{ J} = 1\text{ N}\cdot 1\text{ m}$;
- $1\text{ cal} \cong 4.185\text{ J}$;
- $1\text{ kcal} \cong 4185\text{ J}$;
- $1\text{ Btu} = 1\text{ British thermal unit} \cong 1055\text{ J}$;
- $1\text{ kWh} = 3.6\times 10^6\text{ J}$;
- $1\text{ e}\cdot\text{V} = 1.6\times 10^{-19}\text{ J}$;
- $1\text{ erg} = 10^{-7}\text{ J}$.

2. DIMENSIONAL ANALYSIS

The physical quantities also have a characteristic called ‘dimension’, symbolized with square brackets. Determining the dimension of a physical quantity is not the same thing as finding its unit of measurement. There are 7 dimensions identified in uppercase letters, corresponding to the 7 fundamental physical quantities: $[\ell]$ = L (length), [m] = M (mass), [t] = T (time), $[T_a] = T_a$ (absolute temperature), [I] = I (intensity of electric current), $[I_{cd}] = I_{cd}$ (light intensity). The derived quantities have combined dimensions, as given by their mathematical definition. Say velocity, defined as the distance travelled during the unit time, has the dimension of length divided by time: $[v] = \frac{[L]}{[T]} = \frac{L}{T} = L\,T^{-1}$. The method of dimensional analysis has interesting applications and can also be useful in finding empirical formulae of certain physical quantities without applying their physical definitions, or to check the validity of certain equations resulted from empirical calculations.

Problem 1

Apply the dimensional analysis to the following physical quantities: force (F), mass density (or *specific gravity*) (ρ), mechanical work (W), electric charge (Q), electric potential (V), electric resistance (R), electric capacitance (C), Young’s modulus of elasticity (Y).

Solution

Force	$F = m{\cdot}a$	$[F] = [m]{\cdot}[a] = [m]\,\frac{[\text{distance}]}{[\Delta t]} = M\frac{L}{T} = M\,L\,T^{-2};$
Density	$\rho = \frac{m}{V}$	$[\rho] = \frac{[m]}{[V]} = \frac{M}{L^3} = M\,L^{-3};$
Work	$W = F{\cdot}d$	$[W] = [F]{\cdot}[\text{distance}] = M\,L\,T^{-2}\,L = M\,L^2\,T^{-2};$
Power	$P = \frac{W}{\Delta t}$	$[P] = \frac{[W]}{[\Delta t]} = \frac{M\,L^2\,T^{-2}}{T} = M\,L^2\,T^{-3};$
Electric charge	$Q = I{\cdot}\Delta t$	$[Q] = [I]{\cdot}[\Delta t] = I\,T;$
Electric potential	$V = \frac{W}{Q}$	$[V] = \frac{[W]}{[Q]} = \frac{ML^2T^{-2}}{I\,T} = M\,L^2\,I^{-1}\,T^{-3};$
Electric resistance	$R = \frac{U}{I} = \frac{\Delta V}{I}$	$[R] = \frac{[\Delta V]}{[I]} = \frac{[V]}{[I]} = \frac{M\,L^2\,T^{-3}\,I^{-1}}{I} = M\,L^2\,T^{-3}\,I^{-2};$
Electric capacitance	$C = \frac{Q}{U} = \frac{Q}{\Delta V}$	$[C] = \frac{[Q]}{[\Delta V]} = \frac{I\,T}{M\,L^2T^{-3}I^{-1}} = M^1L^{-2}T^4I^2;$
Young’s modulus	$Y = \frac{\sigma}{\varepsilon}$	$[Y] = \frac{[\text{stress}]}{[\text{strain}]} = \frac{[\sigma]}{[\varepsilon]} = \frac{\frac{[F]}{[S]}}{\frac{[\Delta l]}{[l_0]}} = \frac{[F]}{[S]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^2.$

Problem 2

Apply dimensional analysis to deduce an empirical equation (up to a dimensionless constant) for determining the period of oscillation T of a gravitational pendulum: a body of mass **m**, hanging on a wire of length ℓ, in the gravitational field of gravitational acceleration **g**.

Solution

Assume that the period of oscillation of the pendulum, T, depends on its mass (m), length (ℓ) and gravitational acceleration (g). We might also assume that the environmental temperature (θ, in ° C) has an influence on the period of oscillation:

$$\mathrm{T} = \mathrm{T}(\mathrm{m}, \ell, \mathrm{g}, \theta).$$

Assume that the period T is given by an equation that considers a proportional constant (k), and the named physical quantities raised at powers α, β, γ, and μ:

$$\mathrm{T} = \mathrm{k} \cdot \mathrm{m}^{\alpha}\, \ell^{\beta}\, \mathrm{g}^{\gamma}\, \theta^{\mu}.$$

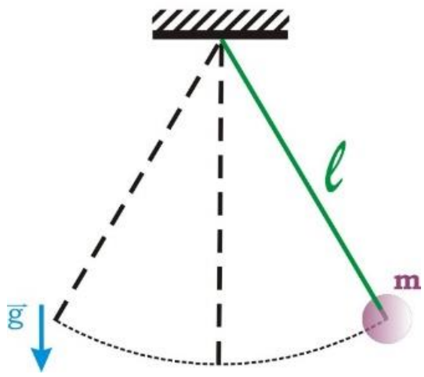


Fig. 3. A gravitational pendulum.

The constant k is dimensionless: [k] = 1. The dimension of the period T should be that of time: [T] = T, and applying dimensional analysis to the above equation, one obtains:

$$[\mathrm{T}] = [\mathrm{k}] \cdot [\mathrm{m}^{\alpha}] \cdot [\ell^{\beta}] \cdot [\mathrm{g}^{\gamma}] \cdot [\theta^{\mu}]\,.$$

Further, substituting the dimension of each quantity, one gets to:

$$[\mathrm{T}] = 1 \cdot \mathrm{M}^{\alpha}\, \mathrm{L}^{\beta} (\mathrm{L}\, \mathrm{T}^{-2})^{\gamma} \theta^{\mu} = \mathrm{M}^{\alpha} \mathrm{L}^{\beta+\gamma} \mathrm{T}^{-2\gamma} \theta^{\mu},$$

since the dimension of the period T is that of time, [T] = T^1, and adding the rest of dimensions at 0:

$$\mathrm{T}^1 = \mathrm{M}^{\alpha} \mathrm{L}^{\beta+\gamma} \mathrm{T}^{-2\gamma} \theta^{\mu} \iff \mathrm{M}^0 \mathrm{L}^0\, \mathrm{T}^1 \theta^0 = \mathrm{M}^{\alpha} \mathrm{L}^{\beta+\gamma} \mathrm{T}^{-2\gamma} \theta^{\mu}$$

And thus, the exponents must fulfil the conditions: $\begin{cases} \alpha = 0, \\ \beta + \gamma = 0, \\ -2\gamma = 1, \\ \mu = 0. \end{cases}$

therefore: $\gamma = -\frac{1}{2}$ and $\beta = \frac{1}{2}$, and the empirical equation of the period of oscillation is: $\mathbf{T} = \mathrm{k}\sqrt{\frac{\ell}{\mathrm{g}}}$.

Problem 3

Find the equation of the power (up to a dimensionless constant) generated by a windmill, if its fan blades have the total surface area **S**, the wind blows at speed **v**, and the air has density **ρ**. Use dimensional analysis.

Solution

Assume that the power generated depends on the surface of the blades exposed to the air current, the wind speed and the air density: $P = P(S, v, \rho)$, through an equation: $P = k\, v^{\alpha} S^{\beta} \rho^{\mu}$, where k is a proportionality dimensionless constant, [k] = 1. Dimensional analysis of the empirical equation leads to:

$$[P] = [k][v]^{\alpha}[S]^{\beta}[\rho]^{\mu} = 1 \cdot (LT^{-1})^{\alpha}(L^{2})^{\beta}(ML^{-3})^{\mu},$$

$$[P] = L^{\alpha+2\beta-3\mu}T^{-\alpha}M^{\mu}.$$

Considering the dimension of power deduced in Problem 1 ($[P] = L^2\, T^{-3} M^1$), one obtains:

$$[P] = L^2T^{-3}M^1 = L^{\alpha+2\beta-3\mu}T^{-\alpha}M^{\mu}.$$

Equating the exponents, one gets:

$$\begin{cases} \mu = 1 \\ \alpha = 3 \\ \alpha + 2\,\beta - 3\,\mu\,\mu = 2 \end{cases} \quad \Rightarrow \quad \beta = 1.$$

Hence, the power generated by the wind turbine becomes: $P = k \cdot v^3 S\, g$, and thus we have shown that it depends on the power 3 of the wind speed, **v**.

Problem 4

Apply the method of dimensional analysis to find an empirical equation for calculating the time interval required for a capacitor of capacitance **C** loaded with the electrical charge $\mathbf{Q_0}$, to discharge across a resistor **R**.

Solution

Assume that the discharge time (**τ**) depends on the initial amount of charge, on the capacitor's capacitance, and on the resistor, hence it is given by the equation: $\tau = \tau\,(Q_0, C, R) = k\, C^{\alpha}\, {Q_0}^{\beta} R^{\gamma}$. Thus, dimensional analysis leads to:

$$[\tau] = [k][C^{\alpha}]\left[{Q_0}^{\beta}\right][R^{\gamma}] = 1 \cdot [C]^{\alpha}[Q_0]^{\beta}[R]^{\gamma},$$

$$[\tau] = (M^{-1}L^{-2}T^{4}I^{2})^{\alpha}(I\,T)^{\beta}(ML^{2}T^{-3}I^{-2})^{\gamma},$$

$$[\tau] = M^{-\alpha+\gamma}L^{-2\alpha+2\gamma}\,T^{4\alpha+\beta-3\gamma}\,I^{2\alpha+\beta+2\gamma}.$$

Given that the discharge time has the dimension of time [τ] = T, the coefficients α, β, γ can be calculated from the exponents of the physical quantities involved:

$$\begin{cases} -\alpha+\gamma = 0 \\ -2\alpha+2\gamma = 0 \\ 4\alpha+\beta-3\gamma = 1 \\ 2\alpha+\beta-2\gamma = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \gamma \\ 4\alpha+\beta-3\gamma = 1 \\ 2\alpha+\beta-2\gamma = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \gamma \\ \alpha+\beta = 1 \\ \beta = 0 \end{cases}.$$

Solving the above system of equations leads to the exponents:

$$\begin{cases}\alpha = 1\\ \beta = 0.\\ \gamma = 1\end{cases}$$

Therefore, the discharging of the capacitor takes place over the time interval:

$$\tau = k\,C\,R,$$

thus, based on dimensional analysis one can deduce that the discharge time of the capacitor does not depend on the total initial charge loaded on the capacitor's plates, Q_0.

Problem 5

Apply dimensional analysis in each member of the following equations to determine if they are consistent (Yes/No ?).

Solution

a) $v^2 = a\,t \quad \Rightarrow \quad [v^2] = L^2T^{-2} \neq [a\,t] = LT^{-1} \quad \Rightarrow$ No

b) $x = v\,t + \frac{at^2}{2} \quad \Rightarrow \quad [vt] + \left[\frac{at^2}{2}\right] = LT^{-1}T + LT^{-2}T^2 = [L+L] = L = [x] \quad \Rightarrow$ Yes

c) $m{\cdot}a{\cdot}x + \frac{mv^2}{2} = F{\cdot}x^2 \quad \Rightarrow [m{\cdot}a{\cdot}x] + \left[\frac{m\,v^2}{2}\right] = M\,L^2T^{-2} \neq [Fx^2] = ML^3T^{-1} \quad \Rightarrow$ No

d) $v^2 = 2{\cdot}a{\cdot}x + \frac{x}{t} \quad \Rightarrow [v^2] = L^2T^{-2} \neq [2ax] + \left[\frac{x}{t}\right] = L^2T^{-2} + LT^{-1} \quad \Rightarrow$ No

e) $v^2 = \frac{F{\cdot}x}{m} \quad \Rightarrow [v^2] = L^2T^{-2} = \left[\frac{F{\cdot}x}{m}\right] = \frac{MLT^{-2}L}{M} = L^2T^{-2} \quad \Rightarrow$ Yes

3. VECTORS

Problem 6

The positions of two points P_1 and P_2 in a 3D system of coordinates are given by the position vectors:

$$\vec{r_1} = 2\vec{i}+5\vec{j}\ (\mathrm{m}) \text{ and } \vec{r_2} = 2\vec{j}+3\vec{k}\ (\mathrm{m}).$$

Answer the following tasks:

a) Draw a diagram of the two vectors in the (x, y, z) system of spatial coordinates.
b) Calculate the magnitude of the segments OP_1 and OP_2.
c) Determine the distance between the points P_1 and P_2 using the difference between the two vectors: $\Delta\vec{r} = \vec{r_2} - \vec{r_1}$. Compare to the difference $\vec{r_1} - \vec{r_2}$.
d) What is $\cos\alpha = ?$ If α is the angle between the two position vectors.
e) Determine the product vectors $\vec{R} = \vec{r_1}\times\vec{r_2} = ?$ and $\vec{R}' = \vec{r_2}\times\vec{r_1} = ?$
f) What is the area of the triangle ΔOP_1P_2?
g) Determine the unit vector of the direction perpendicular to the plane of the triangle ΔOP_1P_2.

Solution

a) The vectors diagram is shown in Figure 4.

b) The magnitude can be calculated using the relationship:

$$r = \sqrt{\vec{r}^2} = \sqrt{x^2 + y^2 + z^2}.$$

In particular one have:

$$OP_1 = r_1 = \sqrt{4+25} = \sqrt{29}\ (\mathrm{m}),$$

$$OP_2 = r_2 = \sqrt{4+9} = \sqrt{13}\ (\mathrm{m}).$$

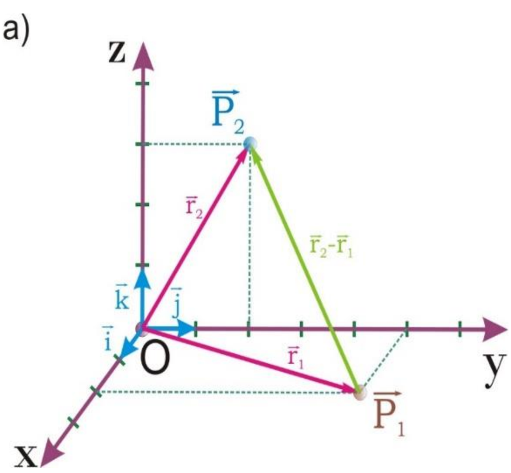


Fig. 4. The vectors $\vec{r_1}$ and $\vec{r_2}$ in Problem 6.

c) $\overrightarrow{P_1P_2} = \vec{r_2} - \vec{r_1} = \Delta\vec{r}$.

The magnitude of the vector $\Delta\vec{r}$ is:

$$P_1P_2 = |\vec{r_2} - \vec{r_1}|.$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2+(y_2 - y_1)^2+(z_2 - z_1)^2}$$

$$P_1P_2 = \sqrt{(0-2)^2+(2-5)^2+(3-0)^2}$$

$$P_1P_2 = \sqrt{(-2)^2+(-3)^2+(3)^2} \Longrightarrow$$

$$P_1P_2 = \sqrt{22} \Longrightarrow P_1P_2 \cong 4.69\ (\mathrm{m}).$$

d) $$\cos(\alpha) = \frac{\vec{r_1}\cdot\vec{r_2}}{r_1\, r_2} = \frac{(2\vec{i}+5\vec{j})(2\vec{j}+3\vec{k})}{\sqrt{29}\cdot\sqrt{13}} = \frac{10}{\sqrt{29}\cdot\sqrt{13}} \Longrightarrow \cos(\alpha) \cong 0.515.$$

e) $\vec{R} = \vec{r_1} \times \vec{r_2} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2(m) & 5(m) & 0(m) \\ 0(m) & 2(m) & 3(m) \end{pmatrix} = 15(m^2)\vec{i} - 6(m^2)\vec{j} + 4(m^2)\vec{k}.$

The absolute value of the vector product $\vec{r_1} \times \vec{r_2}$ is:

$$R = \sqrt{225\ m^4 + 36\ m^4 + 16 m^4} = \sqrt{277 m^4} \Longrightarrow R \cong 16.64\ m^2.$$

$$\vec{R'} = \vec{r_2} \times \vec{r_1} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 3 \\ 2 & 5 & 0 \end{pmatrix} = -15\vec{i} + 6\vec{j} - 4\vec{k} = -\vec{R}.$$

The absolute value of the vector product $\vec{r_2} \times \vec{r_1}$ is: R' = R.

f) The area of the triangle OP_1P_2 is:

$$A_{\Delta OP_1P_2} = \frac{1}{2}|\vec{r_1} \times \vec{r_2}| = \frac{\sqrt{277}}{2} \Longrightarrow A_{\Delta OP_1P_2} \cong 8.32\ (m^2).$$

g) The unit vector of the direction perpendicular to the plane of the OP_1P_2 triangle is defined as:

$$\vec{n} = \frac{\vec{r_1} \times \vec{r_2}}{|\vec{r_1} \times \vec{r_2}|} = n_x\,\vec{i} + n_y\,\vec{j} + n_z\,\vec{k},$$

then,

$$\vec{n} = \frac{15\vec{i} - 6\vec{j} + 4\vec{k}}{\sqrt{277}} \Longrightarrow \vec{n} \cong 0.901\,\vec{i} - 0.361\,\vec{j} + 0.240\,\vec{k}\,.$$

Check that the result above represents a unit vector (hint: calculate the squared vector. The result should be equal to unity.)

$$\vec{n}^2 = \frac{15^2 + 6^2 + 4^2}{277} = 1.$$

4. KINEMATICS

Problem 7

The equation of motion for a moving object is: $s(t) = A+B\cdot t+C\cdot t^2+D\cdot t^3$. Find: a) its absolute velocity (v(t) = ?) and b) acceleration (a(t) = ?). c) What do the coefficients A, B, C and D represent?

Solution

a) The velocity is the first derivative of space relative to time:

$$v(t) = \frac{ds}{dt} = B + 2Ct + 3Dt^2.$$

Therefore: if in the above result one considers t = 0, then the initial velocity is:

$$v(0) = B = v_0 \quad \Rightarrow \quad \text{B is the initial velocity.}$$

b) The acceleration is the first derivative of velocity with respect to time:

$$a(t) = \frac{dv}{dt} = 2C + 6Dt.$$

If one substitutes t = 0 in the above equation, this results in: a(0) = 2C $\Rightarrow C = \frac{a_0}{2}$, i.e. the coefficient C is half of the initial acceleration a_0.

c) In the equation of motion (of the space travelled), $s(t) = A+B\cdot t+C\cdot t^2+D\cdot t^3$,

one substitutes t = 0, and thus: $s(0) = A \quad \Rightarrow \quad A = s_0$, therefore, the coefficient A represents the initial coordinate s_0.

To deduce the meaning of the D coefficient, one calculates the derivative of acceleration relative to time:

$\varepsilon(t) = \frac{da}{dt} = 6D = \text{const.} \Rightarrow$ the coefficient D represents the rate at which the acceleration changes in time, and this rate is constant for this object.

Problem 8

The equation of motion of a moving object is:

$$\vec{r}(t) = 2\cos(\pi t)\,\vec{i} + 2\sin(\pi t)\,\vec{j} + t^2\vec{k} \quad \text{(m)}.$$

Determine:

a) The position vector at $t_1 = 0$ and at $t_2 = 1$ s.

b) The average velocity ($\vec{v}_{av} = ?$) between t_1 and t_2.

c) The instant velocity $\vec{v} = \vec{v}\,(t) = ?$

d) What is the position vector and the instant velocity at t_3 = 0.5 s?
e) What is the acceleration of the moving body?
f) Describe the trajectory of this object.

Solution

a) At t_1 =0, the position vector is: $\vec{r_1}(0)=2\vec{i}+0\,\vec{j}+0\,\vec{k}=2\vec{i}\,(\mathrm{m})$.
At t_2 = 1 s, the position vector is:

$$\vec{r_2}(1)=2\cos(\pi)\,\vec{i}+2\sin(\pi)\,\vec{j}+\vec{k}=2\,(-1)\vec{i}+2\cdot 0\vec{j}+\vec{k}=-\,2\vec{i}+\vec{k}\ \ (\mathrm{m}).$$

b) The average velocity is the vector quantity defined as:

$$\vec{v}_{av}=\frac{\Delta\vec{r}}{\Delta t}=\frac{\vec{r_2}-\vec{r_1}}{t_2-t_1}=\frac{-2\vec{i}+\vec{k}-2\vec{i}}{1}\Longrightarrow\vec{v}_{av}=\ -4\vec{i}+\vec{k}\left(\frac{m}{s}\right).$$

The modulus of the average velocity is then:

$$v_{av}=\sqrt{17}\Longrightarrow v_{av}\ \cong 4.123\left(\frac{m}{s}\right).$$

c) The instant velocity is the derivative of the position vector relative to time:

$$\vec{v}(t)=\frac{d\vec{r}}{dt}=-\,2\pi\sin(\pi t)\vec{i}+2\pi\cos(\pi t)\vec{j}+2t\vec{k}\ \ \left(\frac{m}{s}\right).$$

d) For t_3= 0.5 s, one obtains:

- The position vector:

$$\vec{r}_3=\vec{r}(0.5)=2\cos\left(\frac{\pi}{2}\right)\vec{i}+2\sin\left(\frac{\pi}{2}\right)\vec{j}+0.25\,\vec{k}=0+2\cdot 1\cdot\vec{j}\ +0.25\ \vec{k}$$
$$=2\,\vec{j}+0.25\ \vec{k}\ \ (\mathrm{m})$$

With its absolute value: $r_3=\ \sqrt{(4+0.0625)(m)^2}\cong 2.015\ m$

- The instant velocity:

$$\vec{v_3}=\vec{v}\left(\frac{1}{2}\right)=-\,2\pi\sin\left(\frac{\pi}{2}\right)\vec{i}+2\pi\cos\left(\frac{\pi}{2}\right)\vec{j}+2\frac{1}{2}\vec{k}\Longrightarrow\vec{v_3}=-\,2\pi\,\vec{i}+\vec{k}\ \left(\frac{m}{s}\right).$$

With its absolute value:

$$v_3=v\left(\frac{1}{2}\right)=\sqrt{4\pi^2+1}\Longrightarrow v_3\cong 6.362\left(\frac{m}{s}\right)$$

e) The instant acceleration is the derivative of velocity relative to time:

$$\vec{a}(t)=\frac{d\vec{v}}{dt}=-\,2\pi^2\cos(\pi t)\vec{i}-2\pi^2\sin(\pi t)\vec{j}+2\vec{k}\ \left(\frac{m}{s^2}\right).$$

Its absolute value is then:

$$a(t)=\sqrt{4\pi^4[\cos^2(\pi t)^2+\sin^2(\pi t)]+1}=\sqrt{4\pi^4+1}$$
$$a(t)\cong 19.765\ \left(\frac{m}{s^2}\right)\ =\mathrm{const.}$$

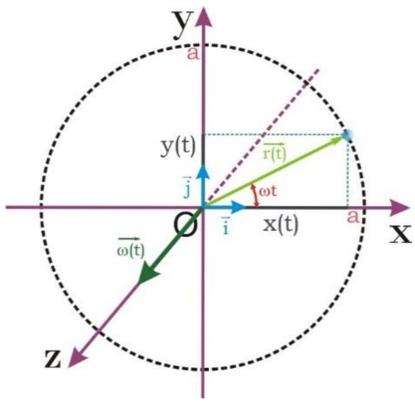


Fig. 5. The trajectory of a circular planar motion.

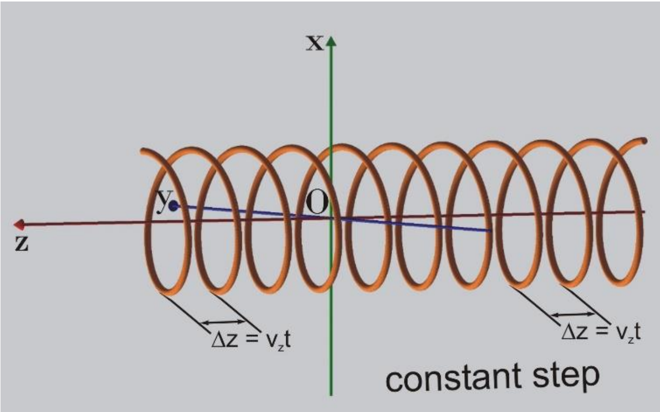


Fig. 6. Helicoidal path, constant step $\Delta z = v_z \cdot t$ [♣].

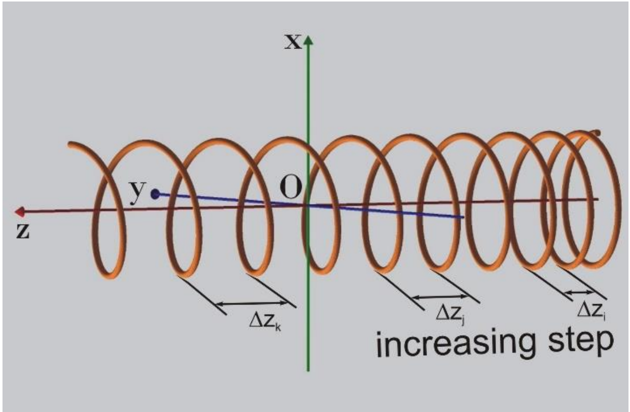


Fig. 7. Helicoidal path, increasing step [♣].

f) To deduce the trajectory of this moving object, let us first examine the equation of motion of a point describing a circle centered on the origin of the xOy system of planar coordinates (see Figure 5):

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} = a\cos(\omega t) + a\sin(\omega t)\ (m).$$

Where: $\vec{r}(t)$ is the position vector at the moment **t**, **a** equals the radius of the circle of equation $a^2 = x^2 + y^2$, and **ω** is the angular velocity:

$$\vec{\omega} = \frac{d\alpha}{dt}\,\vec{i} \times \vec{j}\ \left(\frac{rad}{s}\right).$$

If the point describes a helicoidal path (Figure 6) with constant step $\Delta z = b{\cdot}T$, where **b** is dz/dt, the speed along Oz, and T = $2\pi/\omega$ (the period of the circular motion in the xOy plan), then its projection on the Oz axis moves at constant velocity is $v_z = b{\cdot}t\ \left(\frac{m}{s}\right)$.

Therefore, its equation of motion is:
$\vec{r}(t) = a{\cdot}\cos(\omega t)\vec{i} + a{\cdot}\sin(\omega t)\vec{j} + b{\cdot}t{\cdot}\vec{k}$, where $\vec{k}$ is the unit vector along the Oz axis.

Now looking at the equation of motion given for the mobile point that this problem deals with, its motion along the Oz axis depends on t^2, or else, it is accelerated along Oz, its speed increases with time, and thus it describes a helicoidal motion along Oz, with increasing step, as shown in Figure 7.

Problem 9

A luggage that drops from a plane in flight has the following equation of motion, with respect to an observer on the Earth, considered at rest:

$$\vec{r}(t) = 40t\,\vec{i} + (4000 - 5\,t^2)\vec{j}\ \ (m).$$

Find:

a) The equation of the trajectory;

b) The height (H) where the object was released and the position where it hits the ground (D);

c) The velocity and the acceleration of the object;

d) How long does it take for the object to hit the ground;

e) The vectorial equation of motion of the object with respect to the plan.

Solution

a) $\begin{cases} x = 40\,t \\ y = 4000 - 5t^2 \end{cases} \quad \Rightarrow \quad t = \frac{x}{40}.$

Then the equation of the trajectory is:

$$y = 4000 - 5\,\frac{x^2}{40^2} \Longrightarrow y = 4000 - \frac{5}{1600}x^2\,,$$

is that of a parabolic function:

$$y = y(x) = a + b\,x^2.$$

b) For x = 0 $\Rightarrow$ $y = a = 4000\ (\mathrm{m})$.

For y = 0 $\Rightarrow$ $x = \sqrt{\frac{-a}{b}} = \sqrt{\frac{-4000}{-\frac{5}{1600}}}$

$$x = \pm 800\,\sqrt{2} \Longrightarrow x \cong 1131.37\ (\mathrm{m}).$$

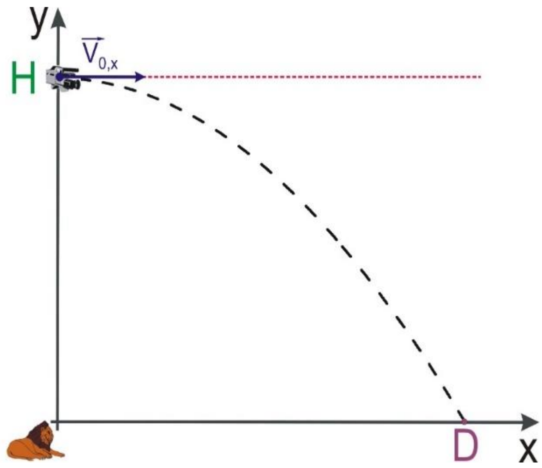


Fig. 8. An object is dropped from a plane moving at velocity $\overrightarrow{V_0}$ relative to the ground.

The height at which the object is dropped:

$$H = 4000\ \mathrm{m}.$$

The distance travelled by the object along Ox until reaching the ground:

$$D \cong 1131.37\ (\mathrm{m}).$$

c) The instant velocity of the object:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \Longrightarrow \vec{v}(t) = 40\vec{i} - 10\,t\,\vec{j}\left(\frac{m}{s}\right).$$

With the velocity components:

$$\begin{cases} v_x = 40\ \left(\frac{m}{s}\right) \\ v_y = -\,10\,t\ \left(\frac{m}{s}\right); \qquad (v_{y,0} = 0) \end{cases}.$$

The total acceleration of the falling object:

$$\vec{a} = \frac{d\vec{v}}{dt} = -\,10\,\vec{j}\ \left(\frac{m}{s^2}\right).$$

The acceleration components:

$$\begin{cases} a_x = 0 \\ a_y = -\,10\ m/s^2 \end{cases}.$$

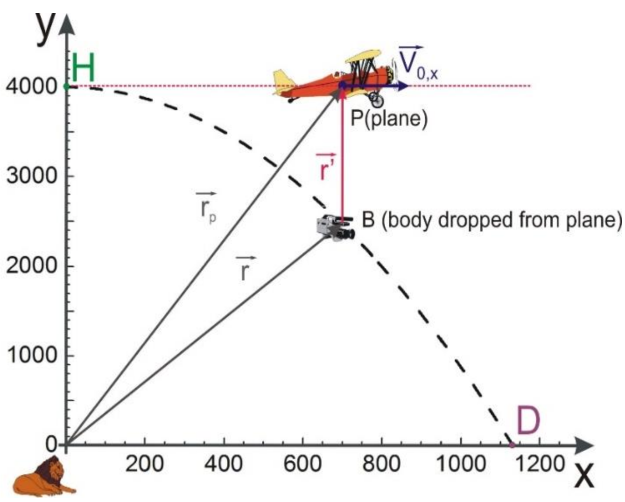


Fig. 9. Motion of the object dropped from plane, relative to the plane.

d) The duration of the fall: t = ?

For y = 0 $\Rightarrow$

$$4000 - 5\,t^2 = 0 \Rightarrow t = \sqrt{800}\ (s) \Longrightarrow$$
$$t = 20\sqrt{2} \Longrightarrow t \cong 28.28\ (s).$$

e) We consider that the resistance force between the object and air is negligible. As the object falls to the ground, it still moves horizontally with the same velocity as the plane that it was dropped from. Therefore, the object falls vertically, being always situated right under the plane (See Figure 9), if the plane continues at the same velocity as it was moving with, when the object was dropped from the plane. Then, $\vec{\mathbf{r}'}$ is the position of the body relative to the plane in motion, $\vec{\mathbf{r}}$ is the position of the body relative to the ground at rest, $\vec{\mathbf{r}_P}$ is the position of the plane relative to the ground at rest.

Then, the position vector of the object, relative to the plane, is given by $\vec{r'} = \vec{r} - \vec{r}_P$, with the terms: $\begin{cases} \vec{r}_P = 40t\,\vec{i} + 4000\,\vec{j} \\ \vec{r} = 40t\,\vec{i} + (4000 - 5t^2)\,\vec{j} \end{cases}$

$$\vec{r'} = -5\,t^2\vec{j}.$$

Problem 10

A body is in motion with the velocity components: $v_x = 4t + 4t^3$ (m/s) and $v_y = 4t$ (m/s). Find the equation of motion of the trajectory if at t = 0 its position is at the point P (1, 2).

Solution

The equation of motion along the Ox axis:

$$v_x = \frac{dx}{dt} \Rightarrow dx = v_x\,dt \Leftrightarrow \int_1^x dx' = \int_0^t (4t' + 4t'^3)dt' \Longrightarrow$$

$$x'|_1^x = \left(\frac{4t'^2}{2} + \frac{4t'^4}{4}\right)\Bigg|_0^t,$$

$$x - 1 = 2\,t^2 + t^4 \Leftrightarrow x = 1 + 2t^2 + t^4 \Rightarrow x = (1+t^2)^2.$$

The equation of motion along the Oy axis:

$$v_y = \frac{dy}{dt} \Rightarrow dy = v_y\,dt, \quad \Rightarrow \quad \int_2^y dy' = \int_0^t 4t'\,dt',$$

$$y - 2 = 2\,t^2, \quad \Rightarrow \quad y = 2 + 2\,t^2 = 2(1+t^2),$$

therefore, the equation of the trajectory can be deduced as: $x = (1+t^2)^2 = \left(\frac{y}{2}\right)^2$, and finally can be written as: $x(y) = \frac{1}{4}y^2$.

Problem 11

A body is in motion, and the equation of its velocity is: $v(t) = 2 - 2\,t + 3\,t^2$ (m/s). Determine:

a) What is its acceleration a(t) = ?
b) What is the distance travelled during the 10th second (from t_1 = 9 s to t_2 = 10 s)?

Solution

a) The instant acceleration is the derivative of speed relative to time:

$$a(t) = \frac{dv}{dt} = -2 + 6t \ \left(\frac{m}{s^2}\right).$$

b) The distance (Δx) travelled during the time $\Delta t = t_2 - t_1$ is found by integrating the instant speed:

$$\Delta x = x_2 - x_1 = \int_{t_1}^{t_2} v(t)dt.$$

For t_1 = 9 s and t_2 = 10 s, one gets the distance travelled during the 10th second, as:

$$\Delta x = x_{10} - x_9 = \int_9^{10} v(t)dt = \left(2t - 2\frac{t^2}{2} + 3\frac{t^3}{3}\right)\Bigg|_9^{10} \Longrightarrow$$

$$\Delta x = (2t - t^2 + t^3)\Big|_9^{10} = (20 - 100 + 1000) - (18 - 81 + 729) \Longrightarrow$$

$$\Delta x = 1080 - 666 = 414 \ \ (m).$$

Problem 12

A ship is tracked by the navigation GPS (global positioning system). With respect to a coordinate origin, the position of the ship is found at point A_1 of coordinates x_1 = 2 km (West) and y_1 = 1.6 km (South), at the time t_1 = 0.3 h. Later, at t_2 = 0.6 h, the ship is at position A_2, and its coordinates are: x_2 = 6.4 km (W) and y_2 = 6.5 km (N).

a) Determine its average speed in terms of its components.
b) Find the direction and the magnitude of the average velocity?

Solution

a) The average velocity components:

$$\bar{v}_x = \frac{\Delta x}{\Delta t} = \frac{(-6.4+2)\ km}{0.3\ h},$$

$$\bar{v}_x = \frac{-4.4}{0.3}\frac{km}{h} = -14.67\frac{km}{h} \text{ (to West)}$$

$$\bar{v}_y = \frac{\Delta y}{\Delta t} = \frac{(6.5+1.6)\ \text{km}}{0.3\ \text{h}} = \frac{8.1}{0.3}\frac{\text{km}}{\text{h}}$$

$$\bar{v}_y = 27\ \frac{\text{km}}{\text{h}}\ \text{(to North)}.$$

Therefore, the total average velocity has the magnitude:

$$\bar{v} = \frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta t} = \sqrt{\bar{v}_x{}^2+\bar{v}_y{}^2} \Rightarrow$$

$$\bar{v} = \sqrt{215.21+729} = \sqrt{944.21}\ \left(\frac{\text{km}}{\text{h}}\right).$$

Thus, finally, the average speed is:

$$\bar{v} \cong 30.73\ \left(\frac{\text{km}}{\text{h}}\right).$$

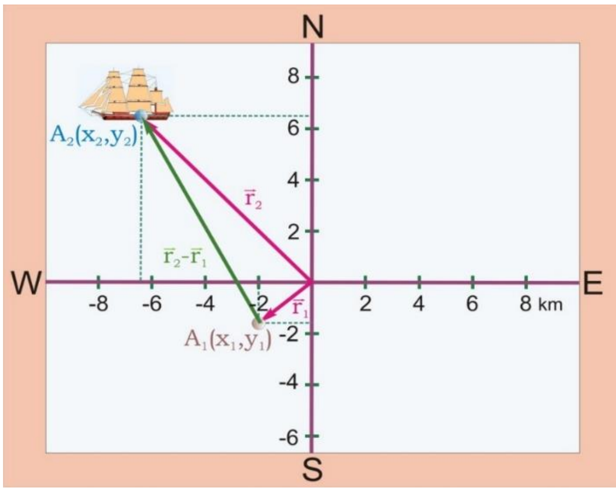


Fig. 10. A ship moves between points A_1 and A_2.

b) Finding the direction of the average velocity:

$$\vec{v}_{av} = \frac{\vec{r_2}-\vec{r_1}}{t_2-t_1} = \frac{(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}}{t_2-t_1}\ \left(\frac{\text{km}}{\text{h}}\right) \Rightarrow \vec{v}_{av} = \left(-14.67\vec{i}+27\vec{j}\right)\frac{\text{km}}{\text{h}}.$$

The direction is to the N-W, along the displacement vector $\vec{r_2}-\vec{r_1}$, as shown in Figure 10.

Problem 13

The trace of a cosmic ray particle in a photographic plate emulsion is found empirically to be described by the relationship: $\vec{r}(t) = (3t^2-6t)\vec{i}+(5-8t^4)\vec{j}\ (\text{m})$. Determine:

a) the total velocity of the particle;
b) the acceleration of the particle.

Solution

a) The velocity is the derivative of the position vector with respect to time:

$$\vec{v} = \frac{d\vec{r}}{dt} = (9\,t^2-6)\vec{i}+(-32t^3)\vec{j}\left(\frac{\text{m}}{\text{s}}\right),$$

b) The acceleration is the derivative of the velocity with respect to time:

$$\vec{a} = \frac{d\vec{v}}{dt} = 18t\,\vec{i} - 96\,t^2\,\vec{j}\quad\left(\frac{\text{m}}{\text{s}^2}\right).$$

5. DYNAMICS

5.1 Dynamics of translation

Problem 14

A body of mass **m** falls in air, experiencing a drag or resistive force $\vec{R} \cong C\,\vec{v}$ (an approximation that applies at small speeds **v**). Find the equations: a) of the velocity and, b) of motion, if the body starts its motion from rest, i.e. its initial velocity is $v_0 = 0$ at $t_0 = 0$.

Solution

a) The dynamic equation of motion can be written as:

$$m\,\vec{a} = \vec{G} + \vec{R} \Rightarrow m\frac{dv}{dt} = mg - C\,v \Leftrightarrow$$

$$\frac{dv}{dt} = g - \frac{C}{m}\,v.$$

Then solve the above equation for v = v(t).

$$dv = \left(g - \frac{C\,v}{m}\right)dt \Leftrightarrow \frac{dv}{g - \frac{C\,v}{m}} = dt\,,$$

$$\int \frac{1}{g - \frac{C}{m}v}\,dv = \int dt \Leftrightarrow$$

$$-\frac{m}{C}\ln\left[\left(g - \frac{C}{m}v\right)k\right] = t + t_0\ .$$

As the initial moment is zero, $t_0 = 0$:

$$\ln\left[\left(g - \frac{C}{m}v\right)k\right] = -\frac{C\,t}{m} \Leftrightarrow$$

$$\left(g - \frac{C\,v}{m}\right)k = e^{-\frac{C}{m}t}\,,$$

$$g - \frac{C\,v}{m} = \frac{1}{k}e^{-\frac{C}{m}t}\,,$$

$$v = v(t) = -\frac{m}{k\cdot C}e^{-\frac{C}{m}t} + g\frac{m}{C}.$$

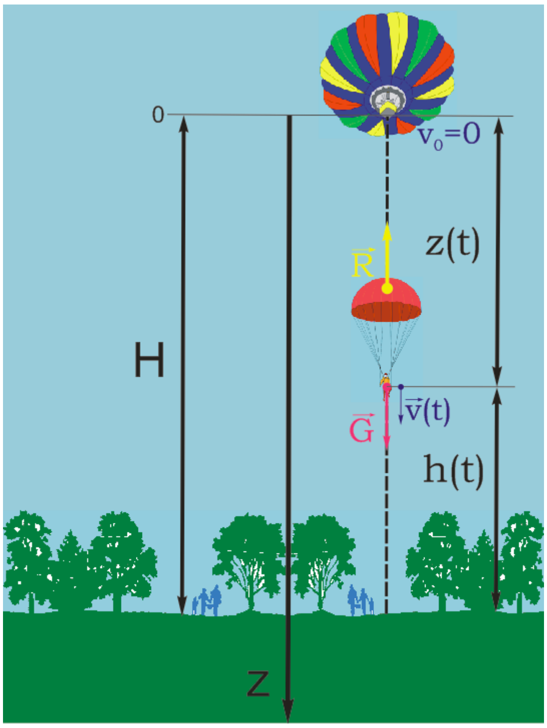


Fig. 11. An object falls in air experiencing the forces $\vec{G}$ (weight) and $\vec{R}$ (air resistance).

The constant of integration k can be determined based on the initial conditions:

$$\begin{cases} t_0 = 0 \\ v_0 = 0 \end{cases} \Rightarrow$$

$$v(0) = v_0 = -\frac{m}{k\,C} + g\frac{m}{C}.$$

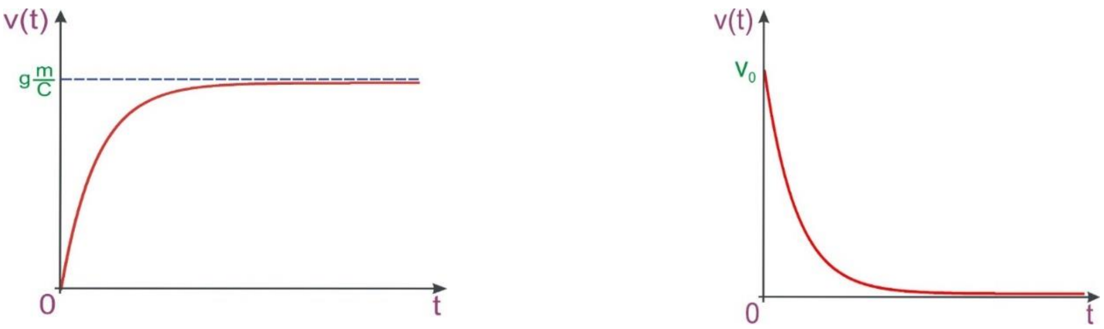


Fig. 12 a) The speed versus time for an object of mass m under the gravitational force (G = mg) and the force of friction with air (R = Cv) (Problem 14); b) The vehicle speed, starting with v_0 (initial speed) decreases exponentially with time after its engine stops working (Problem 15).

Then, if v (0) = 0, then the constant of integration becomes: $k = \frac{1}{g}$.

It follows that the equation of the velocity is: $v(t) = \frac{g\,m}{C}\left(1 - e^{-\frac{C}{m}t}\right)$.

Figure 12-a) presents the graph of speed versus time for the mobile. The speed tends asymptotically to **gm/C**.

b) Deducing the equation of motion z(t) = ?

$$\frac{dz}{dt} = g\,\frac{m}{C}\,\left(1 - e^{-\frac{C}{m}t}\right) \Leftrightarrow dz = g\,\frac{m}{C}\,\left(1 - e^{-\frac{C}{m}t}\right)dt,$$

$$\int dz = g\,\frac{m}{C}\,\int\left(1 - e^{-\frac{C}{m}t}\right)dt \Rightarrow$$

$$z(t) = g\,\frac{m}{C}\,t + g\,\frac{m^2}{C^2}\,\left(e^{-\frac{C}{m}t} + K\right),$$

where **K** is another constant of integration.

To determine **K**, we consider the initial condition z(0) = 0, then we have that:

$$z(0) = g\,\frac{m^2}{C^2}(1+K),$$

$$z(0) = 0 \quad \Rightarrow \quad g\,\frac{m^2}{C^2}\,(1+K) = 0 \quad \Rightarrow \quad K = 1,$$

therefore, the equation of motion results:

$$z(t) = g\,\frac{m}{C}\,t + g\,\frac{m^2}{C^2}\,\left(e^{-\frac{C}{m}t} - 1\right).$$

Problem 15

The engine of a vehicle stops when the car is running at speed $\mathbf{v_0}$, at $t_0 = 0$. It will experience a friction force proportional to the speed but directed in the opposite direction, or $\overrightarrow{F_r} = -\,C\,\vec{v}$. What is the equation of motion and that of the speed of this vehicle? (see Figure 12-b)

Solution

After the engine stops, the only force acting on the vehicle is the resistance, or the friction force. Then, the dynamic equation of motion is:

$$m\,\vec{a} = \vec{F_r} = -\,C\,\vec{v}, \text{ or else, after rearrangements: } m\,\frac{dv}{dt} = -\,C\,v$$

The above is a differential equation with separable variables, and after rearrangement, it becomes:

$$m\frac{dv}{v} = -\,C\,dt \qquad \Leftrightarrow \qquad \frac{dv}{v} = -\frac{C}{m}\,dt,$$

Then, by integrating in both members of the equation, one gets:

$$\int \frac{dv}{v} = \int -\frac{C}{m}\,dt = -\frac{C}{m}\int dt \;\Leftrightarrow\; \ln v + k = -\frac{C}{m}\,t.$$

The constant of integration k can be deduced from the initial conditions, considering that at $t_0 = 0$, the velocity is $v(0) = v_0$.

At the moment t = 0, the equation becomes then: $\ln v_0 + k = 0 \;\Rightarrow\; k = -\ln v_0$.

The equation of the speed then becomes:

$$\ln(v) - \ln(v_0) = -\frac{C}{m}\,t,$$

$$\ln\left(\frac{v}{v_0}\right) = -\frac{C}{m}\,t \;\Leftrightarrow\; \frac{v}{v_0} = e^{-\frac{C}{m}t},$$

$$v = v(t) = v_0\,e^{-\frac{C}{m}t}.$$

Figure 12-b) presents the graph of the speed versus time, decreasing as a logarithmic function.

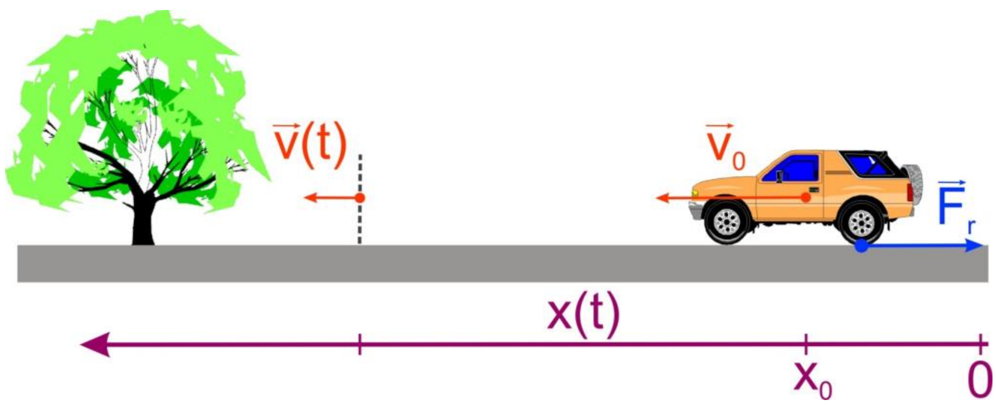


Fig. 13. The driving force of the car is turned off at the time $t_0 = 0$, at coordinate $x(0) = x_0$.

Problem 16

A rigid body is fixed at point O. The force $\vec{F} = -3\vec{i} + \vec{j} + 5\vec{k}\ (\mathrm{N})$, with its origin at a position given by the position vector $\vec{r} = 7\vec{i} + 3\vec{j} + \vec{k}\ (\mathrm{m})$ determines the rotation of the rigid body relative to the point O.

a) Find the torque $\vec{\mathbf{M}}$ of the force $\vec{F} = \overrightarrow{AB}$ (see Figure 14) relative to the point O.

b) What is the angle between the force $\vec{\boldsymbol{F}}$ and the position vector $\vec{\mathbf{r}}$ of its origin A?

Solution

a) The torque of the force is the vector product of the position vector and the force:

$$\vec{M} = \vec{r}\times\vec{F} = \left(x\,\vec{i}+y\,\vec{j}+z\,\vec{k}\right)\times\left(F_x\,\vec{i}+F_y\,\vec{j}+F_z\,\vec{k}\right)$$

$$\vec{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}.$$

With the vectors $\vec{F}$ and $\vec{r}$ given in the problem, the torque becomes:

$$\vec{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix} =$$

$$= \vec{i}(3\times5-1\times1)$$

$$-\vec{j}\big(7\times5-1\times(-3)\big)+\vec{k}\big(7\times1-3\times(-3)\big)$$

$$\vec{M} = 14\,\vec{i} - 38\,\vec{j} + 16\,\vec{k} \quad (\mathrm{N}\times\mathrm{m}).$$

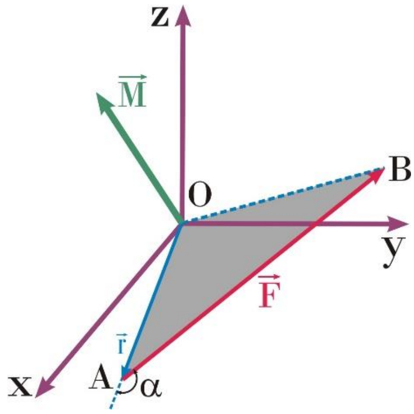


Figure 14. A rigid body fixed at point O is subjected to a force $\vec{F} = \overrightarrow{AB}$ with its origin A at a position given by $\vec{r}$, relative to point O.

The torque has the magnitude equal to the absolute value of the vector:

$$M = \sqrt{14^2+38^2+16^2} = \sqrt{196+1444+256}$$

$$M = \sqrt{1896} = \sqrt{2^3\cdot3\cdot79} = 2\sqrt{474} \implies M=43.54 \ (\mathrm{N}\times\mathrm{m}).$$

Note: the unit of measurement for the torque in SI is $1\ [M]_{SI} = \mathrm{N}\times\mathrm{m}$, resulted from the vector product, and it is not equal to the joule, $1\ \mathrm{J} = 1\ \mathrm{N}\cdot1\ \mathrm{m}$, the latter resulting from the scalar product between the vector force and the vector displacement.

b) To determine the angle between the two vectors $\vec{\mathbf{F}}$ and $\vec{\mathbf{r}}$, one calculates the absolute values of each vector and their scalar product.

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{49+9+1} = \sqrt{59} \implies r \cong 7.68 \ (\mathrm{m}),$$

$$F = \sqrt{9+1+25} = \sqrt{35} \implies F \cong 5.92\ (\mathrm{N}).$$

Their scalar product: $\vec{r}\cdot\vec{F} = r\cdot F\cdot\cos\alpha$. Therefore, being defined by its cosine function, the angle is:

$$\cos\alpha = \frac{\vec{r}\cdot\vec{F}}{r\cdot F} = \frac{(x\,\vec{i}+y\,\vec{j}+z\,\vec{k})\cdot(F_x\,\vec{i}+F_y\,\vec{j}+F_z\,\vec{k})}{r\cdot F} = \frac{x\,F_x+y\,F_y+z\,F_z}{r\cdot F},$$

Finally, substituting with the vector components given in the problem:

$$\cos\alpha = \frac{-21+3+5}{\sqrt{59}\cdot\sqrt{35}} = -\frac{13}{\sqrt{2065}} \Longrightarrow \cos\alpha \cong -0.286 \;\Rightarrow\; \text{the angle } \alpha \cong 118.47° = 118°\ 28'\ 13''.$$

Problem 17

a) What is the torque of the force $\vec{F} = -3\vec{i}+2\vec{j}+2\vec{k}$ (N) if it acts on a rigid body, at distance from the fixed point O, the origin of the reference frame $\vec{r} = 5\vec{i}+3\vec{j}+4\vec{k}$ (m); b) what is the angle $\boldsymbol{\alpha}$ between the two vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$.

Solution

As in the previous problem 16, it can be deduced that the moment $\vec{\mathbf{M}}$ of the force is:

a) $$\vec{M} = \vec{r}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 4 \\ -3 & 2 & 2 \end{vmatrix} = -2\,\vec{i} - 22\,\vec{j} + 19\,\vec{k} \quad (\mathrm{N}\times\mathrm{m}),$$

b) The angle between the two vectors results as :

$$\cos\alpha = \frac{\vec{r}\cdot\vec{F}}{r\cdot F} = -\frac{1}{5\sqrt{34}} \Rightarrow \alpha = \arccos\left(-\frac{1}{5\sqrt{34}}\right) \Longrightarrow \alpha \cong 102.18°\ (or\ 102°\ 11'\ 2.46'')\,.$$

Problem 18

Determine the coefficients α, β and γ, so that the force with components given below is conservative, knowing that the rotor of a conservative force is equal to zero.

$$\begin{cases} F_x = \alpha\cdot x\cdot z + 2y & (\mathrm{N}) \\ F_y = \beta\cdot x - 3y^2\cdot z & (\mathrm{N})\,. \\ F_z = -x^2 + \gamma\cdot y^3 & (\mathrm{N}) \end{cases}$$

Solution

For a force $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ to belong to a conservative field of forces, its rotor must be zero:

$$\nabla\times\vec{F} = 0,$$

$$\left(\frac{\partial\ldots}{\partial x}\vec{i} + \frac{\partial\ldots}{\partial y}\vec{j} + \frac{\partial\ldots}{\partial z}\vec{k}\right)\times\vec{F} = 0 \;\Leftrightarrow\; \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial\ldots}{\partial x} & \frac{\partial\ldots}{\partial y} & \frac{\partial\ldots}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0,$$

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\vec{i} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\vec{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\vec{k} = 0,$$

For the resulting vector to be zero, each of its components must be zero:

$$\begin{cases} \dfrac{\partial F_z}{\partial y} = \dfrac{\partial F_y}{\partial z} \\ \dfrac{\partial F_z}{\partial x} = \dfrac{\partial F_x}{\partial z} \\ \dfrac{\partial F_y}{\partial x} = \dfrac{\partial F_x}{\partial y} \end{cases} \Leftrightarrow \begin{cases} 3\gamma y^2 = -3y^2 \\ \alpha x = -2x \\ \beta = 2 \end{cases} \Leftrightarrow \begin{cases} \alpha = -2 \\ \beta = 2 \\ \gamma = -1 \end{cases} \Leftrightarrow \begin{cases} F_x = -2x\cdot z + 2y \\ F_y = 2x - 3y^2\cdot z \\ F_z = -x^2 - y^3 \end{cases}.$$

Problem 19

Calculate the coefficients α, β and γ, so that the force with the components given below is conservative:

$$\begin{cases} F_x = \alpha\cdot y + 3z \\ F_y = 2\beta\cdot x - 3y\cdot z^2 \\ F_z = 2\beta\cdot x + \gamma\cdot y^3\cdot z \end{cases}.$$

Solution

Following a path similar to that in solving the previous problem, by calculating the rotor of the conservative force and requiring its value to be null: $\nabla \times \vec{F} = 0$, one obtains:

$$\begin{cases} \alpha = 3 \\ \beta = \dfrac{3}{2} \\ \gamma = -2 \end{cases}.$$

5.2 Dynamics of rotation

Problem 20

Determine the moment of inertia of a system made of two identical spheres, each of mass **m**, situated at distance **r** apart, if the spheres rotate about an axis through the centre of mass of the system, C. see Figure 15.

Solution

The centre of mass of a system made of two identical bodies is situated at point C, in the middle of the segment that unites their centres of mass. Therefore, the point C is at distance **r/2** from each of the two spheres of mass **m**.

The moment of inertia of one of the spherical bodies is:

$$I = m\left(\frac{r}{2}\right)^2 = m\frac{r^2}{4}.$$

Then the total moment of inertia equals the sum of the two:

$$I_{total} = I + I = = 2\,I = m\,\frac{r^2}{2} \Rightarrow$$

$$I_{total} = \frac{1}{2}\,m\,r^2.$$

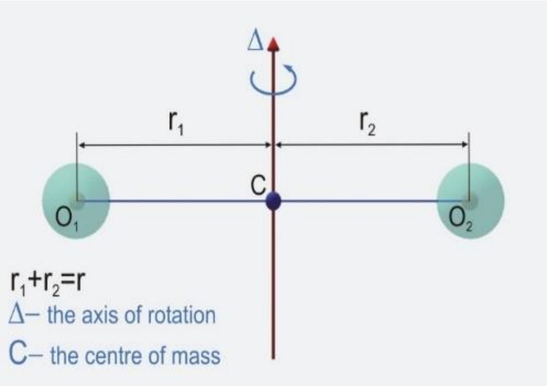


Fig. 15. The centre of mass C of a system consisting of two identical spheres at distance $r = r_1 + r_2$ [♣].

Problem 21

Calculate the moment of inertia (**I**) for a system of two objects of masses **m₁** and **m₂**, situated at distance **r** from each other, measured between their centres of mass, when the system rotates about an axis through the centre of mass of the system, C. See Figure 16.

Solution

$$O_1C = r_1 \quad \text{and} \quad O_2C = r_2$$

To find the position of the centre of mass of the system, one considers that:

$$m_1 \cdot r_1 = m_2 \cdot r_2 \quad \text{and} \quad r_1 + r_2 = r,$$

$$\text{and then: } \begin{cases} r_1 = \frac{m_1}{m_2} r_2 \\ r_1 = r - r_2 \end{cases}.$$

By solving the equation with one variable:

$$r - r_2 = \frac{m_1}{m_2}\,r_2 \quad \Rightarrow \quad r_2 = \frac{m_1}{m_1 + m_2}\,r\,,$$

and thus: $r_1 = \frac{m_2}{m_1 + m_2} r$. When rotating about the axis through C, the centre of mass of the system, the two bodies have the moments of inertia:

$$I_1 = m_1 r_1^2 \text{ and } I_2 = m_2 r_2^2.$$

Finally, the total moment of inertia of the system when it rotates about the axis through its centre of mass C, is found by adding the moments of inertia of the bodies that the system consists of:

$$I = I_1 + I_2 = m_1 r_1^2 + m_2 r_2^2, \text{ thus:}$$

$$I = m_1 \frac{m_2^2 r^2}{(m_1 + m_2)^2} + m_2 \frac{m_1^2}{(m_1 + m_2)^2} \Rightarrow$$

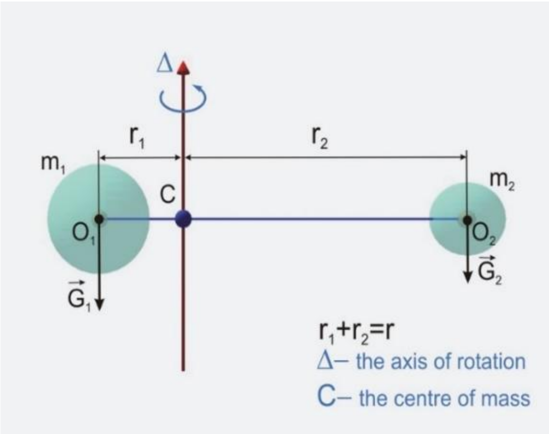


Fig. 16. The centre of mass of a system made of two bodies (m_1 and m_2) at distance $r=r_1+r_2$ [☘].

$$I = \frac{m_1 m_2}{m_1 + m_2} r^2 .$$

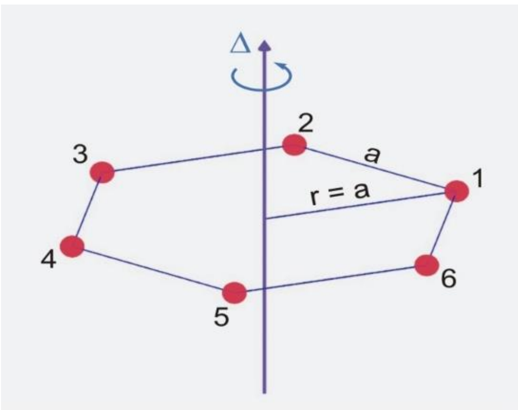
Problem 22

Find the moment of inertia of a molecule made of identical atoms, each of mass **m**, placed at the vortexes of a regular hexagon of side **a**, when the molecule rotates about an axis (see Figure 17):

a) Perpendicular to the hexagonal plane, through its centre of mass C,
b) Parallel to the hexagon, through the centre of mass and through two opposite vortexes,
c) Parallel to the hexagon plane, through the middles of two opposite sides.

Solution

a) The moment of inertia of the molecule rotating around the axis Δ perpendicular to the plane of the hexagonal molecule: $I = 6\, m\, a^2$



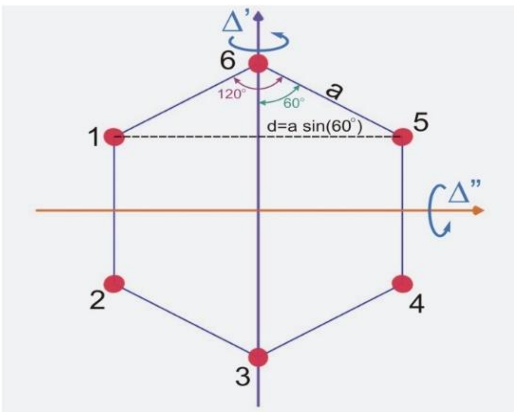


Fig. 17. Calculating the moments of inertia of a hexagonal molecule, rotating about different axes [☘].

b) The moment of inertia of the hexagonal molecule rotating around an axis Δ’ in the plane of the molecule, through two opposite vortexes:

$$I' = I_1'+I_2' +I_4' + I_5' = 4m\,d^2 = 4m\,a^2\,(\sin 60^\circ)^2 = 4m\,a^2\,\frac{3}{4},$$

$$I' = 3\,m\,a^2 \Rightarrow I' = \frac{1}{2}I.$$

c) The moment of inertia of the hexagonal molecule rotating around an axis Δ” in coplanar with the molecule and through the middles of two opposite sides of the hexagon:

$$I'' = 4m\frac{a^2}{4} + 2m\,a^2 = 3m\,a^2 = \frac{1}{2}I\,.$$

Problem 23

Determine the moment of inertia (**I**) of a hoop (See Figure 18) rotating about:

a) an axis through its centre of mass ($\Delta = \Delta_{CM}$) $I_{\Delta CM}$ = ?

b) an axis through a point of the hoop (Δ’) I_{Δ} = I’ = ?

c) an axis through the centre of mass, along a diameter of the hoop: I” = ?

Solution

a) One considers that the hoop is divided into infinitely small segments, each of mass **dm** (element of mass). When rotating about the axis **Δ** (Figure 18 -a), each such element of mass has the moment of inertia: $dI = R^2 dm$.

Then, by integrating along the whole loop, one obtains the total moment of inertia:

$$I = I_\Delta = \int_0^I dI = \int_0^M R^2 dm = R^2 \int_0^M dm \Rightarrow$$

$$I' = M{\cdot}R^2$$

The same equation applies for calculating the moment of inertia of a hollow pipe with relatively thin wall, when rotating about a central axis of the pipe (or tube).

b) When the rotation takes place around an axis parallel to the main axis or rotation (through the centre of mass of the hoop), at distance **d** from the latter, the parallel axes theorem states that the total moment of inertia equals the sum between the moment of inertia relative to the main axis (**$I_{\Delta'}$**) and the moment of inertia of the hoop (Md^2), considering that its whole mass **M** is concentrated at its centre of mass, situated at distance **d** from the second axis of rotation:

$$I_{\Delta'} = I_\Delta + M{\cdot}d^2 \quad \Longrightarrow$$

$$I_{\Delta'} = M{\cdot}R^2 + M{\cdot}d^2 \Longrightarrow$$

$$I_{\Delta'} = 2M{\cdot}R^2.$$

c) The theorem for the moments of inertia relative to perpendicular axes of rotation states that:

$$I_z = I_x + I_y.$$

Given the symmetry of the hoop, the rotation about the axis Ox is similar to the rotation about the axis Oy. In this case, $I_x = I_y$, and therefore the theorem for perpendicular axes becomes:

$$I_z = 2I_x .$$

As $I_z = I_\Delta = M \cdot R^2$, the moment of inertia of the hoop when rotating about the axis Ox or Oy becomes:

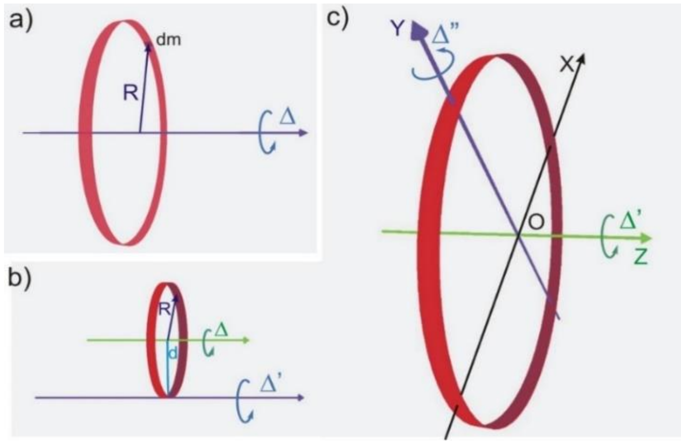


Fig. 18. Axes of rotation of a hoop and calculating the corresponding moments of inertia with respect to the rotation around each axis a)[♣]; b)[♣]; c)[♣].

$$I_x = I_y = \frac{1}{2} M \cdot R^2.$$

Problem 24

Calculate the moment of inertia of a thin, uniform rod of mass **M** and length **L** (See Figure 19), when rotating:

a) About an axis through one of its ends;
b) About an axis through its centre, as shown in Figure 19.

Solution

a) If the linear density of the rod is $\boldsymbol{\beta}$, then the mass of the element of rod is $dm = \beta dx$, at distance **x** from the axis of rotation. Then the moment of inertia of this element of rod is:

$$dI = dm \ x^2 = \beta \ dx \ x^2.$$

To find the total moment of inertia relative to the axis $\boldsymbol{\Delta}$ (through an end of the rod), one integrates along its whole length:

$$I = \int_0^I dI = \beta \int_0^L x^2 \cdot dx = \beta \frac{L^3}{3} = \beta \cdot L \cdot \frac{L^2}{3} \Rightarrow$$

$$I = \frac{ML^2}{3}.$$

b) For the axis that crosses the centre of the rod, one considers the moments of inertia of the two halves rotating about the axis $\boldsymbol{\Delta}$

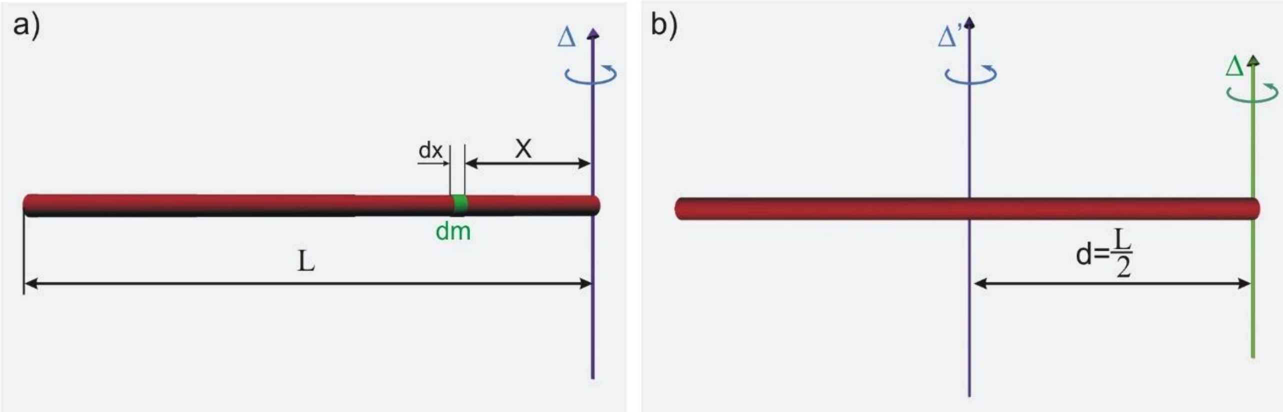


Fig. 19. A rod rotating about an axis through its end (a) and an axis through the centre of mass of the rod a)[♣]; b)[♣].

$$I_{1/2} = \int_0^{L/2} x^2 dm \Longrightarrow$$

$$I_{1/2} = \frac{M\,L^2}{24}.$$

Then, the total moment of inertia is the sum of the inertia of the two halves:

$$I = 2I_{1/2} = \frac{M\,L^2}{12}.$$

Problem 25

Calculate the moment of inertia of a full disk of mass **M** and of radius **R**, when it rotates about its central axis Δ (See Figure 20-a).

Solution

As shown in Figure 20, one considers an element of volume **dV**, with the shape of an imaginary 'pipe' of internal radius *r* and external radius **r + dr** (hence, of thickness *dr*), and of height **H**. The density of the material that the full cylinder is made of:

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2\,H}$$

The element of volume is:

$$dV = \frac{dV}{dr}\,dr$$

$$dV = \frac{d}{dr}(\pi r^2\,H)dr$$

$$dV = 2\pi\,r\,H\,dr\,.$$

The element of mass dm of this cylindrical element of volume **dV** is:

$$dm = \rho\,dV = \rho\,2\pi r\,H\,dr$$

Then the moment of inertia of this element of volume (relative to the axis Δ) becomes:

$$dI = dm\, r^2 = 2\pi\, \rho r^3\, H\, dr,$$

$$dI = 2\pi \frac{M}{\pi R^2\, H} r^3\, H\, dr,$$

$$dI = \frac{2M}{R^2}\, r^3\, dr.$$

Then the total moment of inertia results, by integration along *r*, between *r = 0* and r = R:

$$I = \int_0^I dI = \frac{2M}{R^2} \int_0^R r^3 dr \;\Rightarrow$$

$$I = \frac{M\, R^2}{2}.$$

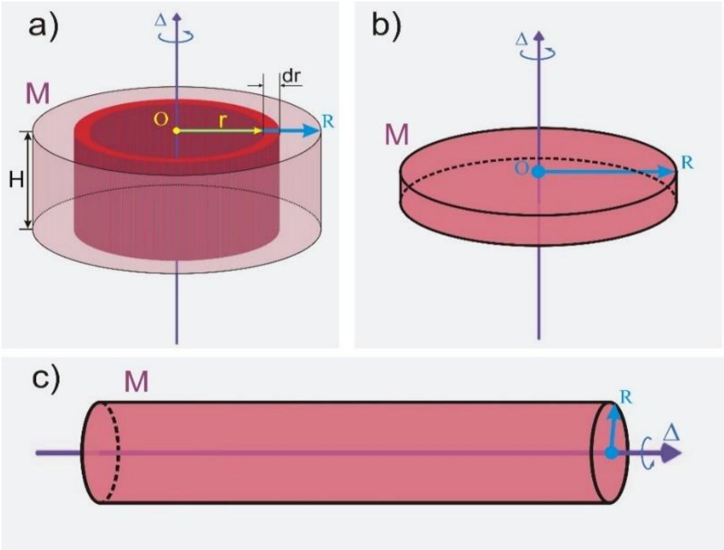


Fig. 20. (a) A full cylinder of radius R. An element of volume with the shape of an empty cylinder ('pipe') shown in grey; (b) A full disk of radius R and mass M. This is a cylinder with its height much smaller than its diameter; (c) A cylinder with its diameter smaller than its length a)[♣]; b)[♣]; c)[♣].

The same result applies to a disc, which is actually also a cylinder with its height smaller that its diameter (Figure 20-b), and to a full cylinder with its diameter smaller than its length (Figure 20-c).

Problem 26

Determine the moment of inertia of a flat disk (mass **M**, radius **R**) rotating about an axis (Δ_{CM}) in its plane, and through its centre of mass (Figure 21).

Solution

The perpendicular axis theorem states that the sum between the moments of inertia about the Ox and Oy axis equals the moment of inertia about an axis (Oz) perpendicular to both other two axes:

$$I_z = I_\Delta = I_x + I_y.$$

Owing to the symmetry of the flat disc, the moments of inertia about the axes Ox and Oy are equal: $I_x = I_y$. Therefore, the moment of inertia around the (Δ_{CM}) axis is: $I_z = I_\Delta = 2I_x$.

The flat disc has the moment of inertia $\mathbf{I_z}$ about the axis $\mathbf{\Delta_{CM}}$ (See Problem 24):

$$I_z = I_\Delta = \frac{1}{2} MR^2.$$

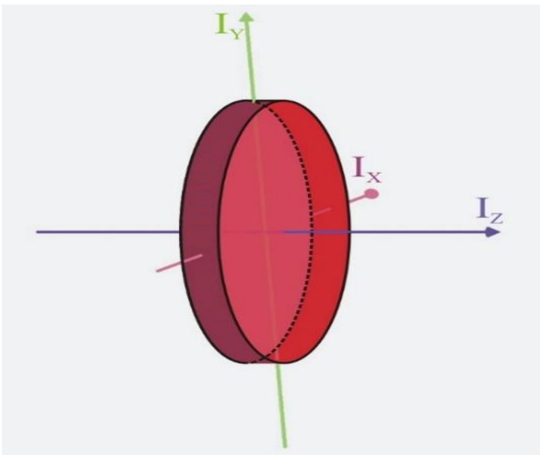


Fig. 21. A disc and three possible axes of rotation [♣].

Then the moments of inertia about the Ox and about Oy axes, results as:

$$I_x = I_y = \frac{I_z}{2} = \frac{1}{4}MR^2.$$

Problem 27

Determine the moment of inertia of a hollow cylinder (or disc) of internal radius $\mathbf{R_1}$ and external radius $\mathbf{R_2}$, and mass **M** (Figure 22).

Solution

Considering that the object has uniform density ρ, the element of volume **dV** has the mass **dm** (in red in Figure 22): $dm = \rho \cdot dV = \rho \cdot 2\pi r \cdot L \cdot dr$.

The element of mass ***dm*** has the elementary moment of inertia **dI** that thus becomes:

$$dI = dm \cdot r^2 = 2\pi\rho \cdot L \cdot r^3 \cdot dr,$$

$$dI = 2\pi \cdot L \frac{M}{\pi\left({R_2}^2 - {R_1}^2\right)L} r^3 dr$$

$$dI = 2\frac{M}{{R_2}^2 - {R_1}^2} r^3 dr.$$

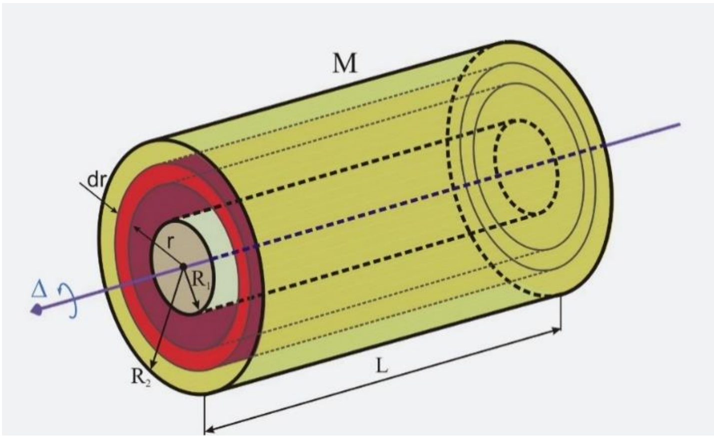


*Fig. 22. A hollow cylinder (in yellow) rotates about its central axis. The element of mass, d**m**, is represented in red [♣].*

Then the total moment of inertia results by integrating the elementary moment of inertia **dI** between R_1 and R_2:

$$I = \int_{R_1}^{R_2} 2\pi \cdot \rho \cdot L \cdot r^3 dr$$

$$I = 2\frac{M}{R_2^2 - R_1^2}\int_{R_1}^{R_2} r^3 dr$$

Then, finally: $I = 2\frac{M}{R_2^2 - R_1^2}\frac{1}{4}(R_2^4 - R_1^4) \Rightarrow I = \frac{M(R_1^2 + R_2^2)}{2}$.

Problem 28

Determine the moment of inertia of a rectangular plate of mass **M**, sides **a** and **b** (see Figure 23 and Figure 24), rotating about:

a) An axis along one of its edges;

b) An axis perpendicular to its plane, through its centre of mass.

Solution

a) Suppose that the surface density of the metal plate is: $\sigma = \frac{M}{a \cdot b}$.

The mas of an element of surface **dS**: $dm = \sigma \cdot dS = \sigma \cdot a \cdot dx$.

Then the moment of inertia of the element of mass dm rotating about the axis **Δa** is:

$$dI = dm \cdot x^2 = \sigma \cdot a \cdot x^2 dx.$$

Therefore, the total moment of inertia I_a, if the plate rotates about an axis **Δa** (along the side **a**) is, by integrating along **dx**:

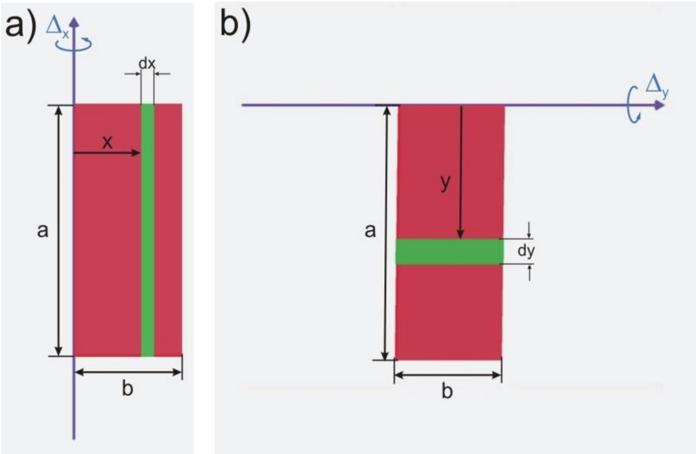


Fig. 23. A rectangular plate (mass M, sides a x b) a) [♣];

$$I_a = \int_0^b \sigma \cdot a \cdot x^2 dx \ = \sigma \cdot a \int_0^b x^2 dx = \sigma \cdot a \frac{b^3}{3} \Longrightarrow$$

$$I_a = \frac{M}{a \cdot b}\frac{b^3}{3} = \frac{M \cdot b^2}{3}.$$

If the plate rotates about an axis **Δb**, along the side **b**, then the moment of inertia $\mathbf{I_b}$ is:

$$I_b = \frac{Ma^3}{3}.$$

b) The parallel axes theorem (Figure 24) states that:

$$I_a = I_y + M\frac{b^2}{4} \Longrightarrow$$

$$I_y = M\frac{b^2}{4} - I_a = M\frac{b^2}{4} - M\frac{b^2}{3} \Rightarrow I_y = M\frac{b^2}{12}.$$

Similarly, along the Oy direction, in the plane xOy of the plate: $\quad I_b = I_x + M\frac{a^2}{4} \Longrightarrow I_x = M\frac{a^2}{4} - I_b$

$$I_x = M\frac{a^2}{4} - M\frac{a^2}{3} = M\frac{a^2}{12}.$$

Then, by applying the perpendicular axes theorem, the moment of inertia for the rotation about an axis perpendicular to the plate, and through its centre of mass, is:

$$I_z = I_x + I_y = M\frac{a^2}{12} + M\frac{b^2}{12} = \frac{M}{12}(a^2 + b^2).$$

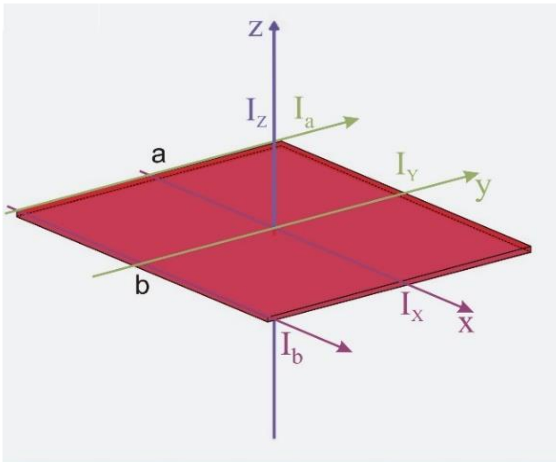


Fig. 24. A rectangular plate rotating about an axis perpendicular to its plane [♣].

Problem 29

Suppose that a car internal mechanism is designed so that, instead of being propelled by an engine with internal combustion, it can be driven by using the kinetic energy of a spinning wheel. This wheel must be heavy and dense (to have a great moment of inertia **I**), not too wide in radius, and the centrifuge force that it is submitted to must not exceed the breaking strength of the wheel material. Else, it could fracture and fragments of it can fly apart. Such a small car could move at a speed of 20 km/h, using approximately 5 HP of power. (Use that 1 HP = 746 W).

a) What amount of energy does this car use for travelling continuously for 1 hour?

b) Suppose that the kinetic energy is stored in a such a spinning wheel of mass 100 kg and radius 60 cm. What frequency will this wheel rotate with?

Solution

a) $E = P\,\Delta t = 5{\cdot}746\text{ W}{\cdot}3600\text{ s} \Longrightarrow E = 13.428\text{ MJ}.$

b) The moment of inertia of the rotating wheel (a disk of mass **M** and radius **R**) is:

$$I = \frac{MR^2}{2} = 18\text{ kg m}^2.$$

If the propelling energy **E** is converted without loss into kinetic energy of rotation:

$$T = \frac{1}{2}I{\cdot}\omega^2 \text{ and } T = E$$

Then we deduce that:

$$\frac{1}{2}I{\cdot}\omega^2 = E \Longrightarrow \omega^2 = 2{\cdot}\frac{E}{I} \Rightarrow \omega = \sqrt{\frac{2E}{I}}.$$

The frequency of the wheel: $\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2{\cdot}13.428\times10^6}{18}} \Longrightarrow \nu = 194.4\text{ s}^{-1}.$

The frequency in turns per minute: $n = 60\,\nu = 11\,664.2\ \text{rot/min}.$

Problem 30

A small car without an internal combustion engine, is propelled by the kinetic energy stored in a rotating wheel of mass M = 240 kg and radius R = 80 cm, spinning at 4000 revolutions per minute. For how long could it provide driving power at a rate of 10 HP?

Solution

$$I = \frac{1}{2}MR^2 = 76.8\ \text{kg}{\cdot}\text{m}^2.$$

The frequency of the wheel: $\nu = \frac{n}{60} = 66.6(6)\ \text{s}^{-1}.$

The angular frequency of the wheel: $\omega = 2\pi\,\nu \cong 418.88\ \frac{\text{rad}}{\text{s}}.$

The energy needed to run for Δt at power P: $E = P\,\Delta t.$

The energy provided by the spinning wheel: $T = \frac{1}{2}I\,\omega^2 = \frac{76.8{\cdot}418}{2}$ and T = E = 6.74 MJ.

The time interval: $\Delta t = \frac{E}{P} = \frac{T}{P} = \frac{6.74\ \text{MJ}}{10 \cdot 746\ \text{W}} \cong 903.17\ \text{s} \cong 15\ \text{min}$.

Problem 31

When the position of a particle of mass m = 2 kg is described by the position vector $\vec{r}(2, 3, -4)$, its velocity is $\vec{v}\,(1, -1, 3)$. What is the angular momentum of the particle relative to the origin of the system of reference?

Solution

The angular momentum of the particle:

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\,\vec{r} \times \vec{v}.$$

$$\vec{L} = m \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -4 \\ 1 & -1 & 3 \end{vmatrix} = 2\left(5\vec{i} - 10\vec{j} - 5\vec{k}\right) \Longrightarrow \vec{L} = \left(10\vec{i} - 20\vec{j} - 10\vec{k}\right)\ \text{kg}\frac{\text{m}^2}{\text{s}}.$$

6. OSCILLATIONS

Problem 32

A body performs harmonic oscillations described by the equation:

$$\psi_1(t) = 0.03 \sin\left(100\pi\, t + \frac{\pi}{2}\right) \quad m.$$

a) Identify the parameters of this oscillation.
b) What is the speed of oscillation?
c) What is the acceleration of the oscillatory motion?

Solution

a) By writing the general expression of the harmonic oscillations one has:

$$\psi_1(t) = A_1 \sin(\omega t + \varphi_0) \quad m$$

The amplitude of the oscillation is: $A_1 = 0.03\ m = 3\ cm$;

The angular frequency: $\omega = 100\pi\ \frac{rad}{s}$;

The period of oscillation: $T = \frac{2\pi}{\omega} = \frac{2\pi}{100\pi} = \frac{2}{100} \Longrightarrow T = 0.02\ s = 20\ ms$;

The frequency of oscillation: $\nu = \frac{\omega}{2\pi} = \frac{100\pi}{2\pi} \Longrightarrow \nu = 50\ (s^{-1}) = 50\ Hz$

The initial phase: $\varphi_0 = \frac{\pi}{3}\ rad$

b) The speed of oscillation: $u_1(t) \stackrel{def}{=} \frac{d\psi_1}{dt} = 3\pi \cos\left(100\pi t + \frac{\pi}{2}\right)$

c) The acceleration of the oscillator is: $a_1(t) \stackrel{def}{=} \frac{d^2\psi_1}{dt^2} = -300\pi^2 \sin\left(100\pi\, t + \frac{\pi}{2}\right) \left(\frac{m}{s^2}\right)$.

Problem 33

Suppose that the oscillator in the previous problem experiences another simultaneous oscillation described by the equation: $\psi_2(t) = 0.04 \sin(100\pi\, t)(m)$, parallel to the oscillation described by $\psi_1(t)$. a) What is the equation of the resulting oscillatory motion of the body when it is simultaneously submitted to both Ψ_1 and Ψ_2? b) What is the elongation $\Psi(t)$, the speed u(t), and the acceleration a(t) of the oscillator at the moment t = 2 s?

Solution

a) The resulting equation of motion: $\psi(t) = \psi_1(t) + \psi_2(t)$.

The resulting amplitude: $A = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2 \cos\frac{\pi}{3}}\ (m) \Longrightarrow A = 0.05\ m = 5\ cm.$

The resulting phase: $\tan\varphi = \frac{A_1 \sin\left(\frac{\pi}{2}\right) + A_2 \sin(0)}{A_1 \cos\left(\frac{\pi}{2}\right) + A_2 \cos(0)} = \frac{3}{4} \Longrightarrow \varphi = \arctan\left(\frac{3}{4}\right)$.

Therefore, the resulting oscillatory motion is described by the equation:

$$\psi(t) = 0.05 \sin(100\pi\, t + \varphi)\ (\mathrm{m}).$$

b) The speed of oscillation: $v(t) \stackrel{\text{def}}{=} \frac{d\psi}{dt} = 5\pi \cos(100\pi\, t + \varphi)\ \left(\frac{m}{s}\right)$.

The acceleration: $a(t) \stackrel{\text{def}}{=} \frac{dv}{dt} = -\,500\pi^2 \sin(100\pi\, t + \varphi)\ \left(\frac{m}{s^2}\right)$

Then when t = 2 s, the above quantities are:

$$\psi(2) = 0.05 \sin(200\pi + \varphi)\ (\mathrm{m}).$$

$$v(2) = 5\pi \cos(200\pi + \varphi)\ \left(\frac{m}{s}\right).$$

$$a(2) = -\ 500\pi^2 \sin(200\pi + \varphi)\ \left(\frac{m}{s^2}\right).$$

Problem 34

A body of mass **m** is attached to a spring, displaced vertically from the equilibrium position, and then released from rest. The mass spring system start oscillating. Assuming there are no energy losses find when are the kinetic and the potential energy of the oscillator equal for the first time?

Solution

The equation of motion of the oscillator, at moment *t* is: $\psi(t) = A\ \sin\omega t$.

The potential energy of the oscillator at *t* is: $U(t) = k\frac{\psi(t)^2}{2} = \frac{k}{2}A^2 \sin{\omega_0}^2$.

The speed of oscillation, at moment *t* is: $v(t) = \frac{d\psi}{dt} = \omega_0 \cos\omega_0 t$

Then the kinetic energy of the oscillator at *t* is: $K(t) = \frac{1}{2}mv(t)^2 = \frac{1}{2}m{\omega_0}^2 \cos{\omega_0}^2$

The characteristic frequency of the oscillator depends on the elastic constant k of the spring and on the mass **m** of the oscillator: $\omega_0 = \sqrt{\frac{k}{m}} \quad \Longrightarrow {\omega_0}^2 = \frac{k}{m} \quad \Leftrightarrow \quad k = m{\omega_0}^2$.

The period of oscillation is the inverse (or the reciprocal) of the oscillator frequency:

$$T_0 = \frac{1}{\nu_0} = \frac{1}{\frac{\omega_0}{2\pi}} = \frac{2\pi}{\omega_0}.$$

If the kinetic and potential energy are equal: $K = U \quad \Leftrightarrow \quad \frac{m}{2}{\omega_0}^2A^2\cos^2(\omega_0 t) = \frac{1}{2}kA^2\sin^2(\omega_0 t)$.

After dividing in both member by $\frac{kA^2}{2}$, then the equation becomes: $\cos^2(\omega_0 t) = \sin^2(\omega_0 t)$.

$$\tan^2(\omega_0 t) = 1 \quad \Longrightarrow \quad \omega_0 t = \frac{\pi}{4}.$$

Then the moment when the kinetic and potential energies are equal can be calculated from:

$$t = \frac{\pi}{4\omega_0} = \frac{1}{8}\frac{2\pi}{\omega_0} \Rightarrow t = \frac{T_0}{8}.$$

Problem 35

A simple pendulum (of length ℓ = 1.5 m) performs 27 oscillations in 60 seconds. What is the value of the gravitational acceleration **g** at this location?

Solution

The frequency of the oscillations: $\nu_0 = \frac{n}{\Delta t} = \frac{27}{60}\ s^{-1} = 0.45\ s^{-1} \Rightarrow \nu_0 = 0.45\ \mathrm{Hz}.$

The period of oscillation of the pendulum: $T_0 = \frac{1}{\nu_0} = 2\pi\sqrt{\frac{\ell}{g}}.$

Therefore: $\nu_0 = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}} \quad \Rightarrow \quad g = 4\pi^2\ \ell^2\nu_0{}^2 \Rightarrow g \cong 9.46\ \mathrm{m/s^2}.$

Problem 36

A damped oscillator of mass **m**, and of elastic constant **k** is subjected to a resistance force $F_r = -C\frac{d\psi}{dt}$ (where **C** is the coefficient of friction). What is the elongation and the speed of oscillation at t = 0?

Solution

The dynamic equation of motion of the oscillator:

$$\vec{F}_{resultant} = \vec{F_e} + \vec{F_r} \quad \Leftrightarrow \quad m\,\vec{a} = k\,\vec{\psi} + \vec{F_r}.$$

In the scalar form, this becomes the differential equation:

$$ma = -k\,\psi - C\,\frac{d\psi}{dt} \Leftrightarrow m\,\frac{d^2\psi}{dt^2} = -k\,\psi - C\,\frac{d\psi}{dt}.$$

Then by rearranging the terms:

$$\frac{d^2\psi}{dt^2} + \frac{k}{m}\psi + \frac{C}{m}\cdot\frac{d\psi}{dt} = 0 \quad \Rightarrow$$

$$\frac{d^2\psi}{dt^2} + 2\delta\,\frac{d\psi}{dt} + \omega_0{}^2\psi = 0.$$

By solving the above equation for Ψ, one obtains:

$$\psi = \psi(t) = e^{-\delta t}\left[A_1 e^{\left(\sqrt{\delta^2-\omega_0{}^2}\right)t} + A_2 e^{-\left(\sqrt{\delta^2-\omega_0{}^2}\right)t}\right],$$

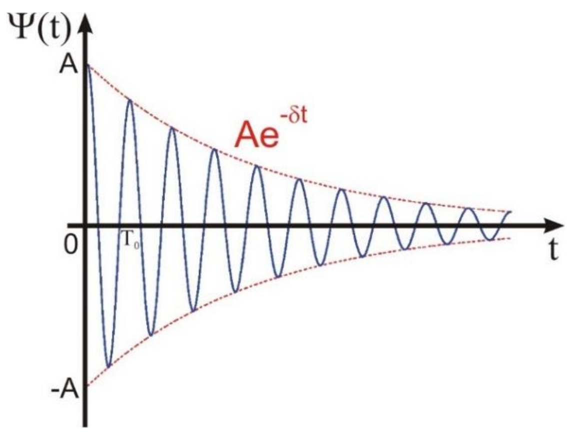


Fig. 25. The elongation Ψ(t) versus time, for a damped oscillator, in the case of small resistance forces and thus, small damping coefficients δ.

where δ is the damping coefficient and

$\omega = \sqrt{\omega_0{}^2 - \delta^2}$ is the angular frequency of the damped oscillator. Thus, the equation of the damped oscillator elongation becomes:

$$\Psi = \Psi\,(t) = e^{-\delta}(A_1 e^{i\cdot\omega t} + A_2 e^{-\,i\cdot\omega t}).$$

At the initial moment, $t_0 = 0$:

The initial position (or elongation):

$$\psi(0) = A_1 + A_2.$$

The initial speed of oscillation:

$$v(0) = \left.\frac{d\Psi}{dt}\right|_{t=0} = v_0.$$

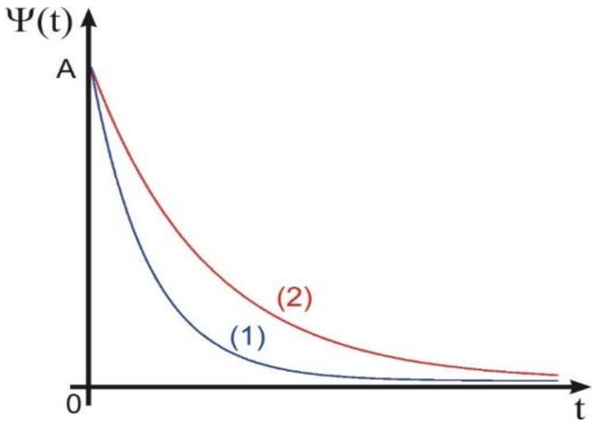


Fig. 26. The elongation of a damped oscillator for large damping coefficients δ: (1) Critically damped (merely no oscillation), and (2) Overdamped, with the damping coefficients: $\delta_2 > \delta_1$.

a) For a small damping coefficient (Figure 25):

$$\delta < \omega_0 \;\Rightarrow\; \sqrt{\delta^2 - \omega_0{}^2} = i\omega.$$

The elongation of the damped oscillator becomes:

$$\psi(t) = e^{-\delta t}(A_1 e^{i\omega t} + A_2 e^{-i\omega t}),$$

$$\psi(t) = A\, e^{-\delta t} \sin(\omega t + \varphi).$$

b) For large resistance force and damping coefficient (Figure 26):

$$\delta > \omega_0 \quad \Rightarrow \quad \gamma = \sqrt{\delta^2 - \omega_0{}^2}.$$

The elongation of the damped oscillator under large resistance forces:

$$\psi(t) = e^{-\delta t}\,(A_1 e^{\gamma t} + A_2 e^{-\gamma t}).$$

When the resistive forces (and the damping coefficient) are very large, the oscillator merely executes one half of an oscillation, slowly moving back to its equilibrium position, as in case (1) in Figure 26. This is called critical damping. If the damping coefficient further increases, the oscillator will get back to its equilibrium position even more slowly, and this is overdamped oscillation. An example of a heavily damped oscillating system is that of a road vehicle with its shock absorbers. These have the role to absorb the oscillation energy of the system, so that when the vehicle hits a bump on the road, it oscillates only very few times or not at all, i.e. so that the oscillations are critically damped or overdamped.

Problem 37

A simple pendulum performs damped oscillations and loses 2 percent of its energy during each oscillation. What is its quality factor, **Q**?

Solution

The damping of a *slightly* damped oscillator can be described by the dimensionless quantity, **Q**, called the quality factor. This is a measure of the fractional energy loss per cycle. If **E** is the total energy of the oscillator, and if it loses an amount of energy **ΔE** during one complete oscillation, then the **Q** factor is defined:

$$Q = 2\pi \frac{E}{|\Delta E|}.$$

After rearranging the terms, one obtains the fractional energy loss per cycle: $\frac{|\Delta E|}{E} = \frac{2\pi}{Q}$.

Given that the oscillator in the problem loses $|\Delta E|/E = 2\ \%$ of its energy during one oscillation, the **Q** factor can be deduced as follows:

$$Q = 2\pi \frac{E}{|\Delta E|} = 2\pi \frac{1}{2\ \%} = 2\pi \frac{1}{0.02} \Longrightarrow Q = 100\ \pi \cong 314.$$

One can observe that the higher the **Q** factor, the smaller the damping, and vice-versa.

Note: Summary of the angular frequencies for harmonic, damped and forced oscillators:

- The angular frequency of an harmonic oscillator (of mass m, and elastic constant k): $\omega_0 = \sqrt{\frac{k}{m}}$.
- The angular frequency of a damped oscillator (**m, k**, coefficient of friction/resistance **C**, damping coefficient δ = C / 2m): $\omega_d = \sqrt{{\omega_0}^2 - \delta^2}$.
- The angular frequency of a forced oscillator (**m, k, C**, external, periodic driving force:
$$F(t) = F_0 \sin(\omega t),\ \text{where } \omega = \omega_f\,.$$
Or else, the frequency of a forced oscillator (ω_f) is equal to the frequency of the external periodic driving force (**ω**).
- The resonance frequency is the frequency of the external force **F(t)**, at which it transfers the maximum amount of energy to the oscillator: $\omega = \omega_r = \sqrt{{\omega_0}^2 - 2\,\delta^2}$.

Problem 38

When will the amplitude of a damped oscillator drop to half of its original value, if the friction coefficient is C = 0.072 kg/s and the spring type oscillator is of mass m = 0.2 kg and elastic constant k = 80 N/m? What is the angular frequency of the damped oscillator?

Solution

The amplitude of the damped oscillations drops in time according to: $A(t) = A_0 e^{-\delta\, t}$ where $\delta = \frac{C}{2m}$.

Requiring the condition $A(t) = \frac{A_0}{2}$, results in:

$$\frac{A_0}{2} = A_0\, e^{-\delta\, t} \;\;\Rightarrow\;\; \frac{1}{2} = e^{-\delta\, t} \;\;\Leftrightarrow\;\; t = \frac{\ln 2}{\delta} = \frac{2m}{C}\, \ln 2 \cong 3.9\ (s).$$

Then the angular frequency of the damped oscillations:

$$\omega_d = \sqrt{\omega_0^2 - \delta^2} = \sqrt{\frac{k}{m} - \left(\frac{C}{2m}\right)^2} \Longrightarrow \omega_d = \frac{\sqrt{4\,k\,m - C^2}}{2\,m}.$$

Problem 39

A metallic beam, part of the structure of a building, oscillates according to the equation:

$$\psi(t) = 0.002\, \sin(\pi t)\ (m).$$

Identify the characteristic parameters of this oscillatory motion.

Solution

The amplitude: A = 0.002 m = 2 mm.

The angular frequency: $\omega_0 = \pi\ \left(\frac{rad}{s}\right)$.

The maximum speed of oscillation: $v_{max} = \omega_0 A = 0.002\pi = 2\pi\ \left(\frac{mm}{s}\right)$.

The maximum acceleration of oscillation: $a_{max} = \omega_0{}^2 A = 0.002\pi^2 \Longrightarrow a_{max} \cong 2\ (cm/s^2)$.

The frequency: $\nu_0 = \frac{\omega_0}{2\pi} = \frac{\pi}{2\pi} \Longrightarrow \nu_0 = \frac{1}{2} = 0.5\ (s^{-1}) \Longrightarrow \nu_0 = 0.5\ (Hz)$.

The period of oscillation: $T_0 = \frac{1}{\nu_0} = 2\ (s)$.

Problem 40

A body with the mass of 0.05 kg is suspended from a spring (of elastic constant k). The mass-spring system (Figure 27-b) is then acted on by an external deformation force (**F**$_d$), and it thus experiences an elongation x = 0.09 m (Figure 27-c), relative to the equilibrium position of the mass-spring system. If the deformation force **F**$_d$ is removed (Figure 27-d), and the system, left under the action of the elastic force **F**$_e$, is set into oscillation, what will be its angular frequency (**ω**), its frequency (**ν**), and period of oscillation (**T**)? *Hint: Assume that $F_r = 0$ (harmonic oscillator) and initial phase zero $\varphi_0 = 0$.*

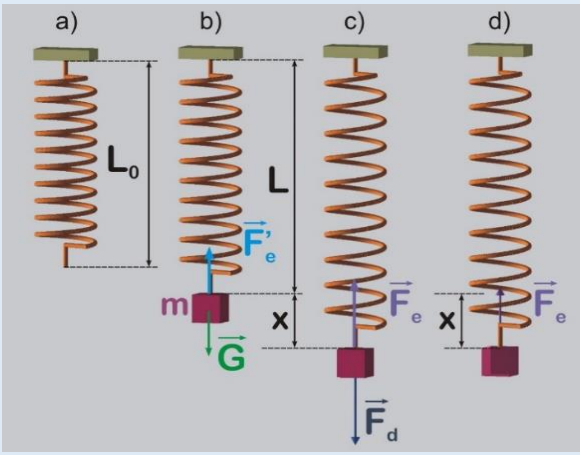


Fig. 27. A mass-spring oscillator [♣].

Solution

If the resistance force is assumed to be zero, then the system performs harmonic oscillations, of equation of motion: $\psi(t) = A\sin(\omega\, t)$. The elastic force that determines the oscillations: $F_e = k x$.

Then the elastic constant is: $k = \frac{F_e}{x} = \frac{mg}{x} = \frac{0.05\ \mathrm{kg} \times 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}}{0.09\ \mathrm{m}} \Longrightarrow k \cong 5.44\,\frac{\mathrm{N}}{\mathrm{m}}$.

The angular frequency of oscillation: $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{x}} = \sqrt{\frac{9.8}{0.09}} \Longrightarrow \omega_0 \cong 10.43\ \left(\frac{rad}{s}\right)$.

The frequency of free (harmonic) oscillations:

$$\nu_0 = \frac{\omega_0}{2\pi} \cong \frac{10.43}{6.28}\mathrm{s}^{-1} = 1.66\ \mathrm{s}^{-1} \Longrightarrow \nu_0 \cong 1.66\ \mathrm{Hz}.$$

The period of harmonic oscillations: $T_0 = \frac{1}{\nu_0} \Longrightarrow T_0 \cong 0.6\ (s)$.

Problem 41

An elastic oscillator (of mass m = 0.5 kg, and elastic constant k = 50 N/m) performs damped oscillations characterized by the logarithmic decrement Δ = 0.125 = 1/8.

a) What is the ratio between the values of the amplitude after one period A(T), and the initial amplitude A_0: $\frac{A(T)}{A_0} = ?$

b) What is the ratio between the value of the speed of the elastic oscillator after one period, and the initial speed of oscillation: $\frac{v(T)}{v_0} = ?$

c) At what moment (t = ?) does the amplitude A(t) drop to half of its initial value, $A_0/2$?

Solution

a) $\frac{A(T)}{A_0} = ?$

For a damped oscillator with the motion described by the equation of the elongation:

$$\psi(t) = A(t)\sin(\omega t + \varphi) = A_0 e^{-\delta t}\sin(\omega t + \varphi),\ \text{with } A(t) = A_0 e^{-\delta t},$$

the logarithmic decrement is defined as:

$$\Delta = \ln\frac{A(t)}{A(t+T)} = \ln\frac{A_0\, e^{-\delta\, t}}{A_0\, e^{-\delta\,(t+T)}} = \ln e^{\delta\, T} = \delta T.$$

Then the required ratio between the amplitudes is:

$$\frac{A(T)}{A_0} = \frac{A_0\, e^{-\delta\, T}}{A_0} = \frac{1}{e^{\delta\, T}} = \frac{1}{e^{\Delta}} = \frac{1}{e^{\frac{1}{8}}} = \frac{1}{\sqrt[8]{e}} \Longrightarrow \frac{A(T)}{A_0} \cong 0.8825.$$

b) The speed of oscillation of the damped oscillator is defined as:

$$v(t) = \frac{d\psi}{dt} = \frac{d\left(A_0 e^{-\delta t} \sin(\omega t + \varphi)\right)}{dt} = -\delta A_0 e^{-\delta t} \sin(\omega t + \varphi) + \omega A_0 e^{-\delta t} \cos(\omega t + \varphi).$$

After one period, T, the speed of oscillation becomes:

$$v(t+T) = -\delta A_0 e^{-\delta(t+T)} \sin[\omega(t+T) + \varphi] + \omega A_0 e^{-\delta(t+T)} \cos[\omega(t+T) + \varphi].$$

Therefore, the ratio of the speed values at moment **t + T** and at **t**, becomes:

$$\frac{v(t+T)}{v(t)} = \frac{A_0 e^{-\delta(t+T)}[\omega \cos(\omega t + \omega T + \varphi) - \delta \sin(\omega t + \omega T + \varphi)]}{A_0 e^{-\delta t}[\omega \cos(\omega t + \varphi) - \delta \sin(\omega t + \varphi)]} \Rightarrow$$

since $\omega T = 2\pi$, one can further deduce that:

$$\frac{v(t+T)}{v(t)} = \frac{A_0 e^{-\delta(t+T)}[\omega \cos(\omega t + \varphi) - \delta \sin(\omega t + \varphi)]}{A_0 e^{-\delta t}[\omega \cos(\omega t + \varphi) - \delta \sin(\omega t + \varphi)]} = e^{-\delta T} = \frac{1}{e^{\delta T}} \Rightarrow \frac{v(t+T)}{v(t)} = \frac{1}{\sqrt[8]{e}}.$$

c) When the amplitude drops to half of its initial value, we have:

$$\frac{1}{2} A_0 = A_0 e^{-\delta t},$$

$$\frac{1}{2} = e^{-\delta t} \Leftrightarrow e^{\delta t} = 2 \Leftrightarrow \delta t = \ln 2 \Rightarrow t = \frac{\ln 2}{\delta}.$$

The damping coefficient δ is related to the logarithmic decrement and the period of the damped oscillator through the equation:

$$\Delta = \delta T_d = \delta \frac{2\pi}{\omega_d} = \delta \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}}.$$

Where T = T_d is the period of the damped oscillator, $\boldsymbol{\omega_0}$ is the angular frequency of the harmonic oscillator (mass m = 0.5 kg, and elastic constant k = 50 N/m): $\omega_0^2 = \frac{k}{m} = \left(10 \frac{rad}{s}\right)^2$.

Therefore, the damping coefficient can be deduced:

$$\Delta^2 = \frac{\delta^2 \, 4\pi^2}{\omega_0^2 - \delta^2} \Rightarrow$$

$$\Delta^2 \omega_0^2 - \Delta^2 \delta^2 = 4\pi^2 \delta^2 \Rightarrow \delta^2 (4\pi^2 + \Delta^2) = \Delta^2 \omega^2 \Rightarrow$$

$$\delta = \frac{\Delta \omega_0}{\sqrt{4\pi^2 + \Delta^2}} = \frac{0.125 \times 10}{\sqrt{40 + 0.125^2}} \cong 0.197 \Rightarrow \delta \cong 0.2\ (s^{-1}).$$

Problem 42

When a body of mass **m** is suspended from a spring (length $\mathbf{L_1}$) of elastic constant $\mathbf{k_1}$, it oscillates at frequency $\nu_1 = 1.5$ Hz. What will be the frequency of oscillation of the spring that results if its length is cut in half? $L_2 = \frac{1}{2} L_1$ (Figure 28-b).

Solution

Hooke's law states that:

$$\frac{F}{S} = E\frac{\Delta l}{L} \Longrightarrow \Delta l = \frac{1}{E}L\frac{F}{S}.$$

With the physical quantities shown in Figure 28:

$$x_1 = \frac{1}{E}L_1\frac{F}{S} \text{ and } x_2 = \frac{1}{E}L_2\frac{F}{S}.$$

As $L_2 = \frac{1}{2}L_1$, therefore:

$$x_2 = \frac{1}{E}\frac{1}{2}L_1\frac{F}{S} = \frac{1}{2}x_1.$$

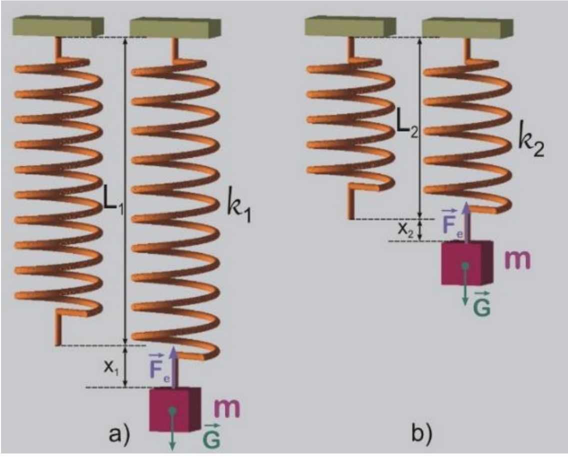


Fig. 28. (a) A mass spring system. (b) The spring is cut in half [♣].

The elongation x_2 equals half of the elongation x_1. The elastic constants of the two springs are given by:

$$k_1 = \frac{F}{x_1} = \frac{mg}{x_1} \text{ and } k_2 = \frac{F}{x_2} = \frac{mg}{x_2} = 2\frac{mg}{x_1}.$$

Then the angular frequencies in the two cases, are: $\omega_1 = \sqrt{\frac{k_1}{m}}$ and $\omega_2 = \sqrt{\frac{k_2}{m}}$.

It follows that the frequencies of oscillation in each of the two cases, are:

$$\nu_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{x_1}} \Longrightarrow \nu_1 = 1.5\ (\mathrm{Hz}).$$

$$\nu_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_2}{m}} = \frac{1}{2\pi}\sqrt{\frac{2g}{x_1}} = \sqrt{2}\times 1.5 \Longrightarrow \nu_2 \cong 2.10\ (\mathrm{Hz}).$$

Problem 43

a) What is the relaxation time (**τ** = ?) of the oscillator described in problem 41?

b) What is the quality factor (**Q** = ?) for the above said oscillator?

Solution

a) The relaxation time (**τ**) of the oscillator is the time required for the amplitude to drop to **1/e** of its initial value:

$$\frac{1}{e}A_0 = A_0 e^{-\delta\tau} \Longrightarrow \ln e = \delta\tau \Longrightarrow \tau = \frac{1}{\delta} \cong \frac{1}{0.2} \Longrightarrow \tau \cong 5\ (\mathrm{s}).$$

In other words, the latter equation gives the physical interpretation of **δ:** it represents the reciprocal of the relaxation time of the damped oscillator.

b) The quality factor for an oscillating system is defined as 2π multiplied by the ratio of the energy stored in the oscillator, and the amount of energy lost over one period.

$$Q = 2\pi \frac{E_{stored}}{(\Delta E)_{lost\ over\ T}} = 2\pi \frac{E}{P{\cdot}T} = \frac{2\pi}{T}\frac{E}{P} = \omega\,\tau = \left(\sqrt{\omega_0^2 - \delta^2}\right)\frac{1}{\delta} \Longrightarrow$$

$$Q = \sqrt{\frac{\omega_0^2}{\delta^2} - 1} = \sqrt{\frac{10^2}{0.2^2} - 1} = \sqrt{\frac{100}{0.04} - 1} \cong \sqrt{\frac{10^4}{4}} \Longrightarrow Q \cong 50.$$

Problem 44

a) A simple pendulum performs damped oscillations and loses 1 % of its energy during each oscillation. What is its quality factor Q ?

b) If the quality factor of an oscillator is Q = 200, by what percentage does the energy decrease during one period of oscillation?

Solution

a) The energy of a damped oscillator decreases in time:

$$\Delta E = E_f - E_i < 0.$$

If: $\frac{|\Delta E|}{E} = 1\ \%$, then the quality factor is: $Q = 2\pi \frac{E}{|\Delta E|} = \frac{2\pi}{1\ \%} = \frac{2\pi}{0.01} = 200\ \pi \Longrightarrow Q \cong 628.$

The quality factor **Q** is a dimensionless number.

b) $Q = 2\pi \frac{E}{|\Delta E|} \Longrightarrow$

The relative drop in energy is: $\frac{|\Delta E|}{E} = \frac{2\pi}{Q} = \frac{2\ \pi}{200}$, and then we obtain:

$$\frac{|\Delta E|}{E} = \frac{\pi}{100} \cong 0.031415 \Longrightarrow \frac{|\Delta E|}{E} = 3.14\ \%.$$

Problem 45

A periodic force $F(t) = 0.1\sin(2\pi t)$ (N) drives an oscillator of mass 0.2 kg, period T_0 = 1 s, and damping coefficient $\delta = \pi/4s^{-1}$.

a) What is the equation of motion of the forced oscillation?

b) Is this system driven at resonance?

Solution

a) If the force $F(t) = F_0\sin(\omega t)$ acts on an oscillator, it determines forced oscillations described by the equation of motion:

$$\psi(t) = A(\omega)\sin(\omega t + \varphi).$$

The amplitude of the forced oscillations depends on the angular frequency of the periodic force:

$$A(\omega) = \frac{F_0}{m\sqrt{({\omega_0}^2 - \omega^2)^2 + 4\,\delta^2\omega^2}},$$

and the phase of the forced oscillation is given by: $\tan\varphi = -\frac{2\,\delta\,\omega}{\omega_0^2 - \omega^2}$.

In this case: F_0 = 0.1 N; ω = 2π (rad/s), and the characteristic angular frequency is:

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi\ (\text{rad/s}).$$

Therefore: $\tan\varphi = \frac{2\delta\omega}{\omega_0^2 - \omega^2} \rightarrow \infty$ and therefore $\Longrightarrow \varphi = \frac{\pi}{2}$ (rad).

The forced amplitude becomes: $A(\omega) = \frac{0.1}{0.2\sqrt{0 + 4\frac{\pi}{16}4\pi^2}} = \frac{1}{2\pi^2} \Longrightarrow A(\omega) \cong 0.05\ (\text{m})$

The equation of motion of the forced oscillator becomes: $\psi(t) = 0.05\ \sin(2\pi t + \varphi)(\text{m})$.

b) An oscillator is stimulated at resonance when the external driving force has the angular frequency $\boldsymbol{\omega_r}$, given by the equation: $\omega_r = \sqrt{\omega_0^2 - 2\delta^2}$.

The characteristic angular frequency of the ideal oscillator is: $\omega_0 = \frac{2\pi}{T_0}$.

The damping coefficient is: $\delta \overset{\text{def}}{=} \frac{C}{2m} = \frac{\pi}{4}\ (s^{-1})$.

Then the resonance frequency is:

$$\omega_r = \sqrt{4\pi^2 - 2\,\frac{\pi^2}{16}} \quad \Longrightarrow \quad \omega_r = \frac{\sqrt{62}}{4}\pi \quad \Longrightarrow \quad \omega_r \cong 6.18\ (\text{rad/s}).$$

This resulting value is different from 2π, which is the angular frequency of the driving force, as it is given in the problem.

7. MECHANICAL WAVES AND ACOUSTICS

7.1 Mechanical waves

Problem 46

What are the coefficients a_0, a_1, a_2, …, and b_0, b_1, b_2, …, of the terms in a Fourier series written for a periodic function $f = f(t)$?

Solution

A periodic function $f(t) = f(t + T)$ can be written as the linear combination of *sin* and *cos* functions with frequencies equal to integer multiples ($\omega_j = j\omega$, where $j = 1, 2, 3, \ldots$) of the fundamental angular frequency ω:

If $f(t)$ is a periodic function: $f(t) = f(t + T)$, where $T = \frac{2\pi}{\omega}$. Then the periodic function is:

$$f(t) = a_0 + \sum_j a_j \sin(j\,\omega t) + \sum_m b_m \cos(m\,\omega t), \qquad \text{with: } \omega = \frac{2\pi}{T}.$$

a) Determining $\mathbf{a_0}$ - by integrating f(t) over one period, T:

$$\int_0^T f(t)\,dt = a_0 \int_0^T dt + \sum_j \int_0^T a_j \sin(j\,\omega t) + \sum_j \int_0^T b_j \cos(j\,\omega t),$$

$$\int_0^T f(t)\,dt = a_0 T + 0 + 0 \implies a_0 = \frac{1}{T}\int_0^T f(t) dt.$$

Integrating the functions $\sin(\omega t)$ and $\cos(\omega t)$ over one period gives 0, as their positive values are cancelled by the negative ones (See also Figure 29).

b) The coefficients a_1, a_2, a_3, … can be determined by integrating $f(t)\sin(m\,\omega t)$ over one period T:

$$\int_0^T f(t)\sin(m\,\omega t)\,dt = a_0 \int_0^T \sin(m\,\omega t)\,dt + \sum_j a_j \int_0^T \sin(j\,\omega t)\,\sin(m\,\omega t)\,dt + \sum_j b_j \int_0^T \cos(j\,\omega t)\sin(m\,\omega t)\,dt\,.$$

In the above equation, the right side member contains terms that depend on the value of 'm', as shown below:

- For $m \neq j \implies$ the second term is null:

$$\int_0^T \sin(j\,\omega t)\,\sin(j\,\omega t)dt = \frac{1}{2}\int_0^T \{\cos[(j-m)\,\omega t] - \cos[(j+m)\,\omega t]\}dt = \frac{1}{2}(0-0) = 0.$$

- For $m=j \implies$ the second term becomes:

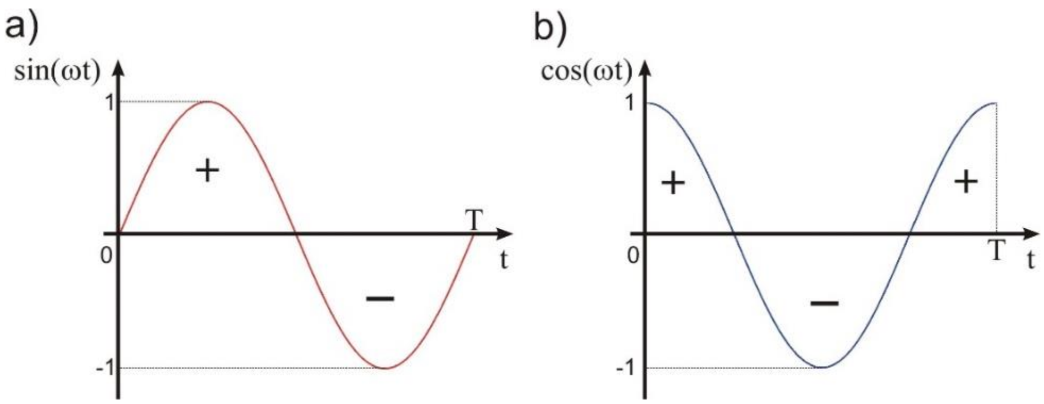


Fig. 29. The values taken by functions $sin(\omega t)$ *and* $cos(\omega t)$ *over one period of oscillation T.*

$$\int_0^T \sin^2(j\,\omega t)\,dt = \frac{1}{2}\int_0^T [1-\cos(2j\omega t)]dt.$$

- For any **m**, equal or different from **j**, the third term becomes null:

$$\int_0^T \cos(j\,\omega t)\,\sin(m\,\omega t)\,dt = \frac{1}{2}\int_0^T \{\sin[(j+m)\,\omega t] + \sin[(j-m)\,\omega t]\}dt = \frac{1}{2}(0+0) = 0.$$

Therefore, the integral of *f*(t) sin (m ωt) over one period **T**, becomes:

$$\int_0^T f(t)\,\sin(j\,\omega t)\,dt = \frac{1}{2}a_j\int_0^T \left(1-\ \cos(2\,j\,\omega t)\right)dt = \frac{T}{2}\,a_j - 0 = \frac{T}{2}\,a_j.$$

This leads to the coefficient a_j of harmonic **'j'** is given by:

$$a_j = \frac{2}{T}\int_0^T f(t)\,\sin(j\,\omega t)\,dt.$$

c) The coefficients b_1, b_2, b_3, …– determined by integrating *f*(t)cos(j ωt) over one period T:

$$\int_0^T f(t)\cos(m\,\omega t)\,dt = a_0\int_0^T \cos(m\,\omega t)\,dt + \sum_j a_j\int_0^T \cos(m\,\omega t)\,\sin(j\,\omega t)\,dt + \sum_j b_j\int_0^T \cos(m\,\omega t)\,\cos(j\,\omega t)\,dt.$$

The terms in the right side member of the above equation depend on the value of the coefficient m, as shown below:

- For any value of **m**, the first term is null, because integrating the cosine function over a period equals zero:

$$a_0\int_0^T \cos(m\,\omega t)\,dt = 0.$$

- For any value of **m**, the second term is null:

$$\int_0^T \cos(m\,\omega t)\,\sin(j\,\omega t)\,dt = \frac{1}{2}\int_0^T \{\sin[(m+j)\,\omega t] - \sin[(m-j)\,\omega t]\}\,dt = \frac{1}{2}(0-0) = 0.$$

- For $m \neq j$, all the members of the sum in the third term are null:

$$\int_0^T \cos(m\,\omega t)\,\cos(j\,\omega t)\,dt = \frac{1}{2}\int_0^T \{\cos[(m-j)\,\omega t] - \cos[(m+j)\,\omega t]\}\,dt = \frac{1}{2}(0-0) = 0.$$

- For $m = j$, the only term different from zero, of the sum in the third term, becomes:

$$b_j\int_0^T \cos^2(j\,\omega t)dt = b_j\frac{1}{2}\int_0^T [1+\cos(2\,j\,\omega t)]dt = b_j\frac{T}{2} + \frac{1}{2}b_j\cdot 0 = \frac{T}{2}\,b_j\,.$$

Therefore, the integral of $f(t)\cdot\cos(m\,\omega t)$ over one period T, becomes:

$$\int_0^T f(t)\,\cos(j\,\omega t)\,dt = \frac{T}{2}\,b_j\,.$$

This leads to the coefficient $\mathbf{b_j}$ of harmonic **'j'**:

$$b_j = \frac{2}{T}\int_0^T f(t)\,\cos(j\,\omega t)\,dt\,.$$

Problem 47

Apply the Fourier analysis to the square function ***f*(t)**, Figure 30, given by the equation below. Calculate the coefficients $\mathbf{a_j}$ of the first three harmonics of the sin(j ωt) function, that can be superimposed to synthesize the square function represented in Figure 30 and/or Figure 31-a, and described by the equation:

$$f(t) = \begin{cases} 1, & \text{if } t \in \left[0, \frac{T}{2}\right) \\ -1, & \text{if } t \in \left[\frac{T}{2}, T\right] \end{cases}.$$

Solution

Determining the free term, the coefficient $\mathbf{a_0}$:

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\left[\int_0^{T/2} 1\,dt + \int_{T/2}^T (-1)\,dt\right] = \frac{1}{T}\left(\frac{T}{2} - \frac{T}{2}\right) = 0.$$

Determining the coefficients of the sin (ωt) harmonics series:

$$a_j = \frac{2}{T}\int_0^T f(t)\,\sin(j\,\omega t)\,dt = \frac{2}{T}\left[\int_0^{T/2} \sin(j\,\omega t)\,dt - \int_{T/2}^T \sin(j\,\omega t)\,dt\right]$$

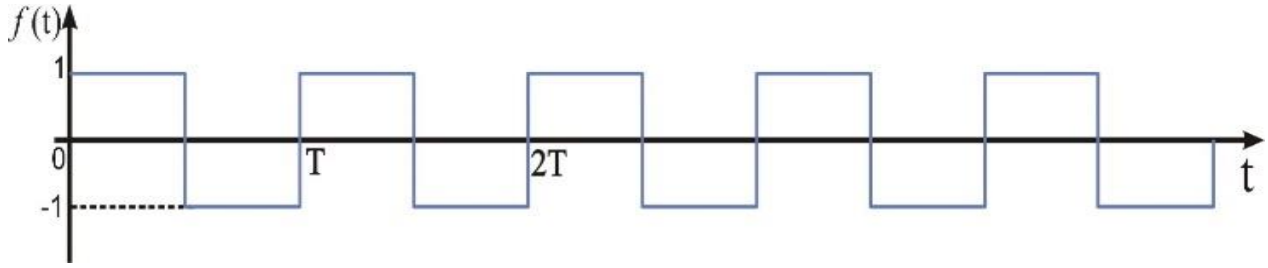


Fig. 30. Graph showing a square function versus time.

$$a_j = \begin{cases} 0 & \text{for even } j = 2, 4, 6, \ldots \\ \dfrac{4}{j \cdot \pi}, & \text{for odd } j = 1, 3, 5, \ldots \end{cases},$$

Therefore, the amplitudes of the **sin** harmonics that are to be superimposed for reconstructing the square wave are given by the following equations:

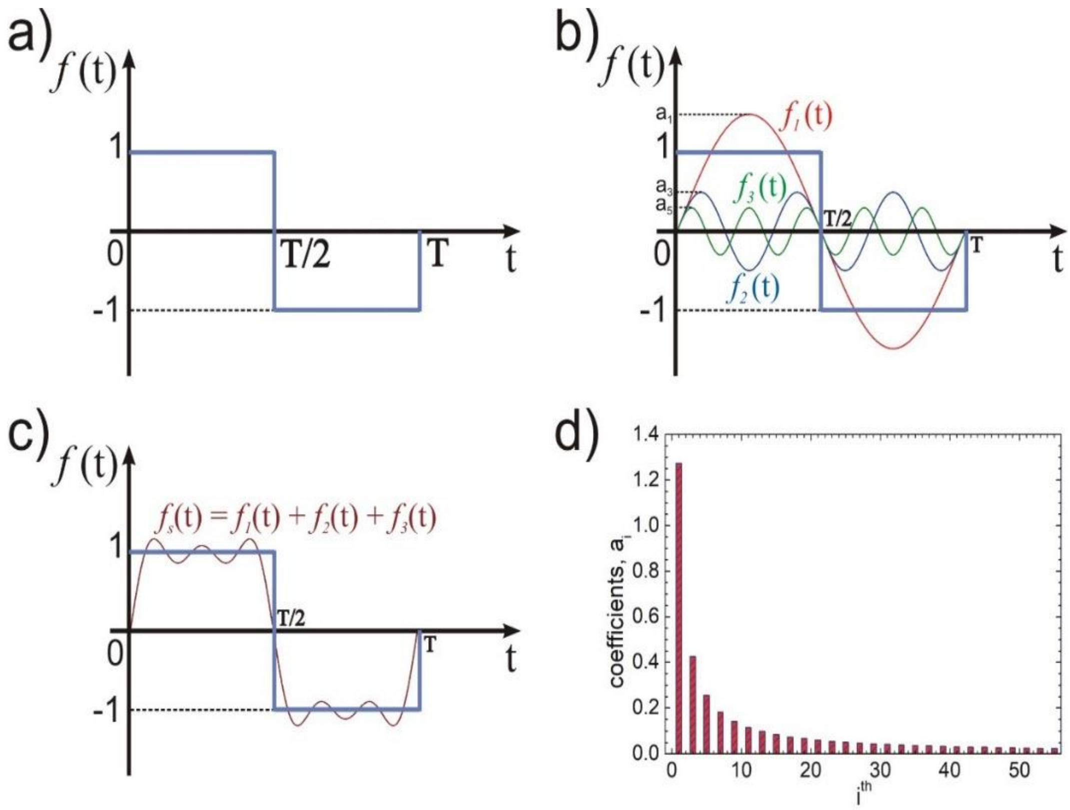


Fig. 31. (a) A square wave over one period T; (b) The first 3 harmonics in the Fourier series of a square wave, over one period; (c) The resultant function $f_s(t) = f_1(t)+f_2(t)+f_3(t)$, the superposition of the first three harmonics, to resynthesize the square function, over one period T; (d) The coefficients a_i or the amplitudes of the oscillations to be added in the Fourier series for synthesizing the square function.

$$a_j = \begin{cases} a_1 = \dfrac{4}{\pi} \rightarrow f_1(t) = \dfrac{4}{\pi}\sin(\omega t) \\ a_3 = \dfrac{4}{3\pi} \rightarrow f_2(t) = \dfrac{4}{3\pi}\sin(3\omega t) \\ a_5 = \dfrac{4}{5\pi} \rightarrow f_3(t) = \dfrac{4}{5\pi}\sin(5\omega t) \\ a_7 = \dfrac{4}{7\pi} \rightarrow f_4 = \dfrac{4}{7\pi}\sin(7\omega t) \\ \qquad a_9, a_{11}, \text{etc.} \ldots \end{cases},$$

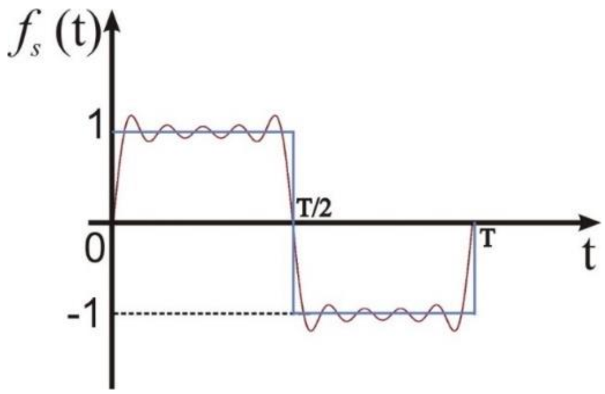


Fig. 32. Adding the 1st to 9th harmonics results in a function $f_s(t)$ that resynthesizes the original square function more accurately than in figure 31-c [♣].

The higher the order ***j*** of the harmonic, the smaller its amplitude $\boldsymbol{a_j}$, and tis contribution to the total function resulted by reconstruction [the re-synthesized wave function $\mathbf{f_s(t)}$], by overlaying the harmonics of the Fourier series:

$$f_s(t) = f_1(t) + f_2(t) + f_3(t) + \cdots$$

If we now consider the first three odd multiple harmonics, the synthesized function becomes:

$$f_s(t) = a_1 \sin(\omega t) + a_3 \sin(3\omega t) + a_5 \sin(5\omega t).$$

By substituting the coefficients with the results obtained, the resynthesized function results as:

$$f_s(t) = \frac{4}{\pi}\sin(\omega t) + \frac{4}{3\omega}\sin(3\omega t) + \frac{4}{5\omega}\sin(5\omega t).$$

The superposition of these three oscillations results in the wave form shown in Figure 31 (c). The more harmonics of higher order are added, the better the synthesized oscillation $\boldsymbol{f_s(t)}$ becomes close to the original periodic function $\boldsymbol{f(t)}$:

$$f_s(t) \cong f(t) = f(t + T).$$

Adding harmonics of higher order (higher pitch) improves the accuracy of the resulting wave, as shown in Figure 32, especially in better reproducing the detailed shapes (the sharp corners) of the original periodic function $f(t) = f(t+T)$.

Problem 48

A string with the linear mass density μ = 480 g/m is stressed with a tension force of 48 N. A transverse wave of frequency 200 Hz and amplitude 4 mm travels down the tensioned string. What is the power of this wave (the rate at which the wave transports the vibrational energy)?

Solution

- The wave speed along the string: $v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{48}{0.48}} \Longrightarrow v = 10 \left(\frac{m}{s}\right)$.

- The angular frequency of the transverse wave: $\omega = 2\pi\,\nu = 2\pi\,200 = 400\pi$ (rad/s).
- Finding the power of the wave as the amount of vibrational (oscillatory) energy carried by the wave during the unit time:

For a segment of the vibratory string, of length **dx** and mass $dm = \mu\cdot dx$, the total oscillatory energy is equal to its maximum potential energy: $E_p = \frac{1}{2}\cdot k\cdot A^2$, where $k = dm\cdot\omega^2 = \mu\cdot dx\cdot\omega^2$, and **A** represents the amplitude of the wave.

By dividing this result by the time interval when this oscillatory energy is transferred between neighboring segments along the string, one obtains:

$$P = \frac{E_p}{dt} = \frac{1}{2}\,\frac{\mu\,dx\,\omega^2 A^2}{dt} \Rightarrow P = \frac{1}{2}\,\mu\,\omega^2 A^2 \frac{dx}{dt}.$$

In the above, we used that $\frac{dx}{dt} = v$ is the speed at which the oscillatory motion propagates along the tensioned string. With the data given in the problem, the power (or the rate at which vibrational energy is carried along the vibrating string) results as:

$$P = \frac{1}{2}\,0.48\times 10^2\,(400\pi)^2 \times (4\,\times 10^{-3})^2 \Longrightarrow P = 61\ (\mathrm{W}).$$

Problem 49

In a mandolin, the string G ('sol') is 0.34 m long and has a linear density μ = 4 g/m. The thumbscrew attached to the mandolin cord is adjusted to provide a tension of 71.1 N. What is then the fundamental frequency of the string?

Solution

The speed of the transverse wave in the tensioned string:

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{71.1}{0.004}} = \sqrt{17775} \Longrightarrow$$

$$v \cong 133.32\ \left(\frac{m}{s}\right).$$

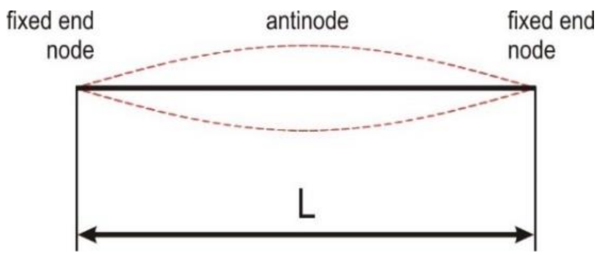


Fig. 33. The fundamental mode (n = 1 loop) of a tensioned string with fixed ends.

The wavelength of the fundamental mode of vibration is twice the length of one loop:

$$\lambda_1 = 2\,L \Longrightarrow \lambda_1 = 0.68\ \mathrm{m}.$$

The frequency of the fundamental mode of vibration:

$$\nu_1 = \frac{v}{\lambda_1} = \frac{133.32}{0.68} \Longrightarrow \quad \nu_1 \cong 196\ (\mathrm{Hz}).$$

Problem 50

For copper, the Young's modulus of elasticity is $Y = 1.4 \times 10^{11}\left(\frac{N}{m^2}\right)$ and the specific gravity (i.e. the mass density) is $\rho = 8.92\left(\frac{g}{cm^3}\right)$.

a) What is the speed of sound (a longitudinal wave) in copper?

b) Calculate the fundamental frequency and the second harmonic for a 2 m long Cu rod, fixed in the middle, and with free ends (Figure 34)?

Solution

The speed of longitudinal mechanical waves in copper (Cu):

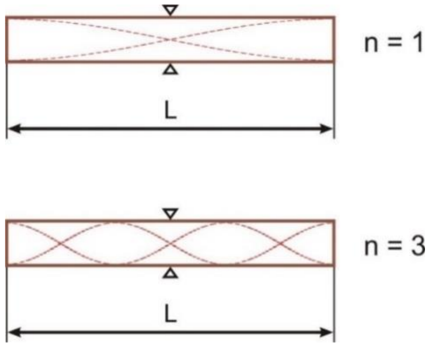


Fig. 34. A rod fixed at the middle, with free ends.

a)

$$v_l = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{1.4 \times 10^{11}}{8900}} \Longrightarrow v_l = 3960 \left(\frac{m}{s}\right).$$

b) The fundamental frequency (for n = 1 loop):

$$\nu_1 = \frac{v}{\lambda_1} = \frac{v}{2L} = \frac{3960\ \frac{m}{s}}{2 \times 2\ m} \Longrightarrow \nu_1 = 990\ \text{Hz}.$$

The second harmonic (n = 3 loops):

$$\nu_3 = \frac{v}{\lambda_3} = \frac{v}{4\ \frac{L}{6}} = 6\ \frac{3960}{4 \times 2}\ \text{Hz} \Longrightarrow \nu_3 = 2970\ \text{Hz} = 3\nu_1.$$

Problem 51

The speed of sound in iron (of *density*, or *specific gravity* ρ = 7.8 g/cm^3) is 5130 m/s.

a) Determine the modulus of elasticity for iron (Fe)?

b) What are the first three modes of vibration (or the resonances) of an iron beam fixed at one end, of length 0.4 m (Figure. 35)?

Solution

a) The density of iron, in SI units:

$$\rho = 7.8\ \frac{g}{cm^3} = 7.8 \times \frac{10^{-3}\ kg}{10^{-6}\ m^3} \Longrightarrow$$

$$\rho = 7800\ \frac{kg}{m^3}.$$

The speed of longitudinal waves in solids:

$$v = \sqrt{\frac{Y}{\rho}}.$$

Therefore, Young's modulus of elasticity for iron (Fe) results as:

$$\mathbf{Y = \rho\, v^2 \Longrightarrow Y = 2.05 \times 10^{11}\,\frac{N}{m^2}}.$$

b) The fundamental frequency for a beam fixed at one end corresponds to the vibration mode (n = 1) with wavelength $\lambda_1 = 4\,L$. The fundamental frequency is then:

$$\nu_1 = \frac{v}{\lambda_1} = \frac{v}{4L} = \frac{5130\,\frac{m}{s}}{1.6\ m} = 3206.25\ s^{-1} \Longrightarrow$$

$$\nu_1 = 3206.25\ \mathrm{Hz}.$$

The next vibration mode possible (n = 3), with a node at the fixed end, and an antinode at the free end, corresponds to the wavelength:

$$\lambda_3 = 4\,\frac{L}{3}\ .$$

Thus, the first harmonic has the frequency:

$$\nu_3 = \frac{v}{\lambda_3} = \frac{v}{4\,\frac{L}{3}} = \frac{3}{4}\,\frac{v}{L} \Longrightarrow$$

$$\nu_3 \cong 9.62\ \mathrm{kHz} = 3\nu_1.$$

The next possible vibration mode n = 5 corresponds to the wavelength $\lambda_5 = 4\,\frac{L}{5}$. Therefore, the frequency of the second harmonic is:

$$\nu_5 = \frac{v}{\lambda_5} = \frac{5\,v}{4\,L} \Longrightarrow$$

$$\nu_5 \cong 16\ \mathrm{kHz} = 5\,\nu_1.$$

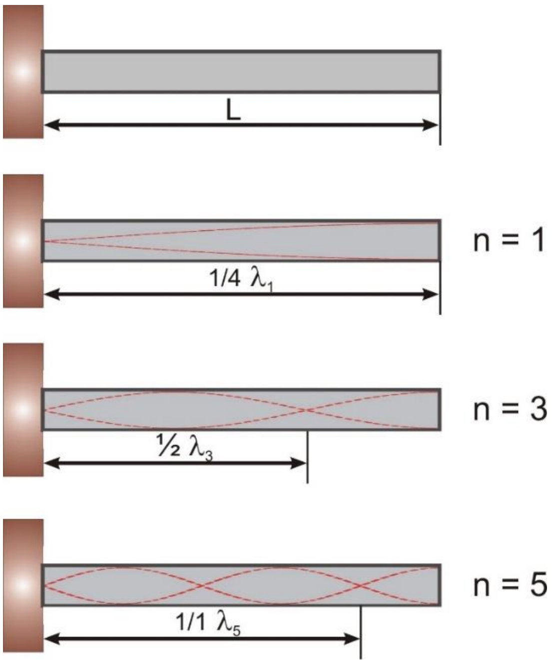


Fig. 35. Harmonics of vibration for a beam with a fixed end

7.2 Acoustics

Brief review of the key equations to be used in Acoustics

- The power of an acoustic wave: $P = \frac{1}{2}\,\rho\,v\,S\,\omega^2 A^2$,
- The intensity of an acoustic wave: $I = \frac{P}{S} = \frac{1}{2}\,\rho\,v\,\omega^2 A^2$,
- Instant pressure of an acoustic wave: $\Delta p(t) = \rho\,v\,\omega\,A\,\sin(\omega t - kx)$,
- Maximum of the pressure variation: $(\Delta p)_0 = \rho\,v\,\omega\,A = \rho\,v\,u_{max}$,
  (where $u(x,t) = \frac{\partial\psi}{\partial t}$ = the speed of vibration of the particles reached by the acoustic wave)
- Therefore, the acoustic wave intensity: $I = \frac{{(\Delta p)_0}^2}{2\,\rho\,v}$.

Problem 52

Normal conversation is carried out at an acoustic level of 60 dB. What is the intensity level that this value corresponds to?

Solution

The acoustic level: $L = 10\,\lg\left(\frac{I}{I_0}\right)$, where the threshold acoustic level is $I_0 = 10^{-12}\,\frac{W}{m^2}$.

Solving for the sound intensity I: $L = 10\,\lg\left(\frac{I}{I_0}\right) \Longrightarrow \frac{L}{10} = \lg\left(\frac{I}{I_0}\right) \Longrightarrow \frac{I}{I_0} = 10^{\frac{L}{10}} \Longrightarrow$

$$I = I_0\,10^{\frac{L}{10}} \Longrightarrow I = 10^{-12}\,\frac{W}{m^2}\,10^{\frac{60}{10}} \Longrightarrow I = 10^{-6}\,\frac{W}{m^2} = 1\,\frac{\mu W}{m^2}.$$

Problem 53

A point like source of sound (S) emits isotropically (vibrational energy released at the same intensity along any direction in space), at a power of 60 W.

a) What is the intensity of the sound wave at 4 m distance from the point like source?

b) What is the sound level (the sound pressure level or the acoustic pressure level)?

Solution

a) The intensity of a wave is the rate at which it transfers energy during its propagation, across the unit surface area (see Figure 36):

$$I = \frac{P}{S_\Sigma} = \frac{P}{4\pi\,R^2} \Longrightarrow$$

$$I = \frac{60\ \text{W}}{4\pi\ 4^2\ \text{m}^2} \Longrightarrow I \cong 0.3\ \frac{\text{W}}{\text{m}^2}$$

b) The sound level, L is defined as:

$$L = 10\ \lg\left(\frac{I}{I_0}\right) = 10\ \lg\left(\frac{0.3}{10^{-12}}\right) = 10\ \lg(3 \times 10^{11}) \Longrightarrow$$

$$L = 10\ (\lg 10^{11} + \lg 3) \Longrightarrow$$

$$L = 110\ \text{dB} + 10\ \lg 3\ \text{dB} \Longrightarrow$$

$$L \cong 114.77\ \text{dB}.$$

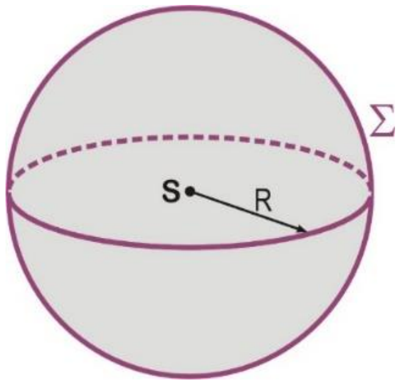


Fig. 36. A point like source of sound S emits sound isotropically in space.

Problem 54

A point like source (S) of sound emits sounds isotropically. At distance $R_1 = 5$ m from the source, the sound level is 90 dB. At what distance $\mathbf{R_2}$ from the source S, will the sound level drop to 50 dB?

Solution

At distance $\mathbf{R_1}$ from S (Figure 37), the sound wave intensity is:

$$I_1 = \frac{P}{4\pi\ R_1^2}.$$

At distance $\mathbf{R_2}$ from S, the sound wave intensity is:

$$I_2 = \frac{P}{4\pi\ R_2^2}.$$

The sound levels at distance $\mathbf{R_1}$ and $\mathbf{R_2}$, respectively:

$$L_1 = 10\ \lg\left(\frac{I_1}{I_0}\right) \text{ and } L_2 = 10\ \lg\left(\frac{I_2}{I_0}\right),$$

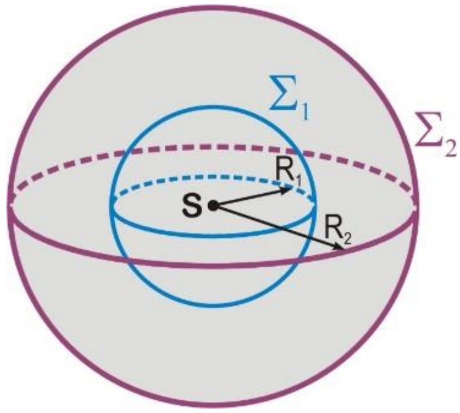


Fig. 37. A point like source S emits sound in the 3D space around it.

therefore:

$$L_1 - L_2 = 10\ \lg\left(\frac{I_1}{I_0}\right) - 10\ \lg\left(\frac{I_2}{I_0}\right) = 10\ \lg\left(\frac{I_1\ I_0}{I_0\ I_2}\right) \Longrightarrow$$

$$L_1 - L_2 = 10\ \lg\left(\frac{I_1}{I_2}\right) \Longrightarrow \frac{L_1 - L_2}{10} = \lg\left(\frac{I_1}{I_2}\right) \Longrightarrow$$

$$\frac{I_1}{I_2} = 10^{\frac{L_1 - L_2}{10}} = 10^{\frac{90-50}{10}} \Longrightarrow \frac{I_1}{I_2} = 10^4\ .$$

On the other hand, the ratio of the sound wave intensities at distance $\mathbf{R_1}$ and $\mathbf{R_2}$ from S, is:

$$\frac{I_1}{I_2} = \frac{\mathrm{P}}{4\pi\, R_1^2}\,\frac{4\pi\, R_2^2}{\mathrm{P}} = \left(\frac{\mathrm{R}_2}{\mathrm{R}_1}\right)^2.$$

Then:

$$\left(\frac{\mathrm{R}_2}{\mathrm{R}_1}\right)^2 = 10^4.$$

The distance from S where the sound level drops to $L_2 = 50$ dB is then given by:

$$\mathrm{R}_2 = 10^2 \mathrm{R}_1 = 500 \text{ m}.$$

Problem 55

A source of sound S radiates uniformly in all directions. What is the drop in the sound level, **ΔL** (in dB) when the distance from the source doubles?

Solution

At distance $\mathbf{d_1}$ from the source, the sound intensity and the sound level are:

$$\mathrm{I}_1 = \frac{\mathrm{P}}{4\pi\, d_1^2} \qquad \text{and} \qquad \mathrm{L}_1 = 10 \lg\left(\frac{\mathrm{I}_1}{\mathrm{I}_0}\right).$$

At distance $d_2 = 2\, d_1$ from the source S, the sound intensity and the sound level are:

$$\mathrm{I}_2 = \frac{\mathrm{P}}{4\pi\, \mathrm{d}_2^2} = \frac{\mathrm{P}}{4\pi\, (2\mathrm{d}_1)^2} = \frac{1}{4}\frac{\mathrm{P}}{4\pi\, \mathrm{d}_1^2} \Longrightarrow \mathrm{I}_2 = \frac{\mathrm{I}_1}{4},$$

and

$$\mathrm{L}_2 = 10 \lg\left(\frac{\mathrm{I}_2}{\mathrm{I}_0}\right) = 10 \lg\left(\frac{\mathrm{I}_1}{4\, \mathrm{I}_0}\right).$$

Then the drop in the sound level when the distance to the source doubles, is:

$$\Delta\mathrm{L} = \mathrm{L}_1 - \mathrm{L}_2 = 10 \lg\left(\frac{\mathrm{I}_1}{\mathrm{I}_2}\right) \Longrightarrow \mathrm{L}_1 - \mathrm{L}_2 = 10 \lg(4).$$

$$\Delta\mathrm{L} \cong 6.02 \text{ dB}.$$

Problem 56

Two sources emit sound at powers $P_1 = 100$ W and $P_2 = 200$ W, respectively.

a) What is the resulting sound level at point A, at 10 m distance from S_2 and 20 m from S_1.

b) What is the sound level ($\mathbf{L_{A1}}$ and $\mathbf{L_{A2}}$) generated by each of the sources, at point A?

Solution

a) The sound level of the sound at point A (Figure 38) is defined as:

$$\mathrm{L}_A = 10 \lg\left(\frac{\mathrm{I}_A}{\mathrm{I}_0}\right) \text{ (dB)},$$

where: $\mathbf{I_A}$ - the total sound intensity at point A, and $I_0 = 10^{-12}$ W/m^2 is the audibility threshold intensity:

$$I_A = I_{1A} + I_{2A} = \frac{P_1}{4\pi\, d_1^2} + \frac{P_2}{4\pi\, d_2^2} \Longrightarrow$$

$$I_A = \frac{100\ \mathrm{W}}{4\pi \times 100\ \mathrm{m}^2} + \frac{200\ \mathrm{W}}{4\pi \times 400\ \mathrm{m}^2} \Longrightarrow$$

$$I_A = \frac{1}{4\pi}\left(1 + \frac{1}{2}\right)\frac{\mathrm{W}}{\mathrm{m}^2} \Longrightarrow$$

$$I_A \cong 0.12\ \frac{\mathrm{W}}{\mathrm{m}^2}.$$

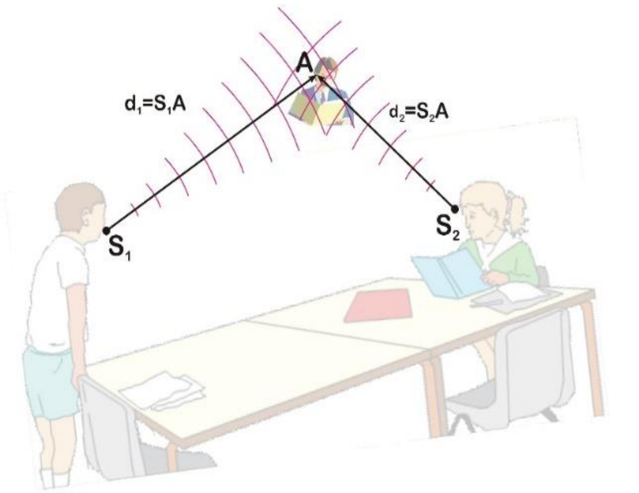


Fig. 38. Two sources of sound emit acoustic waves at sound levels LA_1 and LA_2.

Therefore:

$$L_A = 10\ \lg\left(\frac{I_A}{I_0}\right) = 10\ \lg\left(\frac{12 \times 10^{-2}}{10^{-12}}\right)$$

$$L_A = 10\ \lg(12 \times 10^{10}) \Longrightarrow$$

$$L_A = 10\ (\lg 12 + \lg 10^{10}) = 100 + 10\ \lg(12)$$

$$L_A = 100 + 10\ (\lg 3 + \lg 4)$$

$$L_A = 100 + 10\ (0.47 + 0.6)$$

$$L_A \cong 110.7\ (\mathrm{dB}).$$

b) The source S_1 generates at point A the sound level:

$$L_{A1} = 10\ \lg\left(\frac{I_{1A}}{I_0}\right) = 10\ \lg\left(\frac{\frac{1}{4\pi}}{10^{-12}}\right) = 10\ \lg\left(\frac{0.08}{10^{-12}}\right) = 10\ [\lg(8) + \lg(10^{10})] \Longrightarrow$$

$$L_{A1} = 10\ [3\ \lg(2) + 10] \Longrightarrow L_{A1} \cong 100 + 30 \times 0.3 = 109\ (\mathrm{dB}).$$

The source S_2 generates at point A the sound level:

$$L_{A2} = 10\ \lg\left(\frac{I_{2A}}{I_0}\right) = 10\ \lg\left(\frac{\frac{1}{8\pi}}{10^{-12}}\right) = 10\ \lg\left(\frac{0.04}{10^{-12}}\right) = 10\ \lg(4 \times 10^{10}) \Longrightarrow$$

$$L_{A2} = 100 + 20\ \lg(2) = 100 + 20 \times 0.03 \Longrightarrow$$

$$L_{A2} = 106\ (\mathrm{dB}).$$

One notices that:

$$L_{A1} + L_{A2} = 109\ \mathrm{dB} + 106\ \mathrm{dB} \Longrightarrow$$

$$L_{A1} + L_{A2} = 215\ \mathrm{dB}.$$

Therefore, the total sound level at point A, calculated at point (a) is different from the sum of the individual sound levels of each of the two sources.

Problem 57

A siren sounds on an ambulance vehicle that moves at 90 km/h. If the siren frequency is 300 Hz, what frequency would a cyclist hear the sound when moving towards the siren, at 18 km/h?

Solution

$$\nu_p = \nu_0\left(\frac{v \pm w}{v \mp u}\right);\ \text{where} \begin{cases} \begin{cases} +\,w\text{, if O approaches S} \\ -\,w\text{, if O receds from S} \end{cases} \\ \begin{cases} -\,u\text{, if the source approaches O} \\ +u\text{, if the source recedes from O} \end{cases} \end{cases}.$$

As a result of the relative motion of the source S of sound and of the observer O, the frequency perceived is different from the frequency of the sound released by the source S. This phenomenon is known as the longitudinal Doppler effect. Thus, the frequency of the sound perceived is given by one of the equations:

Where:

- $\boldsymbol{\nu_p}$ is the frequency of the sound perceived by the observer O;
- $\boldsymbol{\nu_0}$ is the frequency of the sound emitted from the point like source S;
- **v** is the speed at which sound propagates in air;
- **w** is the speed of the observer O;
- **u** is the speed of the source S;

In this case, when the observer moves towards the source (+w = 20 km/h), and the source S moves towards the observer (u = 80 km/h), the frequency of the sound perceived by the observer O becomes:

$$\nu_p = \nu_0\left(\frac{v+w}{v-u}\right) = 300\ \text{Hz}\left(\frac{330\,\frac{m}{s} + 5\,\frac{m}{s}}{330\,\frac{m}{s} - 25\,\frac{m}{s}}\right) \Rightarrow \nu_p \cong 329.5\ \text{Hz}.$$

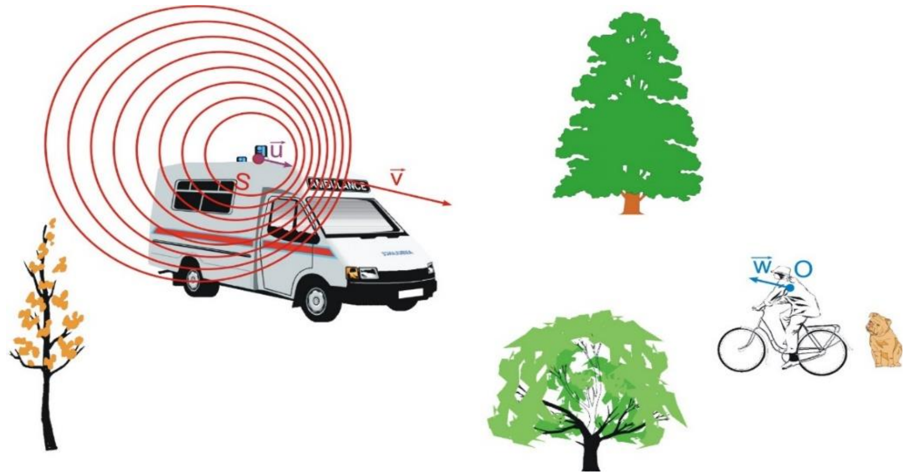


Fig. 39. Sound wave fronts for a source of sound moving at velocity v towards the observer (O).

8. SPECIAL RELATIVITY

Problem 58

A supersonic plane moves with $v_0 = 1000$ m/s (which is approximately the speed of sound in air, typically 1224 m/s) along the Ox axis (see Figure 40). Another plane moves at v' = v_r = 500 m/s relative to the first one. How fast (v = ?) is it moving relative to an observer at rest, on the ground?

Solution

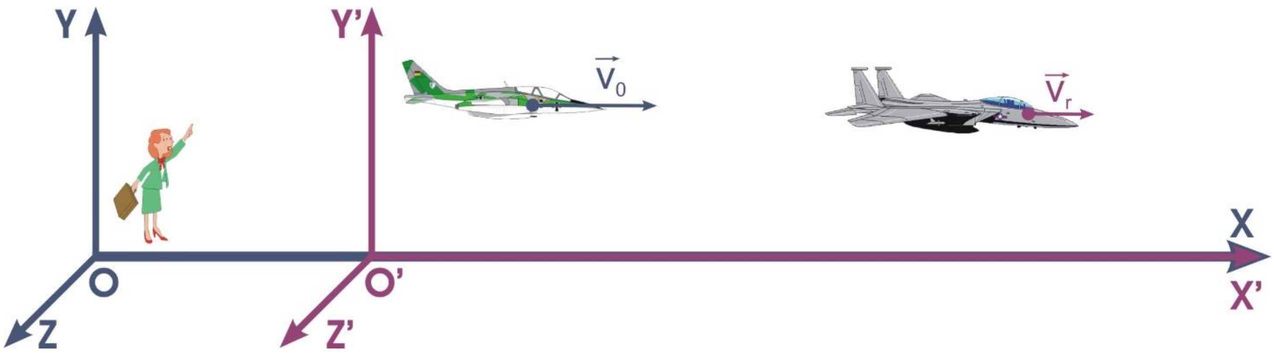


Fig. 40. Two supersonic planes moving along Ox.

For speeds much smaller than the speed of light in vacuum ($c \cong 3 \times 10^8$ m/s), the classical superposition of velocities applies, and relative to plane 1, the relative velocity of plane 2, is:

$$\vec{v}_r = \vec{v} - \vec{v}_0.$$

Where:

- The velocity of plane 1 relative to ground: $\vec{v}_0$;
- The velocity of plane 2 relative to plane 1:

$$\vec{v}_r = \vec{v'};$$

- The velocity of plane 2 relative to the ground (the frame of reference at rest): $\vec{v}$.

Then the absolute velocity of plane 2 relative to the ground results as:

$$\vec{v} = \vec{v}_r + \vec{v}_0.$$

In absolute value, considering the directions of the two parallel vectors (shown in Figure 40):

$$v = v_r + v_0 = 500 + 1000 \implies v = 1500 \text{ m/s}.$$

Problem 59

Calculate the absolute velocity of a particle (B) if it moves at speed $\boldsymbol{v_r} = 0.8\boldsymbol{c}$ relative to another particle (A) that is in motion at speed $\boldsymbol{v_0} = 0.8\boldsymbol{c}$ relative to a system of reference considered at rest, where **c** is the speed of light in free space.

Solution

Classical superposition of velocities:

When applying the classical superposition of velocities to particles A and B that move close to the speed of light in free space, the velocity of B relative to particle A is given by the difference between the absolute velocity of particle B relative to the frame of reference at rest and the velocity $\boldsymbol{v_0}$ of the particle A relative to the frame at rest:

$$\mathrm{v_r} = \mathrm{v}_a - \mathrm{v_0}.$$

In this case, based on the *classical* superposition of velocities, one would obtain that the absolute speed $\boldsymbol{v}$ of particle B, relative to the system considered at rest, would be greater than $\boldsymbol{c}$:

$$\mathrm{v}_a = \mathrm{v_r} + \mathrm{v_0} = 1.6\ \mathrm{c}.$$

Relativistic superposition of velocities:

In elaborating the special relativity theory, Einstein postulated that no particle can move at speeds greater than $\boldsymbol{c}$, the speed of light in free space. When certain elementary particles move at speeds approaching the speed of light, one applies the *relativistic* superposition of velocities. This is obtained (see the course material) from the Lorentz-Einstein equations that describe the space-time coordinates transformation between two frames of reference: one considered at rest, and the other in motion at constant velocity $\boldsymbol{v_0}$ (such as particle A in this problem).

By applying the relativistic superposition of velocities, one obtains the absolute velocity $\boldsymbol{v_a}$ of the second particle, relative to a frame of reference at rest, according to the equation:

$$\mathbf{v_a} = \frac{\mathbf{v_r} + \mathbf{v_0}}{1 + \frac{\mathbf{v_r\,v_0}}{\mathbf{c^2}}} = \frac{\mathbf{0.8\,c + 0.8\,c}}{\mathbf{1} + \frac{\mathbf{0.8^2\,c^2}}{\mathbf{c^2}}} \Rightarrow$$

$$\mathbf{v_a} = \frac{\mathbf{1.6\,c}}{\mathbf{1.64}} \Rightarrow$$

$$\mathbf{v_a} \cong \mathbf{0.9\,c}.$$

Therefore, when one applies the relativistic equations, the absolute velocity of particle B relative to a frame of reference considered at rest, cannot be greater than the speed of light $\boldsymbol{c}$ in free space, even though it moves faster than another particle (A) that is also in motion relative to the frame considered at rest.

Problem 60

An electron of rest mass $m_0 \cong 9.1 \times 10^{-31}$ kg is subjected to a constant force **F**. Determine what is the magnitude of its velocity as a function of time **v = v(t)**, if the electron is initially at rest ($v_0 = 0$)?

Notes

The speed of light $c \cong 3 \times 10^8$ m/s, is a cosmic speed limit. Light happens to move that fast, but particles with mass don't. Light photons, neutrinos, gravitons have the rest mass zero, $\mathbf{m_0} = 0$.

1. The linear momentum in classical mechanics is: $p = m\,v$;
2. The linear momentum in relativistic mechanics is defined as: $p\,(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\,v$. This expression tends to the value $p \cong m_0\,v$, for speeds v << c, when $\left(\frac{v}{c}\right)^2 \cong 0$, as it is in classical mechanics.

Solution

<u>*Classical approach*</u>:

In classical mechanics we have that: $v = a\,t = \frac{F}{m_0}t$, which means that the speed may increase infinitely. Though, experiments with electrons (very light particles, of mass $m_e \cong 9.1 \times 10^{-31}$ kg) accelerated at higher and higher voltages demonstrate that the electrons cannot move faster than a certain limit that they tend asymptotically to. Therefore, at these speeds the classical approximation cannot be applied.

<u>*Relativistic approach:*</u>

The force equals the derivative of linear momentum relative to time: $F = \frac{dp}{dt}$.

This equation can be also written by separating the variables: $dp = F\,dt$.

Then, by integrating, one obtains:

$$\int_0^p dp = \int_0^t F\,dt\,, \qquad \text{and then:} \qquad p = F\,t.$$

In the left side member of the equation, the relativistic linear momentum is: $p = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\,v$.

Therefore, the equation becomes: $\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\,v = F\,t$.

Solving it for the speed v, one obtains:

$$\frac{m_0^2}{1-\frac{v^2}{c^2}}\,v^2 = F^2\,t^2\,.$$

Then: $$m_0^2\,v^2 = F^2\,t^2 - F^2\,t^2\,\frac{v^2}{c^2} \Leftrightarrow \left(m_0^2 + \frac{F^2\,t^2}{c^2}\right) v^2 = F^2\,t^2,$$

And the velocity then becomes: $$v^2 = \frac{F^2\,t^2}{m_0^2+\frac{F^2\,t^2}{c^2}} \Rightarrow v = \frac{F\,t}{m_0\,\sqrt{1+\left(\frac{F\,t}{m_0\,c}\right)^2}}.$$

One can see that, when time *t* tends to infinity, the speed of the particle acted on by the force F tends to a finite limit, which is the speed of light in vacuum ***c**:*

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{F\,t}{m_0 t\sqrt{\frac{1}{t^2} + \left(\frac{F}{m_0\,c}\right)^2}} = \frac{F}{m_0\,\frac{F}{m_0\,c}} \Longrightarrow \lim_{t\to\infty} v(t) = c\,.$$

Therefore: if $t \to \infty\ \Rightarrow$ then $v(t) \to c$, where the result, c, is a finite limit.

Problem 61

At what speed the kinetic energy **T** of a relativistic particle (i.e. a particle that moves at speeds close to the speed of light in free space) is equal to its rest energy $\mathbf{E_0}$?

Solution

The total energy of a relativistic particle (v is very close to the speed of light in free space, c):

$$E = m\,c^2 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\,c^2.$$

The total relativistic energy E equals the sum between the kinetic (T) and the rest energy E_0:

$$E = T + m_0\,c^2\,.$$

If the kinetic energy T equals the rest energy E_0: $T = m_0\,c^2$, and then the total energy becomes:

$$E = 2\,m_0\,c^2.$$

Therefore: $\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\,c^2 = 2\,m_0\,c^2.$

Solving the equation for v, results in: $\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \Leftrightarrow 1-\frac{v^2}{c^2} = \frac{1}{4}$,

$$v^2 = \frac{5}{4}\,c^2 \;\Rightarrow\; v \cong 0.87\ c\,.$$

Problem 62

Deduce the relativistic mass-energy relationship between the mass of a particle and its total energy $E = m\,c^2$, by using the variation of the particle mass with its speed in the special relativity theory.
Hint: *determine the elementary increase in the kinetic energy T of a particle acted on by the force* $\vec{F} = \frac{d\vec{p}}{dt}$*. Consider the total energy E of the particle as the sum between its kinetic and rest energies.*

Solution

a) If:

- $\delta W = \vec{F}\cdot d\vec{r}$ is the elementary amount of work done by a force $\vec{F} = \frac{d\vec{p}}{dt}$, acting on a particle of mass m and linear momentum $\vec{p} = m\,\vec{v} = m\,\frac{d\vec{r}}{dt}$, when the particle is moved along $d\vec{r}$;
- dT is the elementary increase in the kinetic energy of the particle of mass m, as a result of the elementary work δW done upon the particle, then one can write that:

$$dT = \delta W\,,$$

$$dT = \vec{F}\cdot d\vec{r} = \frac{d\vec{p}}{dt}\cdot d\vec{r} = d\vec{p}\cdot\frac{d\vec{r}}{dt} = d(m\,\vec{v})\cdot\vec{v}\ .$$

When the particle performs a rectilinear motion (along one dimension), the above equation becomes:

$$dT = v\,d(mv) = v\,(m\,dv + v\,dm)\,,$$

$$dT = mv\,dv + v^2\,dm\,.$$

b) If the particle has the rest mass $\mathbf{m_0}$, its relativistic mass $\mathbf{m}$ is given by the equation:

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}.$$

By raising to the power 2, multiplying in both members by c^2, and then by rearranging the terms, one obtains:

$$m^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}} \qquad \Rightarrow \quad m^2 - m^2\,\frac{v^2}{c^2} = m_0^2.$$

After regrouping the terms, one obtains:

$$m^2\,c^2 - m^2\,v^2 = m_0^2\,c^2.$$

By applying the differential operator in each of the two members of the equation, one gets:

$$d(m^2\,c^2 - m^2\,v^2) = d(m_0^2\ c^2)\,, \quad \Rightarrow \quad 2m\,dm\,c^2 - 2m\,dm\,v^2 - 2m^2\,v\,dv = 0.$$

The second term of the equation is null, as the product $m_0^2\,c^2 = \text{constant}$, and therefore, its differential is zero. After rearrangements:

$$c^2\,dm = m\,v\,dv + v^2\,dm.$$

c) By comparing the above results from the points a) and b), the right side members of the equations are identical. Therefore, $dT = c^2\,dm$, or else: $dm = \frac{dT}{c^2}$.

In other words, this equation says that an infinitely small amount of kinetic energy **dT** transferred to a particle results in a small increase in its mass **dm**, given by the above equation. The term **dm** is extremely small, as the speed of light in free space is $c \cong 3 \times 10^8$ m/s, so **dm** is found by dividing the amount of the gained kinetic energy **dT**, by $c^2 = 3^2 \times 10^{16}$ $(m/s)^2$. Then this increase **dm** in mass is negligible, compared to the masses of macroscopic objects. It may though become important for subatomic and subnuclear particles, of much smaller masses.

If one integrates the previous equation in both members, this results in:

$$\int_{T_0}^{T} dT = c^2 \int_{m_0}^{m} dm \;\Leftrightarrow\; T - T_0 = c^2\, m - c^2\, m_0\,.$$

If $T_0 = 0$ (the kinetic energy of the particle at rest is null), then the kinetic energy becomes:

$$T = mc^2 - m_0 c^2.$$

Thus, the total energy ($E = m\,c^2$) of a particle is the sum between its rest energy ($m_0\,c^2$) and its kinetic energy (T):

$$m\,c^2 = T + m_0\,c^2.$$

This result means that:

i) A particle of mass **m** has the total energy: $E = m\,c^2$. (The quantity ***m*** is the relativistic mass of the particle and it depends on its speed)

ii) Conversely, a field that stores the energy **E** also has an associated mass $m = \frac{E}{c^2}$; this example is discussed in Problem 63 (a) below.

Problem 63

Based on the relativistic relation between mass and energy, determine:

a) The mass density of the electric field of a Cu sphere (made of copper) of diameter 1 m, charged at potential V = 1 kV.

b) The energy difference resulted from the mass defect in the deuterium nucleus, by comparing the mass of the deuterium nucleus with the sum of masses of the neutron and proton considered separately. Use the fact that the speed of light in free space is $c \cong 2.9979 \times 10^8$ m/s.

c) The average atomic mass unit $u \approx 1.66 \times 10^{-27}$ kg may also be given as: $u = \frac{931\ \mathrm{MeV}}{c^2}$. Demonstrate that the result is correct.

Solution

Note: In order not to be confused with the electrical field **E,** in this problem we used the following symbols: **W** (total energy); **w** (energy density).

a) The energy density *w* of the electric static field created by a charge Q in free space is:

$$w(r) = \frac{1}{2}\varepsilon_0\, E^2(r)\,.$$

Where:

ε_0 is the electric permittivity of free space having the value: $\varepsilon_0 \cong 8.856 \times 10^{-12}$ F/m.

E(r) is the intensity of the electric field at distance r from the centre of the charge **Q** is given by the equation:

$$E(r) = \frac{Q}{4\pi\varepsilon_0\, r^2}\,.$$

If the charge is distributed over the surface of the metallic sphere of radius R = 0.5 m, then the electric field is only between r = R and infinity (there is no electric field inside the metallic closed surface). Thus, the total energy of the electric field produced by Q can be found by integration of the energy density w(r), between R and infinity:

The energy density of the electric field:

$$w(r) = \frac{1}{2}\,\varepsilon_0\,\frac{Q^2}{(4\pi\,\varepsilon_0\,r^2)^2} = \varepsilon_0 \frac{Q^2}{2\cdot 16\,\pi^2\varepsilon_0^2\,r^4}\quad .$$

The element of volume d$\boldsymbol{V}$ is the volume of an elementary sphere of radius r and thickness dr:

$$d\boldsymbol{V} = d\left(4\pi\,\frac{r^3}{3}\right) = 4\pi r^2\,dr\,.$$

Then, the elementary energy within the element of volume d$\boldsymbol{V}$ is found as the product between the energy density and the elementary volume:

$$w(r) \stackrel{\text{def}}{=} \frac{dW}{d\boldsymbol{V}} \Longrightarrow\; dW = w(r)\,d\boldsymbol{V} = \frac{Q^2}{8\pi\varepsilon_0\,r^2}\,dr\,.$$

Thus, the total energy W, of the electric field created by the charge Q distributed over the surface of the metal sphere of radius R = 1/2 m, results:

$$W = \int_R^\infty w(r)\,4\pi\,r^2\,dr = \frac{Q^2}{8\,\pi\,\varepsilon_0}\int_R^\infty \frac{dr}{r^2} = -\frac{Q^2}{8\,\pi\,\varepsilon_0 r}\bigg|_R^\infty \Longrightarrow W = \frac{Q^2}{8\,\pi\,\varepsilon_0}\left(-0+\frac{1}{R}\right)$$

$$W = \frac{Q^2}{8\,\pi\,\varepsilon_0\,R}\quad .$$

The electric charge **Q** is not given, but one knows that the metallic sphere is charged at the potential V = 1 kV. (*From here on, the symbol V represents the electrical potential. The symbol* $\boldsymbol{V}$ *was used for volume only in the first part of the demonstration, and the volume is not needed anymore).* Therefore, the electric charge may be deduced as follows:

$$V \stackrel{\text{def}}{=} \frac{W}{Q} \Longrightarrow V = \frac{Q}{4\pi\,\varepsilon_0\,R} \quad\Rightarrow\quad Q = 4\pi\,\varepsilon_0\,V\,R = 2\pi\varepsilon_0\,V\,D\,.$$

Thus, the energy of the electric static field around the point like charge **Q** is:

$$W = \frac{(4\pi\,\varepsilon_0\,V\,R)^2}{8\,\pi\,\varepsilon_0\,R} = \frac{16\,\pi^2\,\varepsilon_0^2\;V^2\,R^2}{8\pi\,\varepsilon_0\,R} = 2\pi\,\varepsilon_0\,V^2\,R\,.$$

Then, as the relativistic energy is $W = mc^2$, the mass of the electric static field results:

$$m = \frac{W}{c^2} = \frac{2\pi\varepsilon_0\,V^2\,R}{c^2} = \frac{2\cdot 3.14\cdot 8.856\;\times 10^{-12}\,\frac{F}{m}\times 10^6\,V^2\times 0.5\text{ m}}{9\times 10^{16}\,\frac{m^2}{s^2}} \quad\Rightarrow$$

$$m = 3.1\times 10^{-22}\,\text{kg}\,.$$

b) Hydrogen has three isotopes:

- the hydrogen atom ($^1_1$H) has 1 proton in its nucleus:
  - the mass of the hydrogen nucleus (the mass of a proton): m_p = 1.007276 u;
- deuterium ($^2_1$D) contains 2 nucleons, 1 proton and 1 neutron:
  - the mass of the deuterium nucleus: m_d = 2.013553 u;

- tritium is the heaviest hydrogen isotope, and its nucleus consists of 1 proton and 2 neutrons. Tritium is radioactive, it emits beta radiation (i.e. electrons: ${}_{-1}^{0}\mathrm{e}$), and decays into ${}_{2}^{3}\mathrm{He}$:

$${}_{1}^{3}\mathrm{T} \Rightarrow {}_{-1}^{0}\mathrm{e} + {}_{2}^{3}\mathrm{He}.$$

The problem only focuses on the mass difference between the mass of the deuterium nucleus and the sum between the masses of a proton and that of a neutron (where the atomic mass unit is $\mathrm{u} \cong 1.66 \times 10^{-27}$ kg):

- mass of deuterium: m_d = 2.013553 u;
- masses of separate proton and neutron:

$$\mathrm{m_p + m_n = 1.007276\ u + 1.008665\ u = 2.015941\ u}\,;$$

- the mass difference (or the 'mass defect'):

$$\Delta\mathrm{m} = (\mathrm{m_p + m_n}) - \mathrm{m_d} = 0.002388\ \mathrm{u} = 3.964 \times 10^{-24}\ \mathrm{kg}\,.$$

Then the relativistic energy stored within this amount of mass is: $\Delta\mathrm{W} = \Delta\mathrm{m}\ c^2$,

$$\Delta\mathrm{W} = 3.964 \times 10^{-24}\,(3 \times 10^8)^2\ \mathrm{J} = \frac{35.676 \times 10^{-14}\mathrm{J}}{1.6 \times 10^{-19}\ \frac{\mathrm{J}}{\mathrm{eV}}} = 22.2975 \times 10^5\ \mathrm{eV}\,,$$

$$\Delta\mathrm{W} \approx 2.229 \times 10^6\mathrm{eV} = 2.229\ \mathrm{MeV}\,.$$

The reactions of nuclear fusion that constitute the source of the solar radiation are supposed to take part over certain cycles of nuclear fusion reactions, all starting with isotopes of hydrogen (H) (Z = 1) and resulting in isotopes of helium (He) (Z = 2).

c) Calculating the ratio:

$$1\ \mathrm{u} = \frac{931\ \mathrm{MeV}}{\mathrm{c}^2} = \frac{931 \times 10^6 \cdot 1.66 \times 10^{-19}\ \mathrm{J}}{(3 \times 10^8)^2\ \frac{\mathrm{m}^2}{\mathrm{s}^2}} = \frac{931 \cdot 1.66}{9} \cdot \frac{10^6 \times 10^{-19}}{10^{16}}\ \mathrm{kg}$$

$$1\ \mathrm{u} \approx 165.59 \times 10^{-29}\mathrm{kg} \approx 1.66 \times 10^{-27}\ \mathrm{kg}.$$

Problem 64

In classical mechanics, the relationship between the kinetic energy (**T**) and the linear momentum (p = mv) of a particle is $\mathrm{T} = \frac{\mathrm{p}^2}{2\mathrm{m}}$. The total energy of a classical particle is the sum between its potential and kinetic energies: $\mathrm{E} = \mathrm{E_p} + \mathrm{T} = \mathrm{E_p} + \frac{\mathrm{p}^2}{2\mathrm{m}}$. Deduce the relationship between momentum and energy of a relativistic particle (i.e. moving at speed **v**, close to the speed of light in free space, ***c***).

Solution

The total relativistic energy **E** of the particle can be calculated as follows:

- when the particle is at rest: $\mathrm{E} = \mathrm{m_0\ c^2}$, where m_0 = the mass at rest (v = 0);

- when the particle is in motion with velocity v, its total energy becomes: $E = m\,c^2$, where m is the mass of the particle in motion at speed **v** and is given by: $m = m(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$.

Therefore, the total energy **E** of the particle becomes: $E = \frac{m_0\,c^2}{\sqrt{1-\frac{v^2}{c^2}}}$.

Raising to the power 2 each member of the above equation leads to: $E^2 = \frac{m_0^2\,c^4}{1-\frac{v^2}{c^2}}$.

The linear momentum of the relativistic particle is: $p = m\,v = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\,v$.

Raising the linear momentum to the power 2 leads to: $p^2 = \frac{m_0^2\,v^2}{1-\frac{v^2}{c^2}}$.

Multiplication with ***c²*** in both members of the above equation for momentum, yields:

$$p^2\,c^2 = \frac{m_0^2\,v^2\,c^2}{1-\frac{v^2}{c^2}}.$$

Then, by adding the quantity $m_0^2c^4$ both members of the equation:

$$p^2\,c^2 + m_0^2\,c^4 = \frac{m_0^2\,v^2\,c^2}{1-\frac{v^2}{c^2}} + m_0^2\,c^4 \quad \Leftrightarrow$$

$$p^2\,c^2 + m_0^2c^4 = m_0^2\,c^2\left(\frac{v^2}{1-\frac{v^2}{c^2}} + c^2\right),$$

$$p^2\,c^2 + m_0^2\,c^4 = m_0^2\,c^2\,\frac{(v^2+c^2-v^2)}{1-\frac{v^2}{c^2}} \quad \Leftrightarrow \quad p^2\,c^2 + m_0^2\,c^4 = \frac{m_0^2\,c^4}{1-\frac{v^2}{c^2}}.$$

The right-side member of the latter equation is identical to the squared total energy (**E²**) deduced above, and therefore the relationship between the total energy and linear momentum for a relativistic particle, can be deduced as: $E^2 = p^2c^2 + m_0^2\,c^4$.

Note

The above result can be used to deduce *the linear momentum of a photon* of wavelength λ:

- the photon at rest has no mass: $m_0 = 0 \Rightarrow E^2 = p^2\,c^2 \Rightarrow E = p\,c$;
- the mass of the photon in motion is: $m = \frac{E}{c^2} = \frac{p\,c}{c^2} = \frac{p}{c}$
- the energy of a photon of frequency ν is: $E = h\,\nu = h\,\frac{c}{\lambda}$;
- therefore, the linear momentum of the photon is: $p = \frac{E}{c} = \frac{1}{c}h\frac{c}{\lambda} \Rightarrow p = \frac{h}{\lambda}$.

Problem 65

An electron with the rest energy $E_0 = m_0 c^2 = 0.511\ \text{MeV}$ moves at speed **v** = 0.8 ***c***. Find its total energy (**E**), the kinetic energy (**T**) and its linear momentum (**p**).

Solution

- the total energy of the electron in motion at speed v = 0.8 c can be written as:

$$E = mc^2 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{E_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{0.511\ \text{MeV}}{\sqrt{1-\frac{0.8^2\ c^2}{c^2}}} = \frac{0.511}{\sqrt{0.36}}\ \text{MeV} = \frac{0.511}{0.6}\ \text{MeV} \Longrightarrow E = 0.8533\ \text{MeV}.$$

- the kinetic energy is then:

$$T = E - E_0 = 0.342\ \text{MeV}.$$

- the linear momentum can then be calculated as:

$$p = m{\cdot}v = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}{\cdot}v = \frac{m_0 c^2}{c^2\sqrt{1-\frac{v^2}{c^2}}}{\cdot}v = \frac{0.511 \times 10^6{\cdot}1.6 \times 10^{-19}{\cdot}9 \times 10^{16}}{c^{\,2}\sqrt{1-\frac{v^2}{c^2}}}{\cdot}0.8\ c$$

$$p \cong 9.81 \times 10^3\ \ \text{kg}\ \frac{\text{m}}{\text{s}}.$$

Problem 66

Suppose that one finds the means to convert the whole relativistic energy of a grain of dust of mass **1 g**, at rest, into electric energy. How long would this energy supply a 100 W filament light bulb to emit light?

Solution

The relativistic rest energy of the grain of dust:

$$E_0 = m_0\ c^2 = 10^{-3}\text{kg} \times 9 \times 10^{16}\frac{\text{m}^2}{\text{s}^2} \Longrightarrow E_0 = 9 \times 10^{13}\ \text{J}.$$

The electric energy needed by the 100 W light bulb to emit light for the time Δt:

$$E = P\ \Delta t.$$

Therefore, if the means are found to fully convert the rest energy of the grain of dust into electric energy ($E_0 = E$) for the light bulb to consume while emitting light, the time interval that the bulb would emit light for results as:

$$\Delta t = \frac{E_0}{P} = \frac{m_0\ c^2}{P} = \frac{9 \times 10^{13}}{100}\ \text{s} = 9 \times 10^{11}\text{s} \cong 28\ 500\ \text{years}.$$

Note that we used: $1\ \text{year}\ \cong 365\ \text{days}\ \cdot 24\frac{\text{h}}{\text{day}} \cdot 3600\frac{\text{s}}{\text{h}} = 3.16\ \times 10^7\ \text{s}.$

9. ELECTROMAGNETIC PHENOMENA

9.1. Electrostatics

Problem 67

A constant force F = 0.032 N moves a charge q = 420 μC in an electric field, between two points 25 cm apart. What is the potential difference between these two points?

Solution

The work of the force that moves the charge q from the point A (of electric potential V_A) to point B (of electric potential V_B) is:

$$W = \vec{F} \cdot \vec{d} = F \cdot d \ \cos(0^o) = F \cdot d.$$

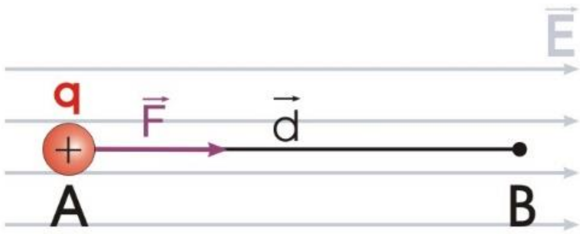


*Fig. 41. A force **F** moves a charge **q** from A to B.*

The electric potential V is defined as the ratio between the electrostatic energy, $\mathcal{E}$ of interaction between the electric field and the electric charge:

$$V = \frac{\mathcal{E}}{q}.$$

Thus, the energy of the charge q moving in the electrical field between points A and B increases with an amount equal to this work, W:

$$W = \Delta\mathcal{E} = q\,(V_B - V_A) \Leftrightarrow F \cdot d = q\,(V_B - V_A) = q \cdot \Delta V.$$

Therefore, the potential difference between the two points, A and B, is:

$$\Delta V = \frac{F \cdot d}{q} = \frac{0.032 \cdot 0.25}{420 \times 10^{-6}}\ (V) \Longrightarrow \Delta V \cong 19\ (V).$$

Problem 68

In a particle accelerator, in vacuum, an electron is submitted to a potential difference of 5 million volts (or 5 MV). What is the energy gained by the electron?

Solution

The energy gained by the electron equals the work that the electric field of acceleration performs on the electron:

$$\Delta E = W = q\,\Delta V = e\,\Delta V = e \cdot 5\ MV = 5\ MeV = 5 \times 10^6 \cdot 1.6 \times 10^{-19}\ J \Longrightarrow \Delta E = 8 \times 10^{-13}\ J.$$

Problem 69

What is the potential gradient between two parallel plates 5 cm apart charged to a potential difference of $1.2\ \mathrm{kV}$? (for gradient, use the symbol 'Nabla': ∇=grad).

Solution

The *gradient* of electric potential: $\nabla\ \mathrm{V} = \frac{\Delta \mathrm{V}}{\Delta \mathrm{r}} \cdot \frac{\vec{\mathrm{r}}}{|\vec{\mathrm{r}}|}$.

The absolute value of the potential gradient: $|\nabla\ \mathrm{V}| = \frac{\Delta \mathrm{V}}{\Delta \mathrm{r}} = \frac{1.2 \times 10^{3}\ \mathrm{V}}{5 \times 10^{-2}\ \mathrm{m}} \Longrightarrow |\nabla\ \mathrm{V}| = 24 \frac{\mathrm{kV}}{\mathrm{m}}$.

Problem 70

Compare the forces of electrostatic repulsion and of gravitational attraction between 2 electrons, at distance **r** apart. Use that the electron charge is $\mathrm{e} \cong -1.6 \times 10^{-19}\ \mathrm{C}$ and the electron mass is $\mathrm{m_e} \cong 9.1 \times 10^{-31}\ \mathrm{kg}$.

Solution

The repelling force between the two electric charges of the electrons at distance **r** apart is:

$$\vec{\mathrm{F}}_{\mathrm{e}} = \frac{\mathrm{e}^2}{4\pi\, \varepsilon_0\, \mathrm{r}^2} \cdot \frac{\vec{\mathrm{r}}}{\mathrm{r}}.$$

The attraction between the electrons owing to their masses (**K** is the Cavendish constant) is:

$$\vec{\mathrm{F}}_{\mathrm{g}} = \mathrm{K}\, \frac{\mathrm{m}_0^2}{\mathrm{r}^2} \cdot \frac{\vec{\mathrm{r}}}{\mathrm{r}}.$$

The ratio of their absolute values:

$$\frac{\mathrm{F_e}}{\mathrm{F_g}} = \frac{\mathrm{e}^2}{4\pi\, \varepsilon_0\, \mathrm{K}\, \mathrm{m}_0^2} = \frac{(1.6 \times 10^{-19})^2}{4\pi \cdot 8.856 \times 10^{-12} \cdot 6.76 \times 10^{-11} \cdot (9.1 \times 10^{-31})^2}$$

$$\frac{\mathrm{F_e}}{\mathrm{F_g}} \cong 10^{42} \iff \mathrm{F_e} = 10^{42} \times \mathrm{F_g}\,.$$

In other words, the force of electrostatic repulsion is <u>10^{42} times stronger</u> than the gravitational attraction between two electrons.

Problem 71

What is the electric field E_1 at a point of coordinates (2, 2), and respectively E_2 at (1, 2) in the xOy frame of reference, where the electric potential is given by: $\mathrm{V(x,y)} = \mathrm{x \cdot y} + \mathrm{y}^2$?

Solution

The electric static field is the opposite of the potential gradient: $\vec{E}\,(x,y) = -\nabla V(x,y)$

$$\vec{E}\,(x,y) = -\nabla V = -\frac{\Delta V}{\Delta r}\cdot\frac{\vec{r}}{r} = -\left(\frac{\partial V}{\partial x}\,\vec{\imath} + \frac{\partial V}{\partial y}\,\vec{\jmath}\right) \Rightarrow \vec{E}(x,y) = -y\,\vec{\imath} - (x+2y)\,\vec{\jmath}\,.$$

$\vec{E}_1 = \vec{E}(2,2) = -2\vec{\imath} - 6\vec{\jmath}\,\left(\frac{V}{m}\right)$, with magnitude: $E_1 = \sqrt{4+36} = 2\sqrt{10} \Longrightarrow E_1 \cong 6.325\left(\frac{V}{m}\right)$.

$\vec{E}_2 = \vec{E}(1,2) = -2\vec{\imath} - 5\vec{\jmath}\,\left(\frac{V}{m}\right)$, with magnitude: $E_2 = \sqrt{4+25} = \sqrt{29} \Longrightarrow E_2 \cong 5.385\left(\frac{V}{m}\right)$.

Problem 72

Two small spheres of mass 0.25 g each, are hanging from the roof, along insulating threads (*ℓ = 50 cm*), so that they just touch. When they equally share a charge, in their new position the threads make an angle $\alpha = 45^\circ$ with the vertical. What is the electric charge on each sphere?

Solution

Note: Consider that $g \cong 10\ \frac{m}{s^2}$.

Given the sketch in figure 42: $\tan\alpha = \frac{F_e}{G}$,

$$\alpha = 45^\circ \ \Rightarrow \tan\alpha = 1 \ \Longrightarrow \ell^2 = 2\left(\frac{d}{2}\right)^2 = \frac{d^2}{2},$$

Then with: $F_e = G \ \Rightarrow \frac{Q^2}{4\pi\varepsilon_0\cdot d^2} = mg \ \Rightarrow$

$$Q^2 = mg\cdot 4\pi\varepsilon_0\cdot d^2 = mg\cdot 4\pi\varepsilon_0\cdot 2\ell^2,$$

and knowing that $\frac{1}{4\pi\varepsilon_0} \cong 9\times10^9$, the electric charge Q on each of the spheres is:

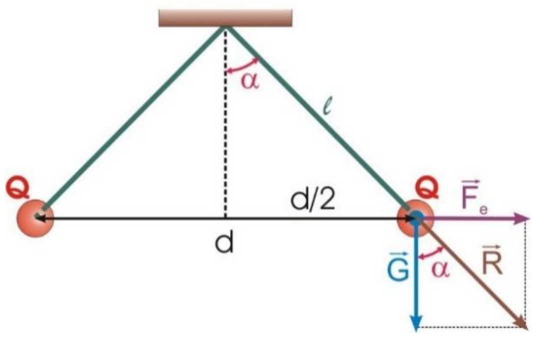


Fig. 42. Electric charges on insulating threads

$$Q = \ell\sqrt{2mg\,4\pi\varepsilon_0} = 0.5\sqrt{\frac{2\cdot0.25\times10^{-3}\cdot10}{9\times10^9}} = \frac{0.5}{3\times10^4}\sqrt{2\cdot2.5\times10^{-4}} \Rightarrow$$

$$Q \cong 3.73\times10^{-7}\ (C).$$

Problem 73

What is the force of electrostatic attraction between the proton (mass $m_p = 1.67\times10^{-27}$ kg, and charge $e^+ = 1.6\times10^{-19}$ C) and the electron (mass $m_e = 9.1\times10^{-31}$ kg, and charge $e^- = -1.6\times10^{-19}$ C), in a H atom of radius $r_0 \cong 0.53\times10^{-10}$ m?

Solution

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{|e^+\cdot e^-|}{r_0^2} \cong 9\times 10^9\,\frac{1.6^2\times 10^{-38}}{0.53^2\times 10^{-20}}\ (\mathrm{N}),$$

$$F_e \cong 8.2\times 10^{-8}\ \mathrm{N}.$$

Problem 74

a) What is the total electrical charge **Q** of all the electrons that are in a sphere of mass 1 g, made of iron (atomic number Z = 26; atomic mass A = 55.85 a.m.u.).

b) What is the force of attraction between the spheres if one of them has lost 1/1000 of its electrons, and the other one gained 1/1000 more electrons, if they are 1 m apart?

c) Is this experiment *practically* possible?

Solution

a) If A is the atomic mass, ν is the number of moles ($\nu = m/A$) and N_A is the Avogadro number, then the total number of atoms in 1 g of pure iron can be calculated as:

$$N = \nu\, N_A = \frac{m}{A}\, N_A = \frac{1\ \mathrm{g}}{55.85\,\frac{\mathrm{g}}{\mathrm{mol}}}\,\frac{6.023\times 10^{-23}\mathrm{atoms}}{\mathrm{mol}},$$

$$N = 1.08\times 10^{22}\ \text{atoms of Fe}.$$

The number of electrons in this sphere of iron of mass 1 g:

$$N_e = N\,Z = N\,26 = 1.08\times 10^{22}\cdot 26 \Longrightarrow N_e = 2.8\times 10^{23}\ \text{electrons}$$

b) The fraction 1/1000 of these electrons:

$$N' = \frac{N_e}{1000} = 2.8\times 10^{20}\ \text{electrons}.$$

The electric charges on each of the spheres:

$$Q_1 = +Q = N'\cdot e = +\,2.8\times 10^{20}\cdot 1.6\times 10^{-19}\mathrm{C} = +\,4.48\times 10\ \mathrm{C} \Longrightarrow Q_1 = +\,44.8\ \mathrm{C}.$$

$$Q_2 = -\,Q = \,-\,N'\cdot e \Longrightarrow Q_2 = -\ 44.8\ \mathrm{C}.$$

As the centres of the spheres are said to be 1 m apart, the electric force of attraction between them is:

$$F_{attr} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1\,Q_2}{d^2} \cong 9\times 10^9\,\frac{44.8^2}{1^2}\ \mathrm{N} \Longrightarrow F_{attr} = 1.8\times 10^{13}\ \mathrm{N} \Rightarrow \text{this is an enormous force!}$$

c) To assess the practical possibility of such an experimental setting, one calculates the *electric potential* of one sphere that would bear the electric charge **Q** (positive or negative):

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \cong 9\times 10^9\,\frac{44.8}{10^{-2}} \Longrightarrow V = 4.032\times 10^{13}\ (\mathrm{V}).$$

This is huge, it is *practically impossible* to produce and to handle such a huge electric potential.

Problem 75

What is the electric charge **Q** on two spheres, each of mass $1\ \mu g$, at distance $1\ cm$ apart, so that the electrostatic force of repulsion between them is *equal* to the weight of each of the spheres? (use the gravitational acceleration: $g = 9.8\ m/s^2$).

Solution

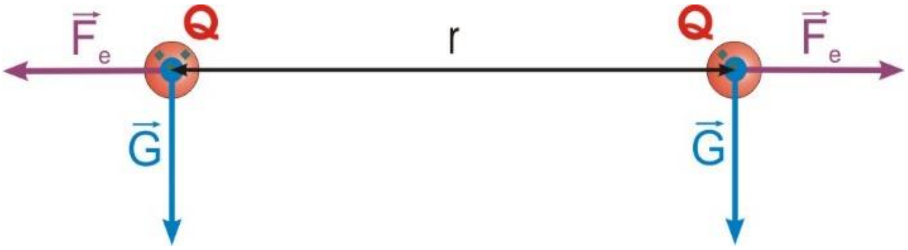


*Fig. 43 Two spheres with electric charge **Q** (each) situated at the distance **r** between their centres.*

The electrostatic force:

$$F_e = \frac{Q^2}{4\pi\ \varepsilon_0 \cdot r^2} \cong 9 \times 10^9\ \frac{Q^2}{r^2}.$$

The weight of one of the spheres:

$$G = m \cdot g.$$

If the two forces above are equal, then:

$$\frac{Q^2}{4\pi\ \varepsilon_0 r^2} = mg \quad \Leftrightarrow \quad Q^2 = 4\pi\ \varepsilon_0 mg\ r^2 \Longrightarrow Q = \sqrt{4\pi\ \varepsilon_0 mg\ r^2}.$$

Then the electric charge on one of the spheres is:

$$Q = \sqrt{\frac{1 \times 10^{-6} \cdot 9.8 \cdot (10^{-2})^2}{9 \times 10^9}} = \sqrt{\frac{9.8}{9} \times 10^{-19}} \Longrightarrow$$

$$Q \cong \sqrt{10.889 \times 10^{-18}} \Longrightarrow$$

$$Q = 3.30 \times 10^{-9}\ C\,.$$

Problem 76

Two identical spheres are hung by two insulating threads, each of length ℓ. In free space, the angle between the threads is α = 90°. In a dielectric fluid of density equal to one third of the spheres density ($\rho_d = \frac{1}{3}\ \rho$), the angle becomes $\beta = 60^o$ (See Figure 44). What is the relative electric permittivity of the dielectric ε_r =? (*Use that*: ρ_d = density of the dielectric; $\mathbf{V_d}$ = the volume of dielectric fluid replaced by the volume $\mathbf{V}$ of the small sphere; ρ= density of the sphere; mass of the sphere m = ρV; $V_d = \mathbf{V}$, i.e. the sphere dislocates a volume of dielectric liquid equal to the volume of the sphere).

Solution

In air, each of the spheres is under two forces: the electrostatic force (F_e) and its weight (G). Therefore, the tangent of the angle between the thread and the vertical is:

$$\tan(45^o) = \frac{F_e}{G} \Leftrightarrow F_e = G \Leftrightarrow$$

$$9 \times 10^9 \cdot \frac{q^2}{2\ell^2} = m \cdot g \Rightarrow$$

$$q^2 = \frac{\rho\, V\, g\, 2\, \ell^2}{9 \times 10^9}.$$

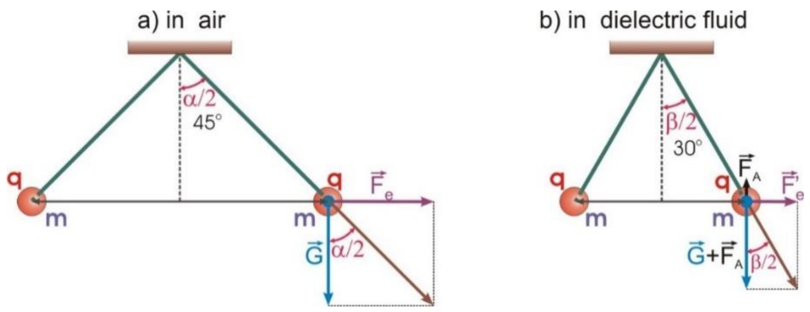


Fig. 44. Two charged spheres hung by insulating threads (a) in air and (b) in a dielectric fluid.

In a dielectric medium, the threads make a different angle with the vertical, as the spheres are now at distance d', experiencing another electrostatic force, and also the contribution of Archimede's force (also called the *buoyancy* force) on the vertical: $\tan(30^o) = \frac{F'_e}{G - F_A}$.

With:

- the force of electrostatic repulsion: $F'_e = \frac{q^2}{4\pi \cdot \varepsilon_0 \varepsilon_r \cdot (d')^2}$
- the weight of the sphere: $G = m \cdot g$
- Archimede's force (also called *buoyancy*): $F_A = \rho_d \cdot V_d \cdot g$

Substituting these expressions of the forces, and considering that $\frac{1}{4\pi \cdot \varepsilon_0 \left(\frac{F}{m}\right)} \cong 9 \times 10^9 \left(\frac{m}{F}\right)$, the trigonometric function *tangent* becomes: $\tan(30^o) \cong 9 \times 10^9 \cdot \frac{q^2}{\varepsilon_r \cdot (d')^2 \cdot (mg - \rho_d V_d \cdot g)}$

Substituting $\tan(30^0) = \frac{\sqrt{3}}{3}$, and factoring out ***g*** at the denominator, one obtains:

$$\frac{\sqrt{3}}{3} = 9 \times 10^9 \frac{q^2}{\varepsilon_r \cdot (d')^2 \cdot (m - \rho_d V_d) \cdot g} \Leftrightarrow 3\, q^2 \cdot 9 \times 10^9 = \sqrt{3}\, \ell^2 \left(\rho - \frac{1}{3}\rho\right) V_d \cdot g \cdot \varepsilon_r.$$

Now by substituting q^2 with the result obtained in the previous equation at point (a), the above equation becomes:

$$3\left[mg\, \frac{2\ell^2}{9 \times 10^9}\right] \cdot 9 \times 10^9 = \sqrt{3}\, \ell^2 \frac{2}{3}\, \rho V g\, \varepsilon_r,$$

And then, as the mass is the product between density and volume, $m = \rho \cdot V$, the previous equation becomes:

$$3\left[\rho g\, V\, \frac{2\,\ell^2}{9 \times 10^9}\right] \cdot 9 \times 10^9 = \sqrt{3}\, \ell^2 \frac{2}{3} \rho g\, V\, \varepsilon_r.$$

Therefore, the relative electric permittivity of the dielectric results as:

$$\varepsilon_r = \frac{9}{\sqrt{3}} = 3\sqrt{3} \implies \varepsilon_r \cong 5.196.$$

Problem 77

An electric dipole is a system made of two opposite charges $+\mathbf{q}$ and $-\mathbf{q}$, at a distance **d** apart. As a physical quantity, the electric dipole is defined as $\vec{p} = q \cdot \vec{d}$.

a) What is the electric field E_A at a point A situated on the line that connects the two charges, outside the dipole (as shown in Figure 45)? What is the electric field intensity at very distant points, for **r** >> **d**?

b) What is the electric field intensity E_B at a point B situated at distance **r** from the dipole centre, on the perpendicular that crosses the centre of the dipole (see Figure 46)?

c) What is the torque experienced by an electric dipole $\vec{p}$ when introduced in an electric static field of intensity $\vec{E}$ (see Figure 47)?

d) Determine the potential energy of the dipole when introduced in an electrostatic field.

Solution

a) $\vec{E}_A = \vec{E}_+ + \vec{E}_-$

$$\left.\begin{array}{l} E_+ = k\dfrac{q}{\left(r - \frac{d}{2}\right)^2} \\ E_- = k\dfrac{q}{\left(r + \frac{d}{2}\right)^2} \end{array}\right\} \Rightarrow$$

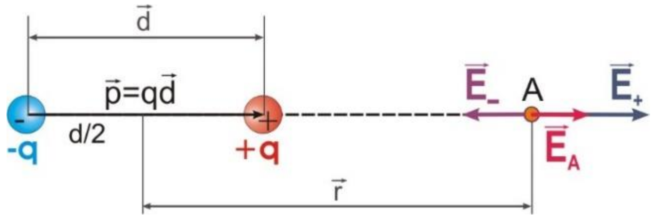


Fig. 45. An electric dipole $\vec{p} = q \cdot \vec{d}$. The electric field intensity at point A, on the dipole axis.

$$E_A = k\frac{q}{\left(r - \frac{d}{2}\right)^2} - k\frac{q}{\left(r + \frac{d}{2}\right)^2} \Rightarrow E_A = k \cdot q\frac{2d \cdot r}{\left(r^2 - \frac{d^2}{4}\right)^2}.$$

For distances r >> d ⇒ $d^2/4 \ll r^2$, and the previous equation becomes:

$$E_A \approx k\frac{2q \cdot d}{r^3} \approx \frac{2p}{4\pi\varepsilon_0 r^3} \sim \frac{1}{r^3}.$$

This result shows that in the case of an electric dipole, the electric field intensity decreases very rapidly with the distance ***r***, inversely proportional to $\boldsymbol{r^3}$.

b) $\vec{E}_B = \vec{E}_+ + \vec{E}_-$.

$$\left.\begin{array}{l} E_+ = k\dfrac{q}{r_+^2} = k\dfrac{q}{r^2 + \frac{d^2}{4}} = E \\ E_- = k\dfrac{q}{r_-^2} = k\dfrac{q}{r^2 + \frac{d^2}{4}} = E \end{array}\right\} \Rightarrow E_B^2 = E_+^2 + E_-^2 + 2\,E_+\,E_-\,\cos 2\alpha,$$

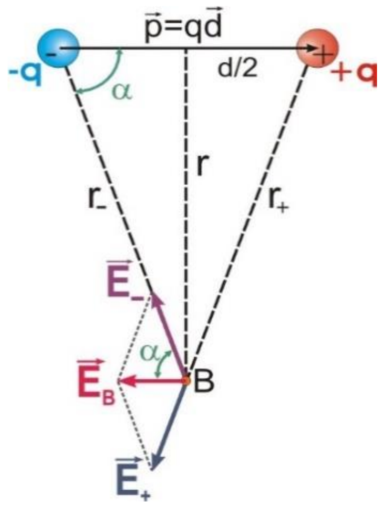


*Fig. 46. The electric field intensity at distance **r** from the dipole axis.*

$$E_B^2 = 2\,\mathrm{E}^2(1+\cos 2\alpha) = 2\,\mathrm{E}^2(1+2\cos\alpha^2 - 1),$$

$$E_B^2 = (2\mathrm{E}\ \cos\alpha)^2 \Rightarrow E_B = 2\mathrm{E}\ \cos\alpha \Rightarrow$$

$$\mathrm{E_B} = 2\mathrm{k}\frac{\mathrm{q}}{\mathrm{r}^2+\frac{\mathrm{d}^2}{4}}\cdot\frac{\frac{\mathrm{d}}{2}}{\mathrm{r}_+} = \frac{2\,\mathrm{k}}{2}\cdot\frac{\mathrm{q\,d}}{(r_+^2)\,\mathrm{r}_+} = \mathrm{k}\frac{\mathrm{p}}{\left(\mathrm{r}^2+\frac{\mathrm{d}^2}{4}\right)^{\frac{3}{2}}}.$$

For distances r >> d ⇒ $d^2 << r^2$, and the resulting intensity of the dipole electric field becomes:

$$\mathrm{E_B} \approx \mathrm{k}\frac{\mathrm{p}}{\mathrm{r}^3} \quad \Rightarrow \quad \mathrm{E_B}\sim\frac{1}{\mathrm{r}^3}.$$

c) The torque on an electric dipole $\vec{\mathrm{p}}$, when introduced in an electric static field of intensity $\vec{\mathrm{E}}$, is determined by the couple of forces ($F_+ = F_- = F$) experienced by the two charges that make up the dipole (See Figure 47).

The torque of the couple of forces $\vec{\mathrm{F}}_{+\mathrm{q}}$ and $\vec{\mathrm{F}}_{-\mathrm{q}}$ at distance $\mathrm{a} = \mathrm{d}\sin\theta$ apart:

$$\tau = \mathrm{F}\cdot\mathrm{a}\,,$$

$$\tau = (\mathrm{qE})\cdot(\mathrm{d}\cdot\sin\theta) = (\mathrm{qd})\cdot\mathrm{E}\sin\theta\,;$$

$$\tau = \mathrm{p}\cdot\mathrm{E}\cdot\sin\theta.$$

The result above represents the absolute value of a *vector* product, the *torque* experienced by the dipole in an external electric field:

$$\vec{\tau} = \vec{\mathrm{p}}\times\vec{\mathrm{E}}.$$

d) The potential energy of the dipole in the external electric static field is given by the *scalar* product of the two vectors, the electric dipole ($\vec{p}$) and the electric field intensity ($\vec{E}$):

$$\mathrm{W_p} = \vec{\mathrm{p}}\cdot\vec{\mathrm{E}} = \mathrm{p}\cdot\mathrm{E}\cos\theta\,.$$

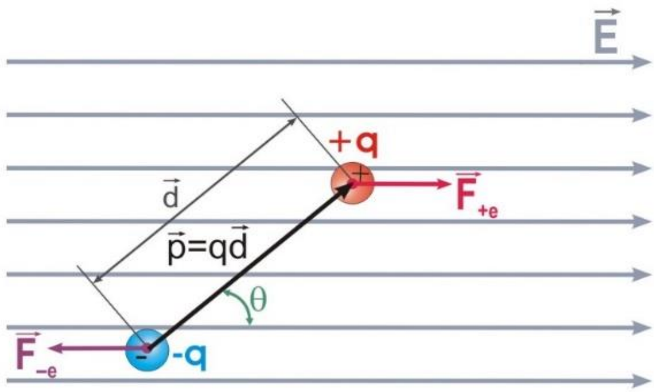


Fig. 47. An electric dipole in an external electric field $\vec{E}$.

Problem 78

Use the superposition principle to determine the electric field in air, on the axis Ox, of the electric charge **Q** uniformly distributed along the segment **L**, at distance **a** from one end of the charge distribution (as shown in Figure 48).

Solution

The linear density of electrical charge, λ is defined as the charge per unit length: $\lambda = \frac{\mathrm{dq}}{\mathrm{dx}}$.

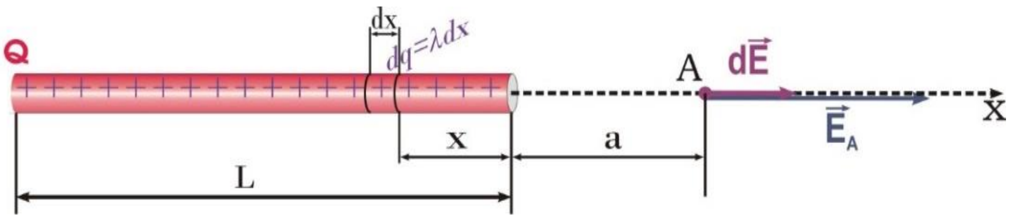


Fig. 48. The intensity of the electric field E_A of a linear distribution of charge, at point A.

The elementary charge on an element of length **dx** can be expressed as:

$$dq = \lambda\, dx = \frac{Q}{L} dx.$$

The elementary field intensity generated by the element of *charge* **dq**, is:

$$dE = \frac{dq}{4\pi\varepsilon_0\,(a+x)^2} \Leftrightarrow dE = \frac{\lambda\, dx}{4\pi\varepsilon_0 (a+x)^2}\ .$$

By applying the superposition principle, the total intensity of the electric field generated by the whole charge distribution can be calculated by adding up all the elementary electric field intensities, i.e. by integration, and is given by:

$$\int_0^E dE' = \int_0^L \frac{\lambda\, dx}{4\pi\varepsilon_0 (a+x)^2} \Leftrightarrow E = \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{dx}{(a+x)^2} = \frac{\lambda}{4\pi\varepsilon_0}\left(-\frac{1}{a+x}\right)\Big|_0^L.$$

Finally: $$E = \frac{\lambda}{4\pi\varepsilon_0}\left(-\frac{1}{a+L}+\frac{1}{a}\right) = \frac{\lambda L}{4\pi\varepsilon_0\, a\,(a+L)} \Longrightarrow E = \frac{Q}{4\pi\varepsilon_0 \cdot a(a+L)}\,.$$

Problem 79

Positive electric charge is distributed uniformly on the Ox direction, between x = − L and x = L. what is the electric field intensity at point B of coordinates (0, 0, z)?

Solution

An element of length **dx**, between (−L, 0) (Figure 49) carries the element of charge $dq = \lambda\, dx$. The elementary charge **dq** generates the elementary intensity of electric field $d\vec{E}$, of components:

$$d\vec{E} = d\vec{E}_x + d\vec{E}_z.$$

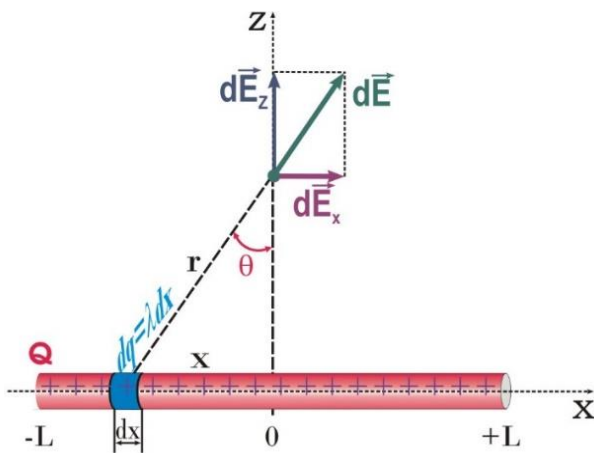


Fig. 49. Intensity of the electric field aside of a segment of electric charge lying between (−L, L) along the Ox axis.

For every elementary charge **dq** on the left of 0, at coordinate − **x**, there is an equal element of charge **dq** on the right side of 0, at coordinate **+x**, which generates an elementary electric field intensity with its Ox component $-d\vec{E}_x$ directed opposite to

$d\vec{E}_x$. Based on the symmetry of the charge distribution in Figure 49, when integrating between (−L, L), the summation of all the $d\vec{E}_x$ components leads to zero, and thus the total component of the electric field along the Ox axis is null: $\vec{E}_x = 0$. The total electric field at the point of coordinates (0, 0, z) has only the $\vec{E}_z$ component, perpendicular to the linear charge distribution. Its magnitude can be calculated as follows:

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = 9\times10^9\frac{dq}{r^2}.$$

The Oz component of the elementary electric field intensity:

$$dE_z = dE\ \cos\theta = \frac{\lambda\,dx}{4\pi\varepsilon_0\,r^2}\cdot\frac{z}{r} = \lambda\,z\,\frac{dx}{4\pi\varepsilon_0\,(x^2+z^2)^{\frac{3}{2}}}.$$

The total electric field intensity on Oz is found by integrating along the segment between (− L, L):

$$E_z = \frac{\lambda\cdot z}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{dx}{(x^2+z^2)^{\frac{3}{2}}} = \frac{\lambda\cdot z}{4\pi\varepsilon_0}\frac{1}{z^2}\left(\frac{x}{\sqrt{x^2+z^2}}\right)\Bigg|_{-L}^{L},$$

$$E_z = \frac{\lambda}{4\pi\varepsilon_0\cdot z}\left(\frac{L}{\sqrt{L^2+z^2}}+\frac{L}{\sqrt{L^2+z^2}}\right) = \frac{2L\,\lambda}{4\pi\varepsilon_0\cdot z\sqrt{L^2+z^2}} = \frac{Q}{4\pi\varepsilon_0\,z\sqrt{L^2+z^2}},$$

The total electric field intensity is only along the Oz axis:

$$E = E_z = \frac{Q}{4\pi\varepsilon_0\,z\sqrt{L^2+z^2}}.$$

Problem 80

The electric charge **Q** is distributed uniformly along a rod. This rod is then bent to form a semicircle of radius **R**. By applying the superposition principle of electric fields, determine what is the electric field intensity at the centre of the semicircle (Figure 50).

Solution

If the total charge Q is uniformly distributed along the rod of length L = πR (linear charge density λ = Q/L), then the elementary charge over the element of arc $d\hat{s}$ is:

$$dq = \lambda\cdot d\hat{s} = \lambda\cdot R\cdot d\theta.$$

The elementary intensity of the electric field in the centre O of the semi-circle has the magnitude:

$$dE = \frac{dq}{4\pi\varepsilon_0\,R^2} = \frac{1}{4\pi\varepsilon_0\,R^2}\frac{Q}{\pi\,R}\,R\,d\theta = \frac{1}{4\pi\varepsilon_0\,R^2}\frac{Q}{\pi}d\theta.$$

This electric field has components along Ox and Oz:

$$d\vec{E} = d\vec{E}_z + d\vec{E}_x,$$

where the components of the elementary electric field are given by:

$$\begin{cases} dE_x = dE\ \sin\theta = \dfrac{Q}{4\pi\varepsilon_0 R^2 \pi}\sin(\theta)\,d\theta \\ dE_z = dE\ \cos\theta \end{cases}.$$

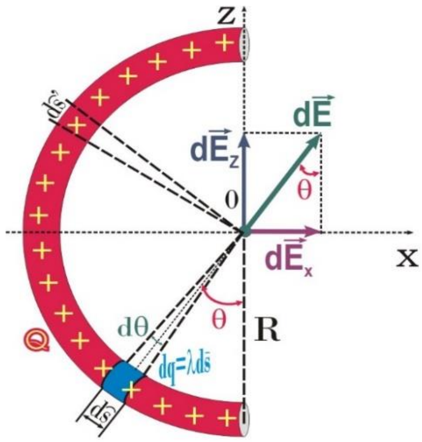


Fig. 50. A continuous distribution of charge in a semi-circular configuration.

Remember that the *superposition principle* states that the *resulting electric field* intensity produced *at point O* in space, by several electric charges, is equal to *the vector summation of the electric fields* produced by each charge *at that point O*. In this case we deal with a continuous distribution of electrical charge along the semicircle. The superposition principle is applied by integrating (adding) the elementary electric field vectors $d\vec{E}$, produced by elementary charges along the charge distribution.

It is useful to consider the perpendicular components of $d\vec{E}$ namely $d\vec{E}_z$ and $d\vec{E}_x$ and integrate separately the scalars dE_z and dE_x. For every element of charge **dq** there is a symmetrical element of charge ($dq' = \lambda\cdot d\widehat{s'}$) that generates an electric field with the Oz component equal and opposed to **dE$_z$**, so that they cancel each other. As a result, after integrating along the charge distribution, the Oz components cancel, and the only component that remains ais long the Ox axis:

$$E_x = \int_0^\pi \frac{Q}{4\pi\varepsilon_0 R^2\pi}\sin(\theta)\,d\theta \Leftrightarrow E_x = \frac{Q}{4\pi\varepsilon_0 R^2\pi}\int_0^\pi \sin(\theta)\,d\theta \Rightarrow$$

$$E_x = \frac{Q}{4\pi\varepsilon_0 R^2\pi}[-\cos(\theta)]\Big|_0^\pi = \frac{Q}{4\pi^2\varepsilon_0\, R^2}[1+1] \implies E_x = \frac{Q}{2\pi^2\varepsilon_0\, R^2}.$$

Problem 81

Electric charge is distributed uniformly (λ is the linear charge density) over the Ox axis, from $-\infty$ to $+\ \infty$. Applying the superposition principle (see Problem 80), determine the intensity of the electric field at the point of coordinates (0, 0, z), as shown in Figure 51.

Solution

The element of charge,

$$dq = \lambda\, dx,$$

at coordinate (+x, 0, 0), generates the elementary electric field, of intensity:

$$dE = \frac{dq}{4\pi\varepsilon_0\, r^2} = \frac{\lambda\, dx}{4\pi\varepsilon_0\, r^2}.$$

With the components:

$$d\vec{E} = d\vec{E}_x + d\vec{E}_z$$

$$dE_x = dE\ \sin\theta = \frac{\lambda\, dx}{4\pi\varepsilon_0\, r^2}\,\frac{x}{r}$$

$$dE_z = dE\ \cos\theta = \frac{\lambda\, dx}{4\pi\varepsilon_0\, r^2}\,\frac{z}{r}$$

For each such element of charge ***dq*** = λdx, at coordinate *(+x, 0, 0)*, there is a corresponding element of charge ***dq'*** = λ dx, at coordinate *(-x, 0, 0*), producing the elementary electric field of intensity:

$$dE' = \frac{dq'}{4\pi\varepsilon_0\, r^2} = \frac{\lambda\, dx}{4\pi\varepsilon_0\, r^2}.$$

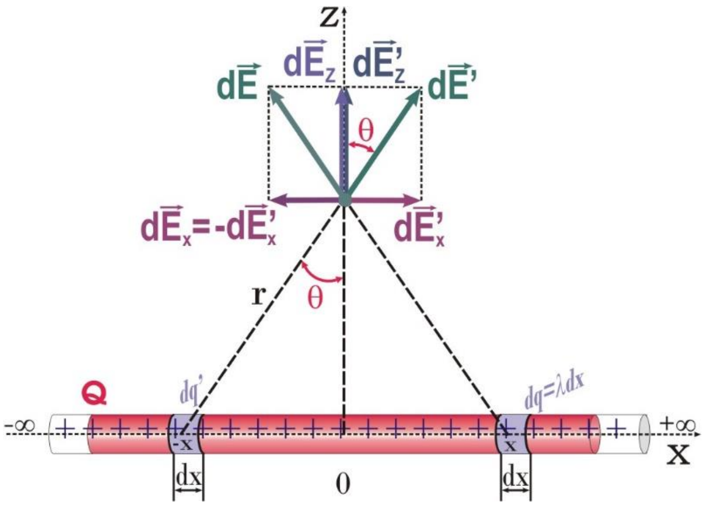


Fig. 51. An infinite line of electric charge along the Ox axis.

Its components are:

$d\vec{E'} = d\vec{E'}_x + d\vec{E'}_z$,   with pairs along Ox that cancel each other: $d\vec{E'}_x + d\vec{E}_x = -d\vec{E}_x + d\vec{E}_x = 0$.

Therefore, when integrating along the whole charge distribution, for $x \in (-\infty, +\infty)$, the total intensity of the electric field generated by an infinite line of charge, at point of coordinates (0, 0, z) is perpendicular to the infinite line of charge and its magnitude is given by:

$$E_z = \frac{\lambda\, z}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+z^2)^{3/2}} = \frac{\lambda\, z}{4\pi\varepsilon_0 z^2}\left(\frac{x}{\sqrt{x^2+z^2}}\right)\Bigg|_{-\infty}^{+\infty} = \frac{\lambda}{4\pi\varepsilon_0\, z}(1+1).$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\,\frac{2\lambda}{z} \Longrightarrow E_z = \frac{\lambda}{2\pi\varepsilon_0\, z}.$$

Problem 82

Electric charge is distributed uniformly, with linear density λ along the positive Ox semi-axis. By applying the superposition principle (as in problem 80), determine the resulting intensity of the electric field at the point of coordinates (0, 0, z).

Solution

The intensity of the elementary electric field produced by the element of charge **dq** at the point of coordinates (0, 0, z):

$$dE = \frac{dq}{4\pi\varepsilon_0\, r^2} = \frac{\lambda\, dx}{4\pi\varepsilon_0(x^2+z^2)},$$

With components: $d\vec{E} = d\vec{E}_x + d\vec{E}_z$, each of magnitude:

$$dE_x = dE\ \sin\theta = \frac{\lambda\, dx}{4\pi\varepsilon_0(x^2+z^2)}\frac{x}{\sqrt{x^2+z^2}}\ ,$$

$$dE_z = dE\ \cos\theta = \frac{\lambda\, dx}{4\pi\varepsilon_0(x^2+z^2)}\frac{z}{\sqrt{x^2+z^2}}\ .$$

Integrating between $(0, +\infty)$ leads to each of the two components of the electric field intensity.

The Ox total component:

$$E_x = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{\infty}\frac{x\, dx}{(x^2+z^2)^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0}\left(-\frac{1}{\sqrt{x^2+z^2}}\right)\Bigg|_0^{\infty}$$

$$E_x = \frac{\lambda}{4\pi\varepsilon_0}\left(-\frac{1}{\infty}+\frac{1}{z}\right) \Rightarrow E_x = \frac{\lambda}{4\pi\varepsilon_0\, z}.$$

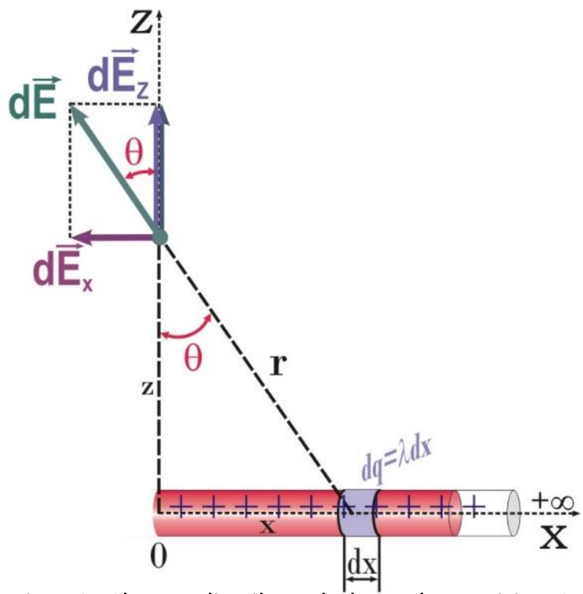


Fig. 52. Charge distributed along the positive Ox semi-axis.

The Oz total component:

$$E_z = \frac{\lambda\, z}{4\pi\varepsilon_0}\int_0^{\infty}\frac{dx}{(x^2+z^2)^{\frac{3}{2}}}$$

$$E_z = \frac{\lambda\, z}{4\pi\varepsilon_0}\frac{1}{z^2}\left(\frac{x}{\sqrt{x^2+z^2}}\right)\Bigg|_0^{\infty} \quad \Rightarrow$$

$$E_z = \frac{\lambda}{4\pi\varepsilon_0\, z}\ (1-\ 0) \Rightarrow E_z = \frac{\lambda}{4\pi\varepsilon_0\, z}.$$

Then the total electric field intensity at point (0, 0, z) is found by the superposition of the two components $\vec{E}_x$ and $\vec{E}_z$ found above:

$$\vec{E} = \vec{E}_x + \vec{E}_z\ \Rightarrow\ E = \sqrt{{E_x}^2 + {E_z}^2} = \sqrt{\left(\frac{\lambda}{4\pi\varepsilon_0\, z}\right)^2 + \left(\frac{\lambda}{4\pi\varepsilon_0\, z}\right)^2} \Rightarrow E = \frac{\lambda\sqrt{2}}{4\pi\varepsilon_0\, z}.$$

Problem 83

Electric charge is distributed uniformly over an infinite planar surface, with the superficial charge density **σ**. Apply the superposition principle to determine what is the resulting intensity of the electric field at point A, a distance **h** from the infinite distribution of planar charge? (Figure 53).

Solution

An elementary charge **dq** (light grey in Figure 53) generates, at distance **r** (height **h**), in point A, an elementary electric field of intensity:

$$d\vec{E} = d\vec{E}_x + d\vec{E}_z.$$

Owing to the symmetry of the configuration, as one integrates over the whole ring (radius R) of elementary area **dS** (blue in Figure 53), the components Ex parallel to the charge distribution cancel each other with that of an element of charge **dq'**, diametrically opposed to **dq**. The whole ring of charge $dQ = \sigma\, dS$ will only produce an electric field of intensity perpendicular to the plan of charge, directed outwards, if the charge is positive:

$$dE = \frac{dQ}{4\pi\varepsilon_0\, r^2}\,.$$

Then, the Oz component of the electric field is:

$$dE_z = dE\ \sin\theta = \frac{\sigma\, dS}{4\pi\varepsilon_0\, r^2}\,\frac{h}{r} = \frac{\sigma\, h\,(2\pi\cdot x\cdot dx)}{4\pi\varepsilon_0\,(x^2+h^2)^{\frac{3}{2}}}.$$

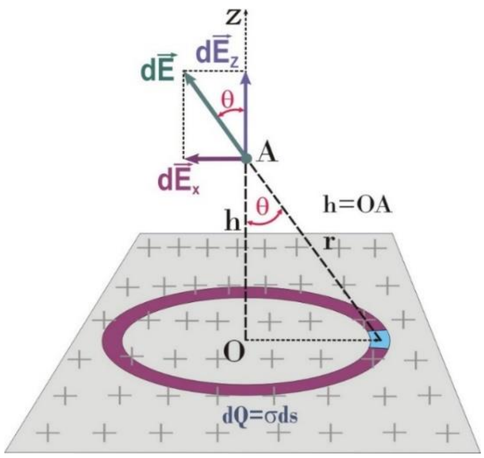


Fig. 53. An infinite planar distribution of charge.

The area of a ring of radius **x** and width **dx** is found by applying the differential to the area of a circle of radius **x**:

$$S = \pi\, x^2 \quad \Rightarrow \quad dS = 2\pi\cdot x\cdot dx$$

By integrating over the radius (x) from O to infinity, the total intensity of the electric field produced by the infinite planar distribution of charge is:

$$E_z = \frac{2\pi\,\sigma\, h}{4\pi\varepsilon_0}\int_0^\infty \frac{x\, dx}{(x^2+h^2)^{\frac{3}{2}}}\ ,$$

$$E_z = -\,\frac{2\pi\,\sigma\, h}{4\pi\varepsilon_0}\ \frac{1}{\sqrt{x^2+h^2}}\Bigg|_0^\infty \ \Rightarrow$$

$$E_z = -\,\frac{\sigma\, h}{2\varepsilon_0}\left(0 - \frac{1}{h}\right) \Rightarrow \qquad E_z = \frac{\sigma}{2\varepsilon_0}\ .$$

The result shows that, for an infinite planar distribution of charge, at distance **h**, the intensity of the electric field generated does not depend on the distance **h**, but it is constant all over the plane, depending only on the superficial charge density **σ**.

Problem 84

The electric charge **Q** is uniformly distributed over a disk of radius **R** (Figure 54). By applying the superposition principle (as in problem 80), determine the electric field intensity at a point A at distance **z**, on the axis that crosses the disk perpendicularly through its centre.

Solution

The superficial density of the charge distributed over the disk:

$$\sigma = \frac{Q}{\pi\, R^2}\ .$$

The point like charge **dq** on an elementary ring of charge **dQ** (over the element of area **dS** of a ring of radius **y** and width **dy**) produces an elementary electric field **dE**:

$$\mathrm{dE} = \frac{\mathrm{dq}}{4\pi\varepsilon_0\, r^2} = \frac{\mathrm{dq}}{4\pi\varepsilon_0\,(R^2+z^2)^2}.$$

This elementary vector has two components, parallel to the disk ($\mathrm{d}\vec{E}_{\parallel}$) and perpendicular to the disk plane $(\mathrm{d}\vec{E}_{\perp})$:

$$\mathrm{d}\vec{E} = \mathrm{d}\vec{E}_{\parallel} + \mathrm{d}\vec{E}_{\perp}\,.$$

With the field components:

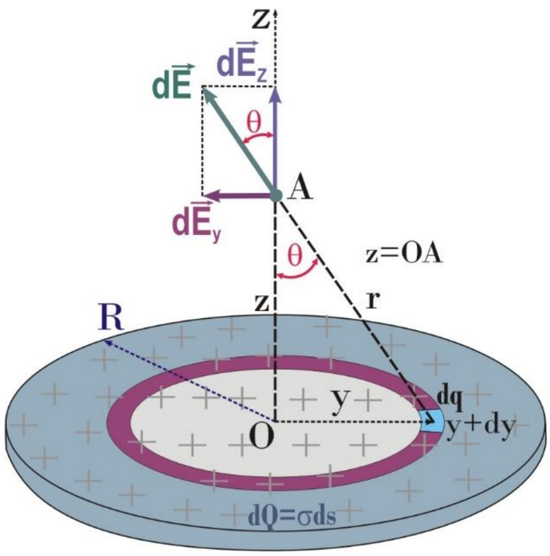


Fig. 54. A disc of electric charge, of radius R.

$$\begin{cases} \mathrm{dE}_{\parallel} = \mathrm{dE}_y = \mathrm{dE}\,\sin\theta = \mathrm{dE}\,\frac{y}{r} = \mathrm{dE}\,\frac{y}{\sqrt{y^2+z^2}} \\ \mathrm{dE}_{\perp} = \mathrm{dE}_z = \mathrm{dE}\,\cos\theta = \mathrm{dE}\,\frac{z}{r} = \mathrm{dE}\,\frac{z}{\sqrt{y^2+z^2}} \end{cases}.$$

For the total charge over the ring of radius *y* and width **dy**, the components parallel to the disk cancel each other, owing to the charge distribution symmetry. Thus, the ring of charge $\mathrm{dQ} = \sigma\,\mathrm{dS}$, only produces the component perpendicular to the disk plan. The area of the ring is:

$$\mathrm{dS} = \mathrm{d}(\pi\, y^2) = 2\pi\, y{\cdot}\mathrm{dy}\,.$$

The charge on the element of surface **dS** (the ring of radius y, width **dy**):

$$\mathrm{dQ} = \sigma{\cdot}\mathrm{dS} = \sigma\,\pi\,2y{\cdot}\mathrm{dy}\,.$$

The component of the electric field intensity generated by the ring of charge, perpendicular to the plan of the disk:

$$\mathrm{dE}_{\perp} = \mathrm{dE}{\cdot}\cos\theta = \frac{2\pi\sigma\, y{\cdot}\mathrm{dy}}{4\pi\varepsilon_0\,(y^2+z^2)^2}\left(\frac{z}{\sqrt{y^2+z^2}}\right).$$

To determine the total electric field intensity generated by the whole disk, one integrates the equation above, between **0** and **R** (the radius of the disk):

$$E_{\perp} = \int_0^R \frac{\sigma{\cdot}y{\cdot}z{\cdot}\mathrm{dy}}{2\varepsilon_0\,(y^2+z^2)^{3/2}} = \frac{\sigma{\cdot}z}{2\varepsilon_0}\int_0^R \frac{\mathrm{dy}}{(z^2+y^2)^{\frac{3}{2}}} = \frac{\sigma{\cdot}z}{2\varepsilon_0}\left(-\frac{1}{\sqrt{y^2+z^2}}\right)\Bigg|_0^R \Rightarrow$$

$$E_{\perp} = \frac{\sigma{\cdot}z}{2\varepsilon_0}(-1)\left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{z}\right) \Longrightarrow$$

$$E_{\perp} = \frac{\sigma{\cdot}z}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right).$$

Problem 85

Determine the electric potential of a dipole at point A, a distance **r** from the centre of the dipole (Figure 55).

Solution

The electric potential generated by the dipole at point A is the algebraic summation of the potentials produced by each of the two charges that make up the dipole:

$$V_A = V_{A(+)} + V_{A(-)}\ ;$$

$$V_A = \frac{q}{4\pi\varepsilon_0\, r_+} + \frac{(-q)}{4\pi\varepsilon_0\, r_-} = \frac{q}{4\pi\varepsilon_0}\,\frac{r_- - r_+}{r_+{\cdot}r_-},$$

$$V_A = \frac{q}{4\pi\varepsilon_0}\,\frac{\Delta r}{r_+{\cdot}\, r_-}.$$

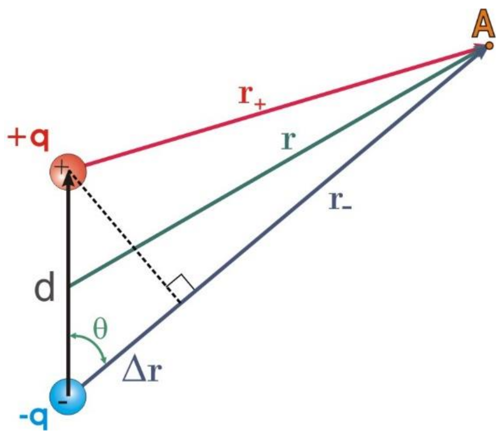


Fig. 55. An electric dipole generates the electric potential V at point A.

If the point A is at distance r >> d ⇒ the distance between A and each of the charges are approximately equal to each other: $r_- \approx r_+ \approx r$. Then: $r_+{\cdot}r_- \approx r^2$.

From Figure 55, in the 90° angle triangle, the segment Δr is given by: $\Delta r = d\cos\theta$. Therefore, the resulting electric potential at point A becomes:

$$V_A = \frac{q}{4\pi\varepsilon_0}\,\frac{d\cos\theta}{r^2} = \frac{q\,d}{4\pi\varepsilon_0}\,\frac{\cos\theta}{r^2} \Longrightarrow$$

$$V_A = \frac{p}{4\pi\varepsilon_0}\,\frac{\cos\theta}{r^2} \sim \frac{1}{r^2}.$$

It results that for an *electric dipole*, the electric potential at distance r from the dipole centre is inversely proportional with the reciprocal of $\mathbf{r^2}$. Note that for a point like charge, the electric potential is inversely proportional to the distance **r**, from the centre of the charge*:* $V \sim \frac{1}{r}$.

Problem 86

A water molecule has an electric dipole moment $\mu_e = 6.1 \times 10^{-30}\ \mathrm{C{\cdot}m}$, playing an important role in determining the specific properties of water, such as its remarkable solvency properties, and in its contribution to food heating, especially in microwave ovens. Find what amount of energy **W** is required to change the orientation of the water dipolar moment from *parallel* to *antiparallel*, relative to an external electric field of intensity $E = 2.4 \times 10^5\ \mathrm{V/m}$ (Figure 56).

Solution

The potential energy of an electric dipole $\vec{\mu}_e$ when placed in an electric field of intensity $\vec{E}$, is equal to the scalar product of the two vectors, with the minus sign:

$$W_p = -\vec{\mu}_e \cdot \vec{E}.$$

For the dipole parallel to the electric field (state 1 in Figure 56), the energy is given by:

$$W_{\parallel} = -\mu_e\, E.$$

The dipole antiparallel with the electric field (state 2 in Figure 56) has a higher potential energy:

$$W_{anti\,\parallel} = \mu_e\, E.$$

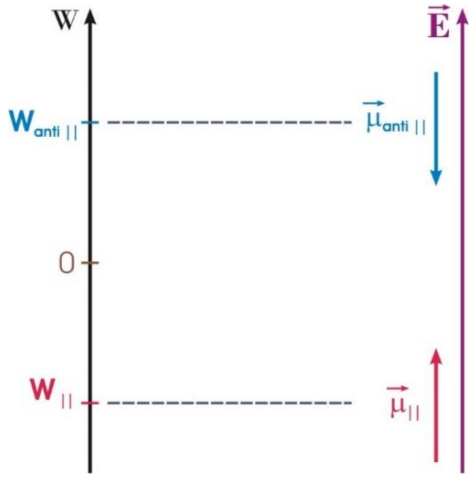


Fig. 56. An electric dipole in an electric field.

The difference between the two potential energy equals the mechanical work necessary to rotate the dipole, from parallel, to anti-parallel to the electric field:

$$W = W_{anti\,\parallel} - W_{\parallel} = 2\,\mu_e\, E;$$

$$W = 2 \times 6.1 \times 10^{-30}\ \mathrm{C{\cdot}m} \times 2.4 \times 10^{5}\,\frac{\mathrm{V}}{\mathrm{m}} = 2.9 \times 10^{-24} J = \frac{2.9 \times 10^{-24} J}{1.6 \times 10^{-19}\,\frac{J}{eV}}$$

$$W \cong 1.83 \times 10^{-5}\ \mathrm{eV} = 0.0183\ \mathrm{meV}.$$

Problem 87

Two capacitors of capacitance $C_1 = 5\ \mu F$ and $C_2 = 8\ \mu F$ are connected in series across a 9 V battery. What is the total energy stored in these capacitors?

Solution

Let us first calculate the capacitance equivalent to the two capacitors connected in series, by applying the equation:

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow$$

$$C_S = \frac{C_1\, C_2}{C_1 + C_2} \approx 3.08\ \mu\mathrm{F}.$$

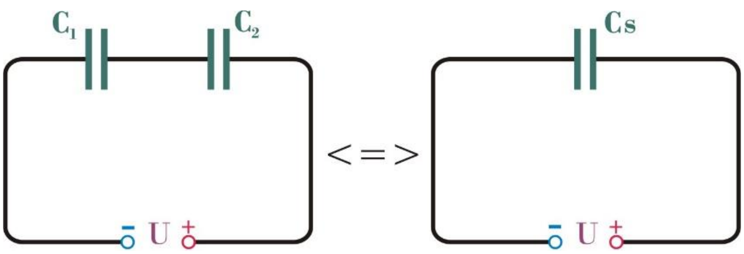


Fig. 57. Two capacitors in series.

Therefore, we can calculate the potential energy $\mathbf{W_p}$ stored in the electric field of the capacitors in series, as that stored in an equivalent capacitor of capacitance $\mathbf{C_S}$, by applying the equation:

$$W_p = \frac{1}{2}\, C_S\, U^2 = \frac{1}{2}\, 3.08\ \times 10^{-6} \times 9^2\ (\mathrm{J}).$$

$$W_p \approx 1.25\ \times\ 10^{-4}\,\mathrm{J} \approx 7.8 \times 10^{14}\ \mathrm{J}.$$

Problem 88

A 6 μF capacitor filled with air is charged by a 12 V battery. Then, it is disconnected from the battery, and the space between the plates is filled with oil (with the dielectric constant $\varepsilon_r = 2.5$).

a) What is the potential difference **U** between the capacitor plates after immersing in oil?

b) What is the free electric charge **Q** on one plate of the capacitor? How many elementary charges make up the total charge **Q**?

c) What is the induced charge $\mathbf{Q_i}$ on the surface of the oil?

Solution

a) The electric charge on the capacitor plates is:

$$Q = C_o\, U_o.$$

Where $\mathbf{C_0}$ and $\mathbf{U_0}$ are the capacitance and the voltage, respectively, for the capacitor filled with air. The electric charge **Q** remains the same after the circuit is interrupted and the oil is inserted between the capacitor plates, now of capacitance **C**:

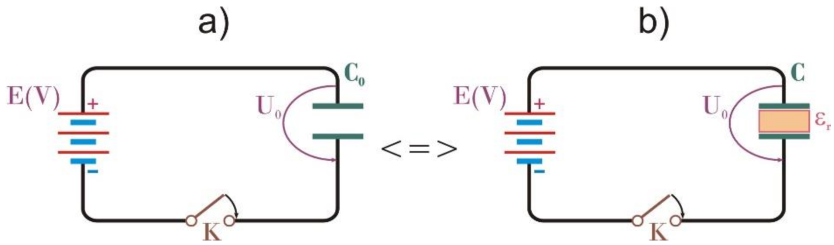


Fig. 58. An RC circuit: a) a capacitor of capacitance C_0 (filled with air); b) a capacitor filled with dielectric oil of relative permittivity ε_r.

$$Q = C\, U.$$

The potential difference at the ends of the capacitor with oil is:

$$U = \frac{Q}{C} = U_0 \frac{C_0}{C} = U_0 \frac{\varepsilon_0 \frac{S}{d}}{\varepsilon_r \varepsilon_0 \frac{S}{d}} = \frac{U_0}{\varepsilon_r} \Longrightarrow U = 4.8\ \mathrm{V}.$$

b) The free electric charge on one plate is given by:

$$Q = C\, U = C_0\, U_0 = 6\ \mu\mathrm{F} \cdot 12\mathrm{V} = 72\ \mu\mathrm{C}.$$

The number of electric charge carriers that make up this total charge Q is then:

$$n = \frac{Q}{e} = \frac{72\ \times 10^{-6}\ \mathrm{C}}{1.6 \times 10^{-19}\ \mathrm{C}} \Longrightarrow n = 4.5\ \times 10^{14}.$$

c) The induced (or fixed) charge Q_i can be deduced as follows, based on Gauss's law that states in its general form that: $\Phi_E = \frac{Q}{\varepsilon_0}$, the electric flux across a closed surface in space equals the ratio between the total charge Q enclosed by that surface and the electric permittivity of free space, ε_0. The electric flux through a closed (or Gaussian) surface Σ is given by the integral:

$$\Phi_E(\text{across } \Sigma) = \iint_\Sigma \vec{E}\cdot d\vec{S} \Rightarrow$$

Then Gauss's law results as: $$\iint_\Sigma \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_r}$$

(1) Consider initially the capacitor with air or empty space between its plates. Then, for a closed surface $\boldsymbol{\Sigma}$ surrounding one of the plates, Gauss's law states that the flux of the electric field $\mathbf{E_0}$ through the gaussian surface equals the total charge enclosed by $\boldsymbol{\Sigma}$ (in this case, **Q**) divided by ε_0. As the electric flux is different from zero only through the part of Σ between the plates of area **S**, the electric flux is: $\Phi_E = E_0 \cdot S$, and thus Gauss's law results:

$$E_0 \cdot S = \frac{Q}{\varepsilon_0}.$$

(2) Now consider that a dielectric (of thickness **d** and dielectric constant $\boldsymbol{\varepsilon_r}$) is placed between the capacitor plates. As a consequence, an electric charge $\mathbf{Q_i}$ is induced on each of the plates, opposed to the charge on the plate, and then the total charge on each plate is reduced by the induced charge amount $(\mathbf{Q} - \mathbf{Q_i})$. Thus, this also causes the electric field to become smaller than the initial ($\mathbf{E_0}$) when no dielectric was between the plates, $\mathbf{E} < \mathbf{E_0}$. Then, for a closed surface $\boldsymbol{\Sigma}$ surrounding one of the plates, Gauss's law states that the flux of the electric field **E** through the whole Gaussian surface equals the total charge enclosed by Σ (in this second case, $\mathbf{Q} - \mathbf{Q_i}$) divided by $\boldsymbol{\varepsilon_0}$ (again, the electric flux is different from zero, i.e. $\Phi_E = E \cdot S$, only through the part of Σ between the plates of area **S**):

$$E \cdot S = \frac{Q - Q_i}{\varepsilon_0}.$$

(3) If we now divide the previous two forms of Gauss's equations member by member, we obtain:

$$\frac{E_0 \cdot S}{E \cdot S} = \frac{Q}{\varepsilon_0} \cdot \frac{\varepsilon_0}{Q - Q_i} \Longrightarrow \frac{E_0}{E} = \frac{Q}{Q - Q_i}.$$

(4) As an effect of the dielectric (of dielectric constant ε_r) being introduced between the capacitor plates, the electric charge Q_i induced on the dielectric surface, of opposite sign with respect to the charge Q on each plate, reduces the total charge to $(Q - Q_i)$, thus also reducing the resulting electric field intensity from its initial value E_0 to $E = \frac{E_0}{\varepsilon_r}$. Therefore, the latter equation becomes: $\varepsilon_r = \frac{Q}{Q - Q_i}$

After regrouping the terms, we obtain: $$Q = \varepsilon_r (Q - Q_i) \Leftrightarrow \varepsilon_r \cdot Q_i = Q\,(\varepsilon_r - 1)$$

$$Q_i = Q\,\frac{\varepsilon_r - 1}{\varepsilon_r} = 72 \times 10^{-6} \times \frac{1.5}{2.5}\ \text{C}$$

And, therefore, the charge induced on the dielectric surface results: $Q_i = 43.2\ \mu\text{C}$.

Problem 89

Determine the electric potential **V** of the field generated at the point A (Figure 59), by the charge **Q** uniformly distributed along the segment of finite length **L**.

Solution

The elementary charge at distance ***x*** from the right end of the segment:

$$dq = \lambda\, dx = \frac{Q}{L}\, dx.$$

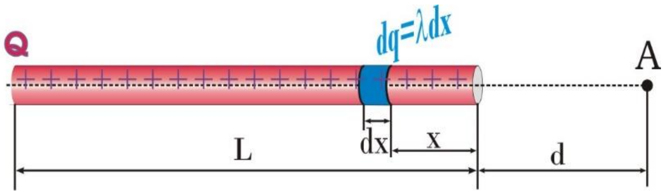


Fig.59. The electric potential of a segment of charge.

This elementary charge dq generates the elementary electric potential **dV** at point A:

$$dV = \frac{dq}{4\pi\varepsilon_0(x+d)} = \frac{\lambda\, dx}{4\pi\varepsilon_0(x+d)}.$$

The total electric potential is found by integrating **dV** over the whole segment of length **L**:

$$\int_0^V dV = \frac{\lambda}{4\pi\cdot\varepsilon_0}\cdot\int_0^L \frac{dx}{x+L}$$

Therefore, the total electric potential of a finite segment of charge results as:

$$V = \frac{Q}{4\pi\varepsilon_0\, L}\ln\left(\frac{L+d}{d}\right).$$

Problem 90

Determine the electric potential of the electric field produced by electric charge distributed on a ring of radius **R**, at distance **h** from the centre of the ring (Figure 60).

Solution

The element of charge: $dq = \lambda\, d\hat{s}$ produces at point A, on the ring axis, at distance **h** from its centre, the elementary electric potential **dV**:

$$dV = \frac{dq}{4\pi\varepsilon_0 r} = \frac{\lambda\, d\hat{s}}{4\pi\varepsilon_0\sqrt{R^2+h^2}}.$$

The total potential at point A is found by integrating over the whole ring of charge:

$$V_A = \int_0^{V_A} dV = \frac{\lambda}{4\pi\varepsilon_0 r}\int_{ring} d\hat{s}\ ;$$

$$V_A = \frac{\lambda\, 2\pi R}{4\pi\varepsilon_0\sqrt{R^2+h^2}} = \frac{\frac{Q}{2\pi R}2\pi R}{4\pi\varepsilon_0\sqrt{R^2+h^2}}$$

$$V_A = \frac{Q}{4\pi\varepsilon_0\sqrt{R^2+h^2}}.$$

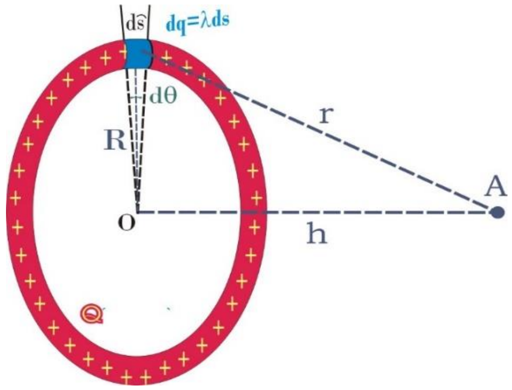


Fig. 60. Charge distributed over a ring of radius R.

At the centre of the ring of charge, **h=0**, and therefore, the potential becomes:

$$V_0 = \frac{Q}{4\pi\varepsilon_0\, R}.$$

Problem 91

An electric charge Q is distributed uniformly over a disk of radius **R**. Determine the potential of the electric field generated at the point B, situated on the disk axis perpendicular to its plan, at distance **z** from the centre of the disk (Figure 61).

Solution

To find the total potential of the disk of charge, let us first consider the elementary charge on a ring of charge, of radius x and width **dx**, over the elementary area dS:

$$\mathbf{dq = \sigma \cdot dS = \sigma\, 2\pi x \cdot dx}$$

This elementary ring of charge produces the elementary electric potential:

$$\mathbf{dV = \frac{dq}{4\pi\varepsilon_0\, r} = 2\pi\,\sigma\frac{x\,dx}{4\pi\varepsilon_0\sqrt{z^2+x^2}}}$$

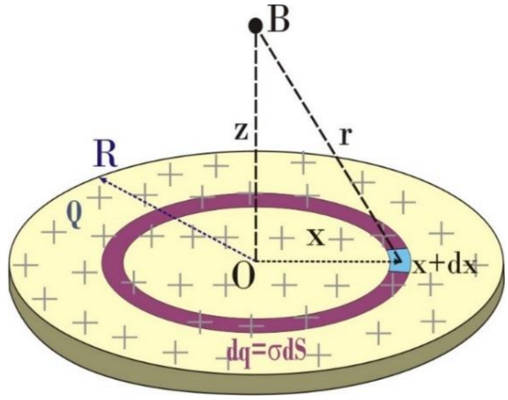


Fig. 61. The electric potential of the field produced at point B by a disk of charge.

The total electric potential of the field generated by the whole disk of charge can be determined by integrating for x taking values between (0,R):

$$V = \int_0^V dV = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_0^R \frac{x\,dx}{\sqrt{z^2+x^2}}\ ;$$

$$V = \frac{\sigma}{2\varepsilon_0}\sqrt{z^2+x^2}\Big|_0^R \Rightarrow V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-z\right].$$

Problem 92

The electric potential of an electric field is given by the equation: $V(x,y) = x{\cdot}y + y^2$. What is the electric field intensity at the point of coordinates (2, 3)?

Solution

The electric field equals minus the gradient of the electric potential:

$$\vec{E} = -\mathrm{grad}\,V = -\nabla V = -\left(\frac{\partial V}{\partial x}\,\vec{\imath} + \frac{\partial V}{\partial y}\,\vec{\jmath}\right) \Longrightarrow \vec{E} = -y\,\vec{\imath} - (x + 2y)\vec{\jmath}\,.$$

For the point of coordinates (2, 3), the intensity of the electric field is:

$$\vec{E}(2,3) = -2\,\vec{\imath} - (2+6)\,\vec{\jmath} \Longrightarrow \vec{E}(2,3) = -2\vec{\imath} - 8\vec{\jmath}\,.$$

Its absolute value then is: $E = \sqrt{E_x^2 + E_y^2}$, and for the point of coordinates (2, 3):

$$E(2,3) = \sqrt{2^2 + 8^2} = \sqrt{68} \Longrightarrow E(2,3) \cong 8.246\left(\frac{V}{m}\right).$$

Problem 93

Represent the electric field lines and the electric field intensity around a point like electric charge (a) positive and (b) negative. Represent the graphs showing the dependence of the electric potential V = V(r), and of the electric field intensity E = E(r) versus **r**, the distance from the point charge Q.

Solution

a) For a positive charge

The magnitude of the *electric field intensity* at distance **r** from the point charge is inversely proportional with **r²**:

$$E(r) = \frac{Q}{4\pi\varepsilon_0\,r^2} \sim \frac{1}{r^2}\,.$$

The *electric potential* at any point a distance **r** from the point of charge is inversely proportional with **r**, being given by the equation:

$$V(r) = \frac{Q}{4\pi\varepsilon_0\,r} \sim \frac{1}{r}\,.$$

b) For a negative charge:

The absolute value of the *electric field intensity* at distance **r** from the point of charge is:

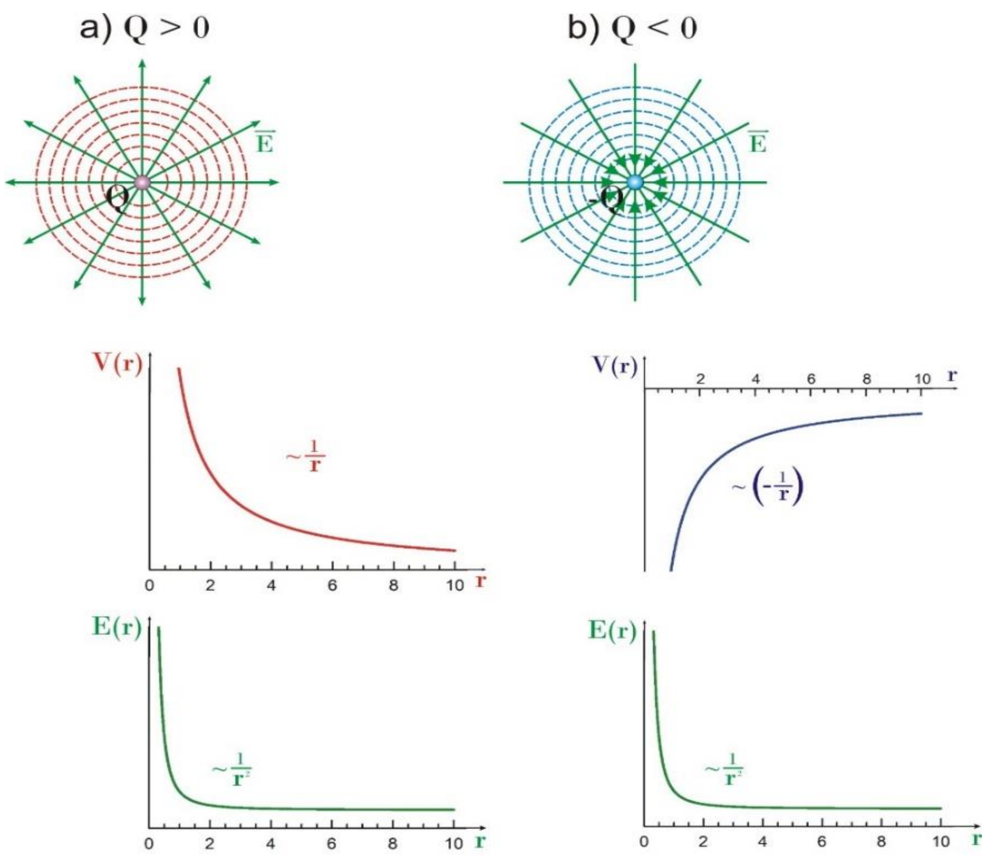


Fig. 62. The electric field of point like charges: (a) Positive and (b) Negative.

$$E(\mathrm{r}) = \frac{\mathrm{Q}}{4\pi\varepsilon_0\,\mathrm{r}^2} \sim \frac{1}{\mathrm{r}^2}.$$

The *electric potential* is:

$$\mathrm{V(r)} = -\mathrm{k}\,\frac{\mathrm{Q}}{\mathrm{r}} \sim \left(-\frac{1}{\mathrm{r}}\right).$$

Problem 94

What is the intensity of the uniform electric field between the capacitor parallel plates at distance d = 0.5 cm, if the capacitor is charged at the potential difference U = 1.2 kV?

Solution

The intensity of the electric field between the plates of a parallel plates capacitor:

$$\mathrm{E} = \frac{\mathrm{U}}{\mathrm{d}} = \frac{1.2 \times 10^3\ \mathrm{V}}{0.5 \times 10^{-2}\ \mathrm{m}} \Longrightarrow$$

$$\mathrm{E} = 2.4\ \times 10^5\ \frac{\mathrm{V}}{\mathrm{m}}.$$

Problem 95

Consider a spherical distribution of total charge **Q**, over the surface of a conducting sphere of radius **R**, as shown in Figure 63-a. Use Gauss's law to determine the intensity of the electric field, E(r) inside and outside the sphere, and the electric potential V(r).

Solution

Let Σ_{ext} be a Gaussian (closed) surface that encloses the sphere of charge Q. Gauss's law states that integral of the scalar product between the electric field intensity and the element of surface, over the whole Gaussian surface equals the total charge enclosed by the surface Σ_{ext} (a sphere of radius ***r***) and the electric permittivity of free space:

$$\oiint_{\Sigma_{ext}} \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}.$$

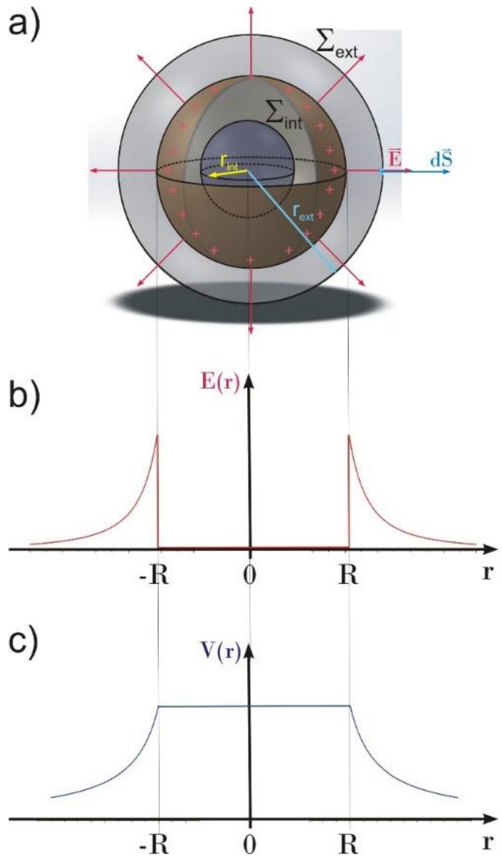


Fig. 63. Electric field intensity and potential of a charge Q distributed over the surface of a metallic sphere.

As the Gaussian surface is spherical (the shape of the Gaussian surface is chosen according to the symmetry of the charge distribution), the intensity of the electric field is constant over the whole surface Σ_{ext}, and therefore the integration becomes:

$$E_{ext} \oiint_{\Sigma_{ext}} dS = \frac{Q}{\varepsilon_0} \Leftrightarrow E_{ext}\, 4\pi\, r^2 = \frac{Q}{\varepsilon_0}.$$

It follows that the external electric field has the intensity

$$E_{ext}(r) = \frac{Q}{4\pi\, \varepsilon_0\, r^2}, \text{when } r > R\,.$$

Inside the metallic (conducting) sphere, there is no charge, all the charge resides on the surface of the sphere. Therefore, for another spherical Gaussian surface, inside the sphere (r < R), the total charge is zero, and thus:

$$\oiint_{\Sigma_{int}} \vec{E}_{int}\cdot d\vec{S} = 0 \Leftrightarrow \vec{E}_{int} = 0.$$

The graph showing *E(r)* inside and outside the sphere with charge **Q** is presented in Figure 63-b.

As for the electric potential, inside the conducting sphere the potential is constant (if it were not, then it would be a transport of charge, until the potential becomes constant).

Outside the sphere (for r > R), the electric potential is:

$$V_{ext}(r) = \frac{Q}{4\pi\,\varepsilon_0\, r}.$$

Applying the condition of continuity of the electric potential inside the conducting sphere, the potential must equal that on the very surface (r = R):

$$V_{int} = \frac{Q}{4\pi\,\varepsilon_0\, R}.$$

The graph of the electric potential of a spherical distribution of charge versus the distance from the centre of the sphere, is shown in Figure 63-c.

Problem 96

The electric charge Q is distributed uniformly ($charge\ density$: $\rho = \frac{Q}{V}$) in the whole volume of a non-conducting sphere of radius **R** (See Figure 64). By using Gauss's law, determine the magnitude of the electric field generated by this charge distribution, both inside and outside the sphere. Represent the graph showing E(r) versus the distance r from the centre of the charge distribution.

Solution

Inside the sphere, where r< R, Gauss's law states that:

$$\oiint_{\Sigma_{int}} \vec{E}_i \cdot d\vec{S} = \frac{Q_{int}}{\varepsilon_0}.$$

All over the sphere that is the Gaussian surface $\mathbf{\Sigma_{int}}$, of radius **r**, the electric field intensity is constant, so it can be factored out of the integration. The charge $\mathbf{Q_{int}}$ is equal to the product between the charge density ρ and the volume of the sphere $\mathbf{\Sigma_{int}}$. Therefore, Gauss's law becomes:

$$E_i \oiint_{\Sigma_{int}} dS = \frac{\rho V_{int}}{\varepsilon_0};$$

$$E_i\, 4\pi\, r_{int}^2 = \frac{\rho \dfrac{4\pi\, r_{int}^3}{3}}{\varepsilon_0};$$

$$E_i = \frac{\rho}{3\,\varepsilon_0}\, r_{int} \Longrightarrow$$

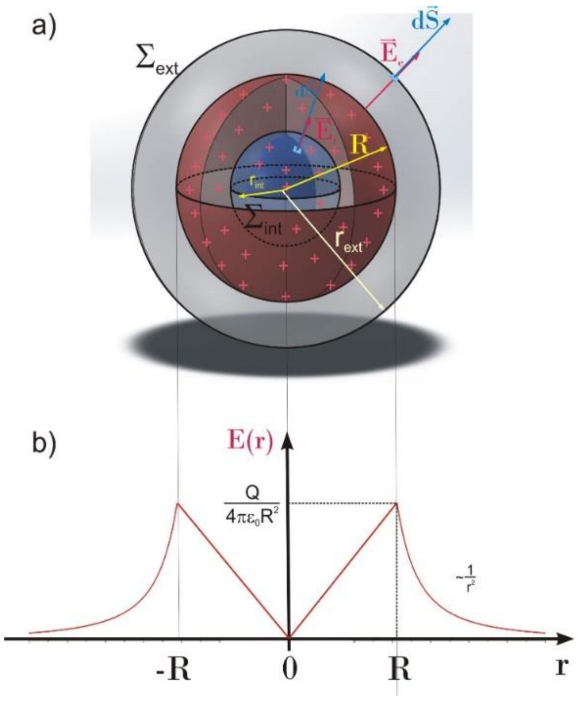


Fig. 64. A non-conducting sphere (R) of charge Q.

$$E_i = \frac{Q}{4\pi\varepsilon_0\, R^3}\, r_{int}.$$

For the region **r** > **R**, outside the sphere of radius **R**, a corresponding Gaussian surface is a sphere Σ_{ext} of radius $\mathbf{r_{ext}}$. This Gaussian surface encloses all the charge Q within the sphere. Therefore, Gauss's law becomes:

$$\oiint_{\Sigma_{ext}} \vec{E}_e \cdot d\vec{S} = \frac{Q}{\varepsilon_0}.$$

Over the spherical Gaussian external surface, the intensity of the electric field is constant, therefore it can be factored out of the integral:

$$E_e \oiint_{\Sigma_{ext}} dS = \frac{Q}{\varepsilon_0}.$$

The intensity of the electric field outside the charge distribution, at distance $\mathbf{r_{ext}}$ from the centre of the sphere:

$$E_e = \frac{Q}{4\pi\, r_{ext}^2}.$$

Therefore, the electric field generated by a uniform distribution of charge with density ρ and of radius **R**, is given by:

$$E(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0\, R^3}\cdot r, \text{for } r < R \\ \dfrac{Q}{4\pi\varepsilon_0\, r^2}, \text{for } r > R \end{cases}.$$

The charge distribution and the graph of **E(r)** versus the distance **r** from the centre of the sphere are shown in Figures 64-a and 64-b.

Problem 97

A non-conducting sphere contains charge distributed so that the charge density depends on the distance **r** to the centre of the sphere: $\rho(r) = \frac{\rho_0}{r}$. Determine the intensity of the electric field $\vec{\mathbf{E}}$ inside and outside the charge distribution (Figure 65).

Solution

Inside the sphere with the total charge Q, Gauss's law for a spherical closed surface of radius $\mathbf{r_i}$:

$$\oiint_{\Sigma_i} \vec{E_i} \cdot d\vec{S} = \frac{Q_i}{\varepsilon_0}.$$

The intensity $\mathbf{E_i}$ has the same value all over the Gaussian surface $\mathbf{\Sigma_i}$, and therefore it can be factored out of the integral:

$$E_i \oiint_{\Sigma_i} dS = \frac{Q_i}{\varepsilon_0};$$

$$E_i\, 4\pi r_i^2 = \frac{Q_i}{\varepsilon_0}.$$

Where the electric charge Q_i enclosed by the Gaussian surface Σ_i is calculated by applying the volume integral:

$$Q_i = \iiint_{V_i} \rho_i(r)\, dV = \int_0^{r_i} \frac{\rho_0}{r} 4\pi r^2 dr = 4\pi\, \rho_0 \int_0^{r_i} r{\cdot}dr = 4\pi\rho_0 \left(\frac{r_i^2}{2} - 0\right);$$

$$Q_i = 2\pi\rho_0 r_i^2.$$

Therefore, Gauss's law becomes:

$$E_i\, 4\pi\, r_i^2 = \frac{1}{\varepsilon_0} 2\pi\, \rho_0 r_i^2.$$

And thus, the intensity of the electric field, at distance $\mathbf{r_i}$ from the centre of the charge distribution given above, does not depend on the value of $\mathbf{r_i}$, as it becomes:

$$E_i = \frac{\rho_0}{2\, \varepsilon_0}.$$

For an external sphere of radius $\mathbf{r_e} > \mathbf{R}$, which encloses the whole spherical distribution of charge, Gauss's law:

$$\oiint_{\Sigma_e} \vec{E}_e{\cdot}d\vec{S} = \frac{Q}{\varepsilon_0}.$$

The intensity of the electric field over the sphere of radius re is constant over the surface of the sphere, as all the points are at the same distance from the charge **Q**, and therefore it can be factored out of the integral:

$$E_e \oiint_{\Sigma_e} dS = \frac{Q}{\varepsilon_0}.$$

$$E_e\, 4\pi\, r_e^2 = \frac{Q}{\varepsilon_0}.$$

Then the intensity of the electric field, outside the distribution of charge **Q**:

$$E_e = \frac{Q}{4\pi\varepsilon_0\, r_e^2}$$

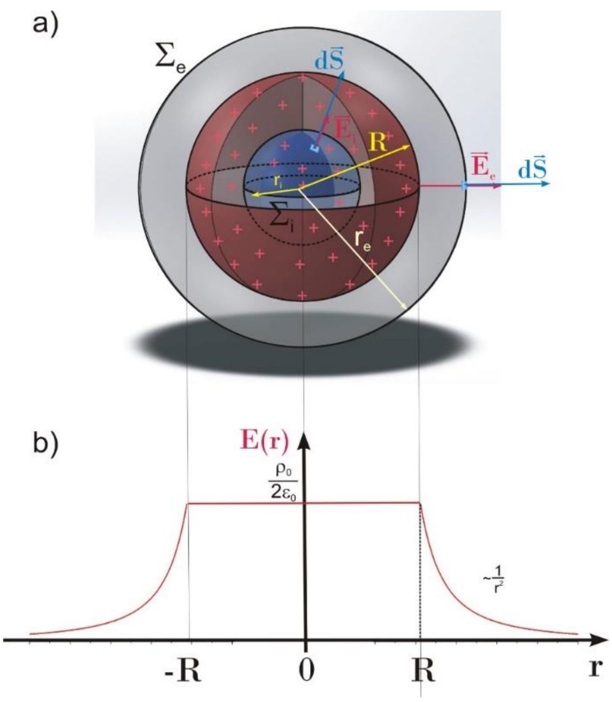


Fig. 65. Spherical distribution (radius R) of charge within a non-conducting material.

In conclusion, the intensity of the electric field generated by a distribution of charge within a non-conducting sphere, with the charge density $\rho(r) = \frac{\rho_0}{r}$ is given by the equations:

$$E(r) = \begin{cases} \dfrac{\rho_0}{2\varepsilon_0}, & \text{for } r < R \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}, & \text{for } r > R \end{cases}.$$

The graph showing the dependence of **E(r)** versus **r**, the distance to the centre of the spherical distribution of charge, is presented in Figure. 65-b.

Problem 98

Electric charge is distributed uniformly (with linear density λ) along an infinite line of charge. Use Gauss's law to determine the intensity of the electric field at distance **z** from the line of charge (Figure 66).

Solution

Let us consider a Gaussian surface of cylindrical shape, of length **L**, with the line of charge being its central axis (see Figure 66). Gauss's law for this closed surface then is:

$$\oiint_{\Sigma} \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}.$$

The charge enclosed by the cylindrical Gaussian surface Σ is: $Q = \lambda L$.

Integrating the scalar product $\vec{E} \cdot d\vec{S}$ over the entire Gaussian surface, and considering the electric flux across the cylinder bases and the cylindrical surface, results in:

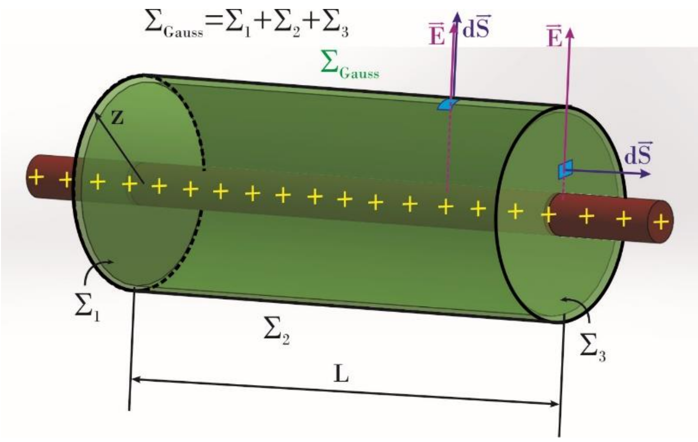


Fig. 66. The electric field intensity E(z) at distance z from an infinite linear distribution of charge

$$\oiint_{\Sigma} \vec{E} \cdot d\vec{S} = \iint_{\Sigma_1} \vec{E} \cdot d\vec{S} + \iint_{\Sigma_2} \vec{E} \cdot d\vec{S} + \iint_{\Sigma_3} \vec{E} \cdot d\vec{S}.$$

Over the cylinder caps, the two circular surfaces Σ_1 and Σ_3, the electric intensity vector is parallel to the surface (hence, no electric field flux), and perpendicular to the elementary surface vector $\mathbf{d\vec{S}}$ (hence, the scalar product between the perpendicular vectors is null). Therefore, there is no electric flux through those areas, and the only surface with electric flux remains the actual cylindrical surface. Thus, Gauss's law results:

$$\iint_{\Sigma_2} \vec{E} \cdot d\vec{S} = \frac{\lambda L}{\varepsilon_0}.$$

The intensity of the electric field has the same value all over the cylindrical surface, as all the points on the cylinder are at the same distance to the line of charge, therefore, **E** can be factored out of the integral:

$$\mathrm{E}\iint_{\Sigma_2} \mathrm{dS} = \frac{\lambda \cdot \mathrm{L}}{\varepsilon_0};$$

$$\mathrm{E} \cdot 2\pi \cdot \mathrm{z} \cdot \mathrm{L} = \frac{\lambda \cdot \mathrm{L}}{\varepsilon_0}.$$

After simplifying with **L,** and regrouping the terms, the intensity of the electric field produced by an infinite linear distribution of charge, at distance **z** from the line of charge, is given by:

$$\mathrm{E(z)} = \frac{\lambda}{2\pi\varepsilon_0 \cdot \mathrm{z}}.$$

Problem 99

Electric charge is uniformly distributed over an infinite non-conducting surface, with the superficial charge density $\boldsymbol{\sigma}$. Apply Gauss's law to determine the intensity of the electric field, at points close to the non-conducting surface of charge (Figure 67).

Solution

Consider a Gaussian surface as shown in Figure 67, a cylinder as a small 'pill-box', of height **h** on each side of the charge distribution, and of circular area **S**. Gauss's law for such a Gaussian surface then can be written:

$$\oiint_{\Sigma_{\mathrm{Gauss}}} \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\mathrm{S}} = \frac{\mathrm{Q}}{\varepsilon_0}.$$

The Gaussian closed surface consists of two caps of circular areas (Σ_1 and Σ_3) and a cylindrical surface (Σ_2):

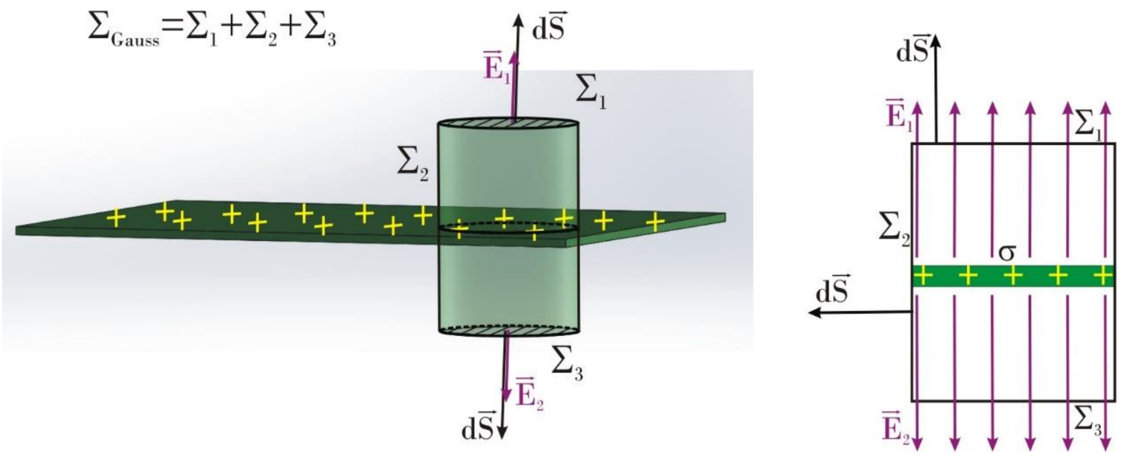


Fig. 67. A non-conducting plan of uniformly distributed charge.

$$\Sigma_{\text{Gauss}} = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

The intensity of the electric field generated by the surface of charge is parallel to the cylindrical surface Σ_2, and thus perpendicular to the element of surface vector $d\vec{S}$. The electric field is perpendicular to the circular surface caps, each of area **S**, and has the same value over each of them. Therefore, the left side member of Gauss's law becomes:

$$\oiint_{\Sigma_{\text{Gauss}}} \vec{E}\cdot d\vec{S} = \iint_{\Sigma_1} \vec{E}\cdot d\vec{S} + \iint_{\Sigma_2} \vec{E}\cdot d\vec{S} + \iint_{\Sigma_3} \vec{E}\cdot d\vec{S} = E\cdot S + 0 + E\cdot S = 2\cdot E\cdot S\,.$$

The charge enclosed by the Gaussian surface**Σ** is:

$$Q = \sigma\cdot S.$$

Substituting in Gauss's law: $2\cdot E\cdot S = \frac{\sigma\, S}{\varepsilon_0}$ Therefore, the intensity of the field close to an infinite insulating plan of electrical charge, is: $E = \frac{\sigma}{2\varepsilon_0}$.

Problem 100

Electric charge is uniformly distributed over an infinite and thick metallic surface, with the superficial charge density **σ**. Apply Gauss's law to determine the intensity of the electric field, at points close to the conducting surface of charge (Figure 68).

Solution

If the charge is distributed over the surface of a metal (or another conducting material), then the electric field lines point only to the exterior (the electric field inside the metal is null, E = 0).

Gauss's law for a closed surface Σ, in the form of a small cylinder of cross section area **S**:

$$\oiint_{\Sigma_{\text{Gauss}}} \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0},$$

Where, as shown in Figure 68, we have that:

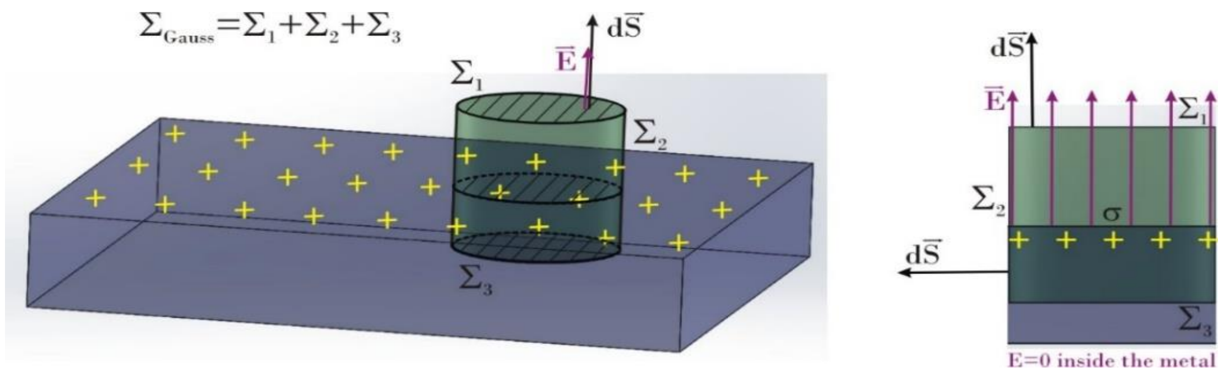


Fig. 68. A conducting plan of charge.

$$\Sigma_{\text{Gauss}} = \Sigma_1 + \Sigma_2 + \Sigma_3\,.$$

The electric field intensity E over the external surface Σ_1 is constant; over the surface Σ_3 inside the metal, there is no electric field; the intensity of the electric field is parallel to the cylindrical surface Σ_2, and perpendicular to the elementary area $d\vec{S}$.

Therefore, Gauss's law becomes:

$$E \iint_{\Sigma_1} d\vec{S} = \frac{\sigma\, S}{\varepsilon_0} \quad \Rightarrow$$

$$E = \frac{\sigma}{\varepsilon_0}.$$

Problem 101

Determine the capacitance **C** of a metal sphere (Figure 69) of radius **R** charged with the electric charge **Q**.

Solution

By definition, the capacitance of an object is equal to the electric charge loaded on that object, under the unit voltage:

$$C = \frac{Q}{V}.$$

The electric field intensity around the sphere charged with electrical charge Q, is:

$$E = \frac{Q}{4\pi\varepsilon_0\, R^2}.$$

The electric potential is then:

$$V = E\cdot R = \frac{Q}{4\pi\varepsilon_0\cdot R}.$$

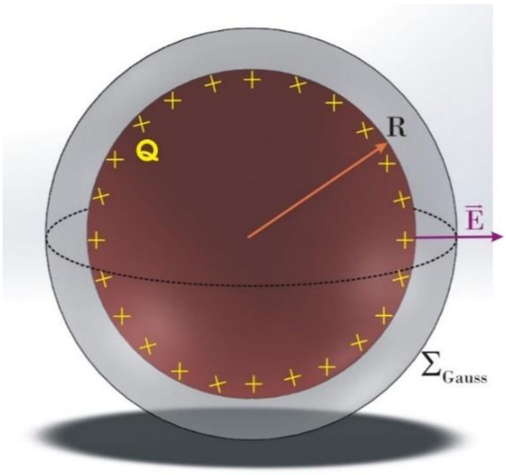


Fig. 69. A sphere of radius R and charge Q.

Then the capacitance becomes as shown below, proportional with the radius of the sphere, R:

$$C = \frac{Q}{V} = 4\pi\varepsilon_0\, R \sim R\,.$$

Problem 102

Considering the result of problem 101, determine:

a) The radius of a sphere charged at the capacitance C = 1 F.

b) The capacitance of the Earth (consider the Earth a sphere of average radius $R_E \cong 6400$ km).

Solution

a) The radius of a sphere that would have the electrical capacitance of 1 F:

$$R = \frac{C}{4\pi\, \varepsilon_0} = \frac{1\ \mathrm{F}}{4\pi{\cdot}8.856 \times \dfrac{10^{-12}\ \mathrm{F}}{\mathrm{m}}} \Longrightarrow$$

$R = 9 \times 10^9\ \mathrm{m} \gg R_{\mathrm{Earth}}$ $\Rightarrow$ this is practically impossible.

b) The electrical capacitance of the Earth:

$$C_{\mathrm{Earth}} = 4\pi\varepsilon_0\, R_{\mathrm{Earth}} = \frac{1}{9\times10^9}\frac{\mathrm{F}}{\mathrm{m}}{\cdot}6400 \times 10^3\mathrm{m} \Rightarrow$$

$$C_{\mathrm{Earth}} = 711\ \mu\mathrm{F}.$$

Problem 103

Apply Gauss's law to determine the capacitance of two concentric metallic spheres of radii R_1 (internal sphere, carrying positive charge) and R_2 (outer sphere, with negative charge).

Solution

From Gauss's law for the Gaussian surface Σ, a sphere of radius $r \in (R_1, R_2)$ (Figure 70):

$$\oiint_{\Sigma} \vec{E}{\cdot}d\vec{S} = \frac{Q}{\varepsilon_0}.$$

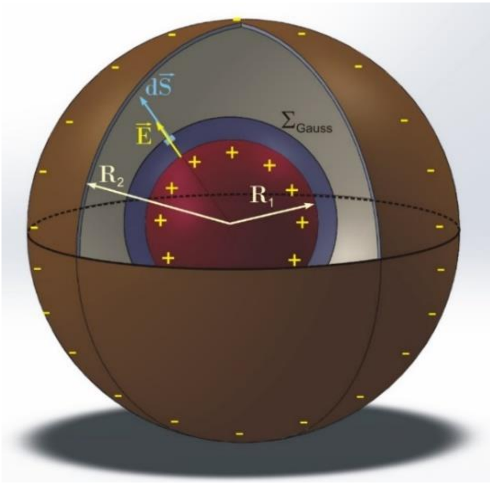


Fig. 70. Two charged concentric spheres.

The intensity of the electric field between spheres of radii R_1 and R_2 is:

$$E(r) = \frac{Q}{4\pi\varepsilon_0\, r^2}.$$

The electric field equals minus the gradient potential between the spheres R_1 and R_2:

$$\vec{E} = -\nabla V \Rightarrow |\vec{E}| = \frac{dV}{dr}$$ and then

$$\Rightarrow dV = E\, dr = \frac{Q}{4\pi\varepsilon_0\, r^2}\, dr.$$

Then the potential difference between the sphere of radius R_1 and the sphere of radius R_2 is:

$$U_{AB} = \Delta V = \int_{R_1}^{R_2} \frac{Q}{4\pi\varepsilon_0\, r^2} dr \ \Leftrightarrow$$

$$U_{AB} = \Delta V = \frac{Q}{4\pi\varepsilon_0} \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_{R_1}^{R_2}.$$

$$U_{AB} = \frac{Q\,(R_2 - R_1)}{4\pi\varepsilon_0 \cdot R_1 \cdot R_2}.$$

From the above equation, expressing the ratio between the electric charge and the potential difference gives the capacitance ***C*** of the system of two concentric metallic spheres:

$$C = \frac{Q}{U_{AB}} = 4\pi\varepsilon_0\,\frac{R_1 \cdot R_2}{R_1 - R_2}.$$

Discussion

1. Suppose that the inner radius is R_1= 2 m, and the second sphere is at 1 mm distance: $R_2 - R_1$ = 1 mm. Then the capacitance of the system is:

$$C = 4\pi\varepsilon_0 \frac{R_1 R_2}{R_2 - R_1} \approx 0.44\ \mu\text{F}$$

2. Calculate the radius of a single sphere that would have the same capacitance:

$$R = \frac{C}{4\pi\ \varepsilon_0} = 9\ \frac{10^9\text{m}}{\text{F}} 0.44\ \times 10^{-6}\text{F} \approx 4\ \text{km}.$$

Problem 104

a) Find the energy stored in the electric field around an electron if one assumes the electron to be spherical and its <u>charge is distributed over its surface</u>. Deduce and evaluate the radius of the electron.

b) Deduce the energy stored in the electric field around an electron if one assumes the electron to be spherical and its <u>charge is distributed uniformly within its volume</u>. Deduce and evaluate the radius of the electron in this case (in both answers, use that the mass of the electron is $m_e \cong\ 9.1 \times 10^{-31}$ kg).

Solution

a) If the electric charge is distributed <u>on the surface of the electron</u> considered a sphere, and if $\mathbf{w_E}$ is the density of electric energy, the elementary amount of energy within the element of volume **dV** (the spherical shell of thickness **dr**) is:

$$dW_E = w_E\ dV = \frac{1}{2}\varepsilon_0\ E^2\ dV.$$

The total energy stored within the whole space around the electron is calculated by integrating $\mathbf{dW_E}$ over the whole space around the electron:

$$W_E = \int_{\substack{\text{space}\\ \text{around e}}} dW_E = \frac{1}{2} \cdot \varepsilon_0 \cdot \int_{\substack{\text{space}\\ \text{around e}}} E(r)^2\ dV.$$

The element of volume of the sphere is:

$$dV = d\left(\frac{4\pi\, r^3}{3}\right) = 4\pi\, r^2\, dr\,.$$

Then the elementary amount of energy stored in the electric static field becomes:

$$dW_E = \frac{1}{2}\cdot\varepsilon_0\cdot\left(\frac{e}{4\pi\varepsilon_0 r^2}\right)^2\cdot 4\pi\, r^2\cdot dr \;\Rightarrow$$

$$dW_E = \frac{e^2}{8\pi\varepsilon_0 r^2}\cdot dr$$

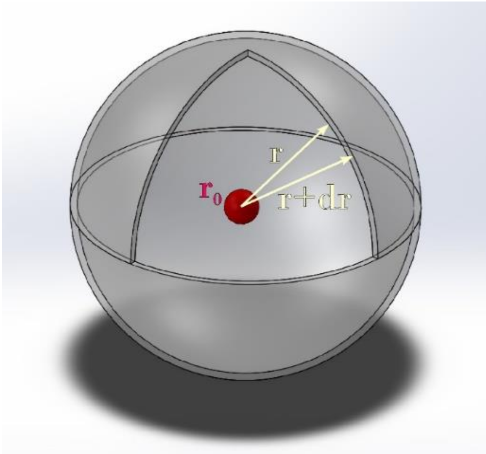


Fig. 71. Deducing the energy stored in the space around an electron.

It follows that the total energy stored in the electric field surrounding an electron can be found by integrating the latter equation from the radius of the electron (**r_e = R**) to infinity:

$$W_E = \frac{e^2}{8\pi\varepsilon_0}\int_R^{\infty}\frac{dr}{r^2} = \frac{e^2}{8\pi\varepsilon_0\cdot R}\,.$$

The relativistic rest energy of the electron is $W = m_e\cdot c^2$, and if this is equal to the energy stored in its electrostatic field, then we have:

$$W = W_E \;\Leftrightarrow\; m_e\cdot c^2 = \frac{e^2}{8\,\pi\,\varepsilon_0\cdot R}$$

Therefore, one deduces that for the electron, **R = r_e**, and using that $\frac{1}{4\pi\varepsilon_0} \cong 9\times 10^9$, the radius of the electron results as:

$$r_e = \frac{e^2}{8\pi\varepsilon_0\cdot m_0\cdot c^2} \cong 9\times 10^9\cdot\frac{(1.6\times 10^{-19})^2}{2\,\times 9.1\times 10^{-31}\cdot(3\times 10^8)^2}\;(m)$$

$$r_e \cong 1.4\,\times 10^{-15}\,m = 1.4\,\times 10^{-5}\,Å$$

b) If the electric charge is distributed uniformly within a spherical volume of radius R, the total energy stored in the electric field is the sum between that stored inside the sphere (W_E^{int}) and that stored in the field outside the sphere (W_E^{ext}):

$$W_E = W_E^{int} + W_E^{ext}.$$

The elementary energy within the spherical element of volume, inside the distribution of charge (r<R) is given by the product of the energy density and the element of volume **dV**:

$$dW_E^{int} = w_E\cdot dV = \frac{1}{2}\cdot\varepsilon_0\cdot E_{int}^2(r)\cdot 4\pi\, r^2\cdot dr = \frac{1}{2}\cdot\varepsilon_0\cdot\left(\frac{Q\cdot r}{4\pi\varepsilon_0\cdot R^3}\right)^2\cdot 4\pi r^2\cdot dr$$

$$dW_E^{int} = \frac{Q^2}{8\pi\varepsilon_0\cdot R^6}\cdot r^4\cdot dr$$

The energy inside the spherical distribution of charge is found by integration:

$$W_E^{int} = \int_0^R \frac{Q^2}{8\pi\varepsilon_0 \cdot R^6} \cdot r^4 \cdot dr = \frac{Q^2}{8\pi\varepsilon_0 \cdot R^6} \cdot \int_0^R r^4 \cdot dr \Rightarrow$$

$$W_E^{int} = \frac{Q^2}{40 \cdot \pi\, \varepsilon_0 \cdot R}$$

Similarly, for the space outside the sphere of radius **R** and of charge **Q**, the elementary energy $\mathbf{dW_E^{ext}}$ within the element of volume **dV** is:

$$dW_E^{ext} = w_E \cdot dV = \frac{1}{2} \cdot \varepsilon_0 \cdot E_{ext}^2(r) \cdot dV = \frac{1}{2} \cdot \varepsilon_0 \cdot \left(\frac{Q}{4\pi\varepsilon_0 \cdot r^2}\right)^2 \cdot 4\pi r^2 \cdot dr$$

$$dW_E^{ext} = \frac{Q^2}{8\pi\varepsilon_0} \cdot \frac{dr}{r^2}$$

The total energy within the space outside the sphere of charge is then:

$$W_E^{ext} = \frac{Q^2}{8\pi\varepsilon_0} \int_R^{\infty} \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0 \cdot R}$$

And the total energy **W** stored within the electric field, both inside and outside the spherical distribution of charge is obtained by adding the two terms deduced above:

$$W = W_E^{int} + W_E^{ext} = \frac{Q^2}{40 \cdot \pi\, \varepsilon_0 \cdot R} + \frac{Q^2}{8\pi\varepsilon_0 \cdot R} = \frac{Q^2}{8\pi\varepsilon_0 \cdot R}\left(\frac{1}{5} + 1\right)$$

$$\Rightarrow W = \frac{3}{5} \cdot \frac{Q^2}{4\pi\varepsilon_0\, R}$$

If the energy W of the electron electric field equals its relativistic rest energy ($W = m_e \cdot c^2$) and if the electron (of charge Q = e) is spherical with its charge uniformly distributed within that sphere, then its radius ($R = r_e$) can be deduced as:

$$r_e = \frac{3}{5} \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{m_e \cdot c^2} \cong \frac{3}{5} \times 2.8 \times 10^{-15}\ (m) = 1.68\ \times 10^{-15}\ (m)$$

Note: In the modern scientific understanding the electron is considered a point charge with no actual spatial volume. Though, for atomic scale phenomena, it is useful to define its size, to characterize the electron interactions with electromagnetic radiation and with other subatomic charged particles. The classical electron radius can be found by considering the energy required to assemble the charge quantity '**e**' into a sphere of a radius $\mathbf{r_e}$. This radius links the classical electrostatic (self-contained) energy of a charge distribution with the relativistic mass-energy of the electron, or else $\mathbf{W_E = m_e c^2}$. The numerical factor 3/5 results, as shown above, in the specific case of a uniform spherical charge density. If one ignores this particular coefficient, the remaining represents the 'classical' electron radius (also called Lorentz radius):

$$r_e = \frac{e^2}{4\pi\varepsilon_0\, m_0 c^2} \approx 2.8 \times 10^{-15}\, m\,.$$

Problem 105

Determine the capacity per unit length (C/L) of a coaxial cable (i.e. two coaxial cylinders, or radii R_1 and R_2, as shown in Figure 72).

Solution

Gauss's law for a closed cylindrical surface, of radius

$r \in (R_1, R_2)$:

$$\oiint_{\Sigma_{Gauss}} \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}.$$

The electric field is parallel to the circular bases of the cylinder, and there the electric flux is null. The electric flux generated by the inner cylinder of charge is different from zero only across the cylindrical area. Then Gauss's law becomes:

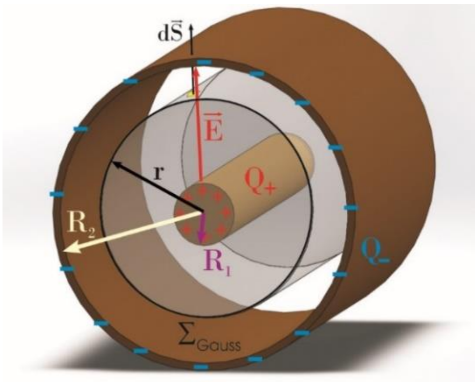


Fig. 72. A short segment of a coaxial cable.

$$E(r) \iint_{Cylinder} dS = \frac{Q}{\varepsilon_0} \Leftrightarrow E(r)\cdot 2\pi r\cdot L = \frac{\lambda\, L}{\varepsilon_0} \Leftrightarrow E(r) = \frac{\lambda}{2\pi\varepsilon_0\, r}.$$

The potential difference between the two charged cylinders, by integration between R_1 and R_2:

$$\Delta V = \int_{R_1}^{R_2} E(r)\, dr = \frac{\lambda}{2\pi\, \varepsilon_0} \int_{R_1}^{R_2} \frac{dr}{r} = \frac{\lambda}{2\pi\, \varepsilon_0} \ln\left(\frac{R_2}{R_1}\right).$$

The capacitance of the coaxial cable over length L:

$$C = \frac{Q}{\Delta V} = \frac{\lambda\, L}{\frac{\lambda}{2\pi\, \varepsilon_0} \ln\left(\frac{R_2}{R_1}\right)} = \frac{2\pi\varepsilon_0\, L}{\ln\left(\frac{R_2}{R_1}\right)}.$$

Then from the above equation the capacitance per unit length becomes:

$$\frac{C}{L} = \frac{2\pi\, \varepsilon_0}{\ln\left(\frac{R_2}{R_1}\right)}.$$

Problem 106

Charging up a capacitor of capacitance **C** through a resistor **R** (Figure 73-a).

a) Find the charge on the capacitor plates and the intensity of the charging electric current. Represent the graphs of **Q**(t) and of **I**(t).

b) Demonstrate the product (CR) has the unit of time in SI.

Solution

a) When closing the switch K, the initial intensity of the charging current is maximum, I_0, and the initial charge on the capacitor plates is $Q_0 = 0$. After a while: $I = 0$, and $Q = Q_{Max}$.

The equation of the energy balance for the circuit:

$$\mathcal{E}\cdot I\cdot t = I^2\cdot t + \frac{q}{C} I\cdot t \;\Big|\times\frac{1}{I\cdot t} \Rightarrow \mathcal{E} = I\,R + \frac{q}{C}\,;$$

$$\mathcal{E} = R\,\frac{dq}{dt} + \frac{q}{C}.$$

The above result is true at any moment t, when the charge on the capacitor is $\mathbf{q} \in (0, Q_{Max})$.

Regrouping the terms leads to:

$$R\,\frac{dq}{dt} = \mathcal{E} - \frac{q}{C}\,.$$

By separating the variables, one can deduce:

$$\frac{dq}{\mathcal{E} - \frac{q}{C}} = \frac{dt}{R} \quad \Big|\times\frac{1}{C} \quad \Rightarrow$$

$$\frac{dq}{q - C\cdot\mathcal{E}} = -\frac{dt}{R\cdot C}$$

$$\Leftrightarrow \int_0^Q \frac{dq}{q - C\cdot\mathcal{E}} = -\int_0^t \frac{dt'}{R\cdot C}\,,$$

$$\ln(q - C\cdot\mathcal{E})\Big|_0^Q = -\frac{t'}{R\cdot C}\Big|_0^t\,,$$

$$\ln(Q - C\cdot\mathcal{E}) - \ln(-C\cdot\mathcal{E}) = -\frac{t}{R\cdot C}\,,$$

$$\ln\frac{Q - C\cdot\mathcal{E}}{-C\cdot\mathcal{E}} = -\frac{t}{R\,C}\,.$$

$$e^{-\frac{t}{R\cdot C}} = \frac{Q - C\cdot\mathcal{E}}{-C\cdot\mathcal{E}} \Rightarrow$$

$$Q - C\cdot\mathcal{E} = C\cdot\mathcal{E}\,e^{-\frac{t}{C\cdot R}}\,,$$

$$Q(t) = C\cdot\mathcal{E}\left(1 - e^{-\frac{t}{C\cdot R}}\right) \Rightarrow$$

$$Q(t) = Q_M\left(1 - e^{-\frac{t}{C\cdot R}}\right).$$

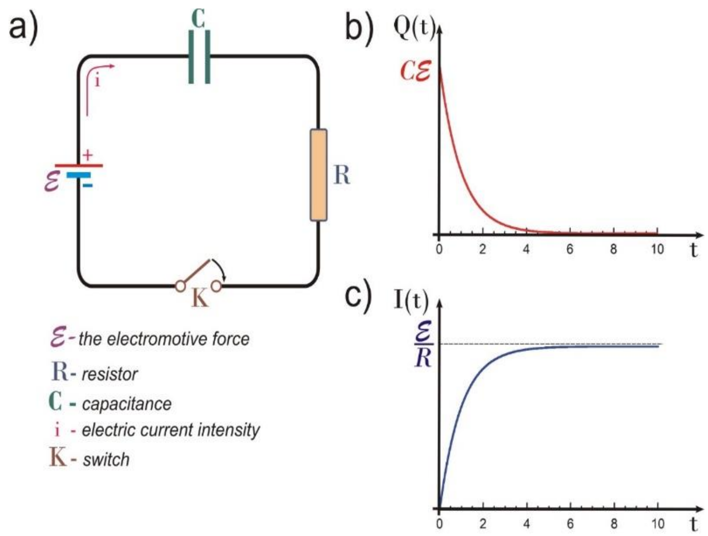


Fig. 73. Charging up a capacitor across a resistor R. a) The electric circuit; b) the electric charge on the capacitor versus time; c) The intensity of the electric current versus time.

The intensity of the electric current while charging the capacitor:

$$I = \frac{dQ}{dt} = -C{\cdot}\mathcal{E}\,\frac{-1}{R{\cdot}C}\,e^{-\frac{t}{C{\cdot}R}}$$

$$I(t) = \frac{\mathcal{E}}{R}\,e^{-\frac{t}{C{\cdot}R}}.$$

In the equations above, the product $CR = \tau$ represents the time constant (relaxation time) of the electric circuit of capacitance C and resistance R.

When t = RC, then: $I = I_0\,e^{-1} = \frac{1}{e}I_0$.

The graphs in Figures 73-b and 73-c show the dependence of Q and of $\mathbf{I}$ versus time, as described by the equations deduced above.

b) The SI unit of the product (C R):

$$[C{\cdot}R]_{SI} = [C]_{SI}\,[R]_{SI} = 1\ F{\cdot}1\Omega$$

$$[C{\cdot}R]_{SI} = \frac{1C}{1V}{\cdot}\frac{1V}{1A} = \frac{1A{\cdot}1s}{1A}$$

$$\Rightarrow\ [C{\cdot}R]_{SI} = 1\ s = [\text{time}].$$

Problem 107

A capacitor of capacitance C = 6 μF is charged from a source of electromotive ‘force’ $\mathcal{E} = 500\ V$ through a resistor of R = 5 kΩ. How long will it take for the capacitor to acquire 99 % of its final charge Q?

Solution

The relaxation time of the circuit, τ (see Problem 106) is:

$$\tau = C{\cdot}R = 6\times10^{-6}F{\cdot}5000\ \Omega \Longrightarrow \tau = 3\times10^{-2}s\,.$$

The electric charge on the capacitor plates is given by (see Problem 106): $Q(t) = Q_M\left(1 - e^{-\frac{t}{\tau}}\right)$.

If Q(t) = 99 % of Q_M, then the latter equation becomes:

$$\frac{99}{100}\,Q_M = Q_M\left(1 - e^{-\frac{t}{\tau}}\right) \Longrightarrow$$

$$e^{-\frac{t}{\tau}} = 1 - 0.99\ \Leftrightarrow\ \frac{t}{\tau} = -\ln(0.01) \Rightarrow t \cong 0.138\ s.$$

Problem 108

Determine what is the energy stored in a sphere of radius **R** in a medium with electrical permittivity **ε** , if it is charged with the electric charge **Q** distributed uniformly over its surface.

Solution

If in the space surrounding the sphere of charge, the electric permittivity is ε, and the electric field the has the intensity **E**, then the density of the electric energy in this field is:

$$\mathrm{w}_E\ \frac{1}{2}\varepsilon\cdot\mathrm{E}^2.$$

The elementary amount of energy **dW$_E$** stored within the element of volume **dV** is:

$$\mathrm{dW}_E=\mathrm{w}_E\,\mathrm{dV}=\frac{1}{2}\varepsilon\,\mathrm{E}^2\,\mathrm{dV}.$$

With the spherical element of volume $\mathrm{dV}=4\pi\mathrm{r}^2\cdot\mathrm{dr}$, the elementary energy becomes:

$$\mathrm{dW}_E=\frac{1}{2}\varepsilon\left(\frac{\mathrm{Q}}{4\pi\varepsilon\,\mathrm{r}^2}\right)^2 4\pi\,\mathrm{r}^2\,\mathrm{dr}=\frac{\mathrm{Q}^2}{8\pi\varepsilon\,\mathrm{r}^2}\,\mathrm{dr}.$$

Then in the space around the sphere of radius **R** with the charge **Q** over is surface, the total energy stored in its electric field can be found by integrating the elementary energy from the sphere surface to infinity:

$$\mathrm{W}_E=\int_{\mathrm{R}}^{\infty}\mathrm{dW}=\frac{\mathrm{Q}^2}{8\pi\varepsilon}\int_{\mathrm{R}}^{\infty}\frac{\mathrm{dr}}{\mathrm{r}^2}\Rightarrow\ \mathrm{W}_E=\frac{\mathrm{Q}^2}{8\pi\varepsilon\,\mathrm{R}}\ .$$

Problem 109

Use the law of Biot-Savart (named after *Jean-Baptiste Biot*, and *Felix Savart*) to determine the magnetic induction of the magnetic field $\vec{B}$ in a medium of magnetic permittivity μ, at distance **R** from a linear conductor carrying the electric current of intensity $\mathbf{I}$.

Solution

The elementary magnetic induction:

$$d\vec{B} = \frac{\mu}{4\pi}\, I\, \frac{\vec{r} \times d\vec{\ell}}{r^3}.$$

With its absolute value:

$$dB = \frac{\mu}{4\pi}\, I\, \frac{\sin(\alpha)\ r\, d\ell}{r^3}.$$

$$dB = \frac{\mu}{4\pi}\, I \frac{R\, d\ell}{r^3} \quad \Rightarrow$$

$$dB = \frac{\mu}{4\pi}\, I \frac{R\, d\ell}{(R^2 + x^2)^{\frac{3}{2}}}.$$

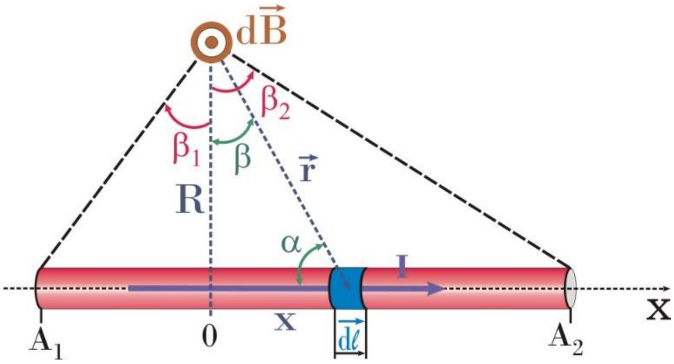


Fig. 74. The magnetic induction $\vec{B}$ of the magnetic field generated by an electric current of intensity I along an infinite conductor.

From the Figure 74, we have:

$x = R\ \tan\beta \quad \Rightarrow \quad$ then the elementary segment along the conductor is: $d\ell = dx = \frac{R}{\cos^2(\beta)}\, d\beta$.

Therefore, the elementary magnetic induction becomes:

$$dB = \frac{\mu}{4\pi}\, I\, \frac{R}{[R^2 + R^2 \tan^2(\beta)]^{\frac{3}{2}}} \frac{R}{\cos^2(\beta)}\, d\beta \Leftrightarrow dB = \frac{\mu}{4\pi}\, I\, \frac{R^2}{\{R^2\, [1 + \tan^2(\beta)]\}^{\frac{3}{2}}} \frac{d\beta}{\cos^2(\beta)}\ ;$$

$$dB = \frac{\mu}{4\pi}\, I\, \frac{R^2}{R^3} \frac{\cos^3(\beta)}{\cos^2(\beta)}\, d\beta \ \Rightarrow\ dB = \mu \frac{I}{4\pi} \frac{\cos(\beta)}{R}\, d\beta.$$

Then by integrating, one finds the total magnetic induction:

$$B = \mu\, \frac{I}{4\pi} \int_{\beta_1}^{\beta_2} \cos(\beta)\, d\beta = \mu\, \frac{I}{4\pi R} [\sin(\beta_2) - \sin(\beta_1)].$$

For an infinite line of current, $\beta_1 \to -\frac{\pi}{2}$ and $\beta_2 \to \frac{\pi}{2}$, and thus: $\sin(\beta_2) \to 1$, and $\sin(\beta_1) \to -1$.

Thus, the magnetic induction is: $B = \mu \frac{I}{2\pi R}$. If the conductor is in air, the magnetic induction is:

$$B_{air} = \mu_0 \frac{I}{2\pi R}$$

Problem 110

Consider the electric current of intensity I circulating along a conducting ring of radius **R**. Using the law of Biot-Savart, determine the magnetic induction $\vec{B}$ of the magnetic field that is thus generated: (a) at point A on the axis perpendicular to the plane of the ring, at distance **h** from the ring centre, and (b) in the centre O of the conducting ring (Figure 75). Assume that the medium around the ring has the magnetic permittivity μ. If this is air, then the permittivity is μ_0, equal to that of free space.

Solution

(a) Let us consider the elementary magnetic induction $d\vec{B}$ (see Figure 75) produced at point A, by an element of length $d\vec{\ell}$ of the ring of current of intensity I, at distance $\vec{r}$, on its axis. The absolute value of the elementary magnetic induction is given by:

$$dB = \frac{\mu I}{4\pi}\frac{d\ell}{r^2}.$$

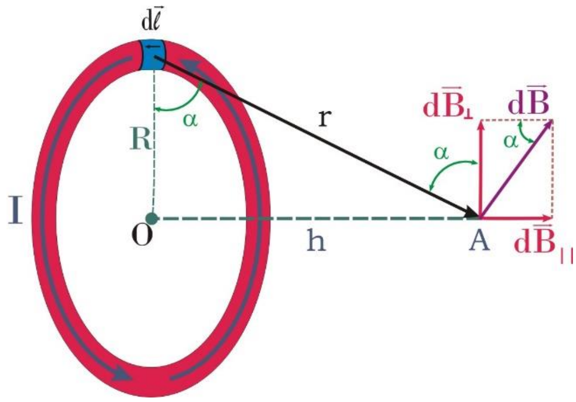


Figure 75: Calculating the magnetic induction B(A) at point A on the ring axis, and B(O) in the centre of a ring of current of intensity I.

This vector can be written as the resultant of two perpendicular components:

- one component perpendicular to the ring axis,
- one component along the axis:

$$d\vec{B} = d\vec{B}_\perp + d\vec{B}_\parallel.$$

The absolute value of the component $d\vec{B}_\parallel$ along the axis of symmetry of the ring can be written: $dB_\parallel = dB\cdot\cos(\propto) = dB\cdot\frac{R}{r} = \frac{\mu I}{4\pi}\cdot\frac{d\ell}{r^2}\cdot\frac{R}{r} \Longrightarrow dB_\parallel = \frac{\mu\,(Id\ell)\,R}{4\pi\,r^3}$, where $(Id\ell)$ is the so called element of current.

To determine the total magnetic induction generated by the ring of current, one integrates $d\vec{B}_\parallel$ along the whole circle of radius R, as well as $d\vec{B}_\perp$. Then one adds up the results.

For the total magnetic induction component perpendicular to the ring axis, $\vec{B}_\perp$, one integrates $d\vec{B}_\perp$ all along the ring. Note that all the perpendicular components of the magnetic induction produced by pairs of diametrically opposed elements of current will cancel one another, because they point in opposite directions to each other. Thus, the total magnetic induction only remains equal to the sum of the components directed along the axis, as they will not cancel, because they all point to the same direction, adding up. The magnetic induction B(A) at point A can then be found by integrating $d\,B_\parallel$ along the ring of radius R:

$$B\,(A) = \oint dB_\parallel = \oint \frac{\mu}{4\pi}\cdot\frac{I}{2R}\cdot\frac{R}{r}\cdot\frac{d\ell}{r^2} = \frac{\mu}{4\pi}\cdot\frac{I\,R}{r^3}\oint d\ell = \frac{\mu}{4\pi}\cdot\frac{I\,R}{r^3}\cdot 2\pi R \Rightarrow$$

$$B(A) = \mu \frac{I\,R^2}{2\,r^3} = \mu \frac{I\,R^2}{2\,(R^2 + h^2)^{\frac{3}{2}}}$$

(b) For the centre of the ring, O, the distance ***r*** becomes equal to the radius of the ring **r = R**, h=0, and then one cand educe that the magnetic induction at the centre of the ring is given by:

$$B(O) = \mu \frac{I}{2R}.$$

If the medium around the ring is air, then the magnetic induction at the centre of the ring is:

$$B(O) = \mu_0 \cdot \frac{I}{2R}.$$

Problem 111

Two parallel metal conductors are in air, separated at 20 cm distance between them (Figure 76) and they carry two electric currents directed in the same direction, I_1 = 5 A, and I_2 = 10 A, respectively.

a) Calculate the force of attraction per unit length between the two conductors.

b) Determine the work per unit length required to pull the conductors apart at 30 cm distance.

Solution

a) The force of attraction between the two conductors of length ℓ, with electric currents flowing in the same direction:

$$F = \mu \frac{I_1\,I_2}{2\pi\,d}\,\ell \Rightarrow \frac{F}{\ell} = \mu \frac{I_1\,I_2}{2\pi\,d},$$

$$\frac{F}{\ell} = 4\pi \times 10^{-7} \frac{50}{2\pi\,20 \times 10^{-2}}\,N = 5 \times 10^{-5}\,N.$$

b) The work per unit length for moving the conductors at 30 cm distance apart:

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \mu \frac{I_1\,I_2\,\ell}{2\pi\,x}\,dx.$$

Then the work per unit length becomes:

$$\frac{W}{\ell} = \mu \frac{I_1\,I_2}{2\pi} \int_{x_1}^{x_2} \frac{dx}{x} = \mu \frac{I_1\,I_2}{2\pi} \ln \frac{x_2}{x_1} \Rightarrow$$

$$\frac{W}{\ell} = 4.05 \times 10^{-6} \frac{J}{m}.$$

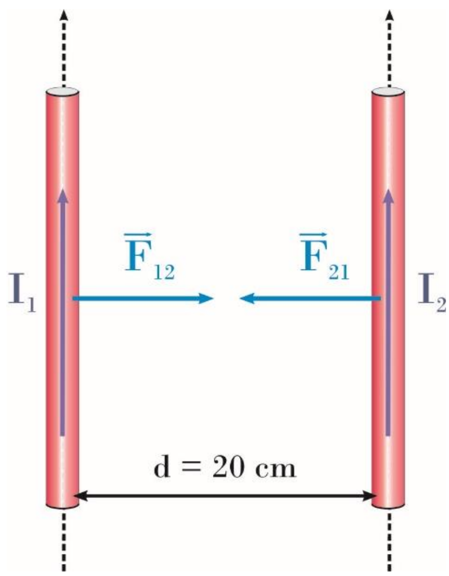


Figure 76. The force between two parallel conductors.

Problem 112

Calculate the mutual induction $\mathbf{M_{21}}$ for two concentric loops of radii $\mathbf{R_1}$ and $\mathbf{R_2}$. The outer loop of radius $\mathbf{R_1}$ (Figure 78) carries an electric current of intensity $\mathbf{I}_1$.

Solution

If M_{21} is the mutual induction between the two loops, then the electromotive force induced by a variable current intensity I_1, in a loop of radius R_2 is:

$$e_{12} = -M_{21}\frac{dI_1}{dt}.$$

The magnetic flux that crosses the area enclosed by the inner loop of radius R_2 is:

$$\phi_{21} = S_2 B_1 = \pi R_2^2\,\mu_0 \frac{I_1}{2R_1}.$$

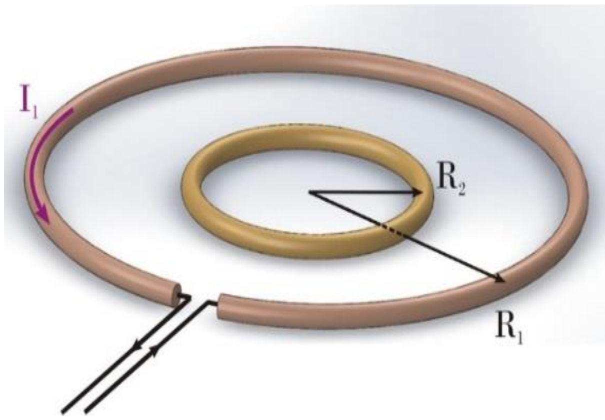


Fig. 77. Two concentric conducting loops.

The electromotive force is also equal to the derivative in time of the magnetic flux:

$$e_{12} = -\frac{d\phi_{21}}{dt} = -\pi\,\frac{R_2^2}{2R_1}\,\mu_0\,\frac{dI_1}{dt} = -M_{21}\frac{dI}{dt}.$$

It follows then that the mutual induction of the two loops is:

$$M_{21} = \pi\mu_0\frac{R_2^2}{2R_1}.$$

Problem 113

A circular coil of conducting wire carries 150 mA of electric current. The coil has 50 loops, each of an area of 2 cm^2. A magnetic field of magnetic induction B_{ext} = 0.3 T is applied parallel to the plane of the coil. Calculate what is the torque that this magnetic field exerts on the coil?

Solution

The magnetic dipole of the whole loop of current:

$$\vec{\mu} = N{\cdot}I{\cdot}\vec{S} = N{\cdot}I{\cdot}A{\cdot}\vec{n}.$$

Where A is the area of the coil, and $\vec{n}$ is the direction normal to this area.

The torque experienced by the loop of current in the external magnetic field is then: $\vec{\tau} = \vec{\mu}\times\vec{B}_{ext}$.

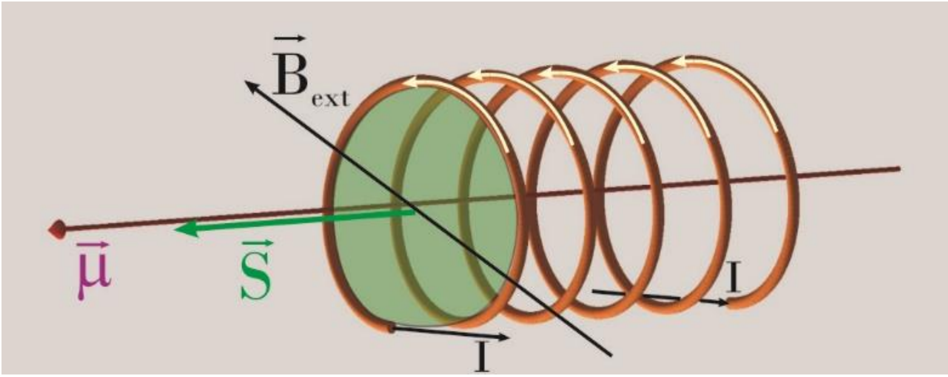


Fig. 78. A coil of current

The absolute value of the torque results as:

$$\tau = \mathrm{N{\cdot}I{\cdot}A{\cdot}B_{ext}{\cdot}\sin(90^o)} \implies$$

$$\tau = 50\ (150 \times 10^{-3}\mathrm{A}){\cdot}(2 \times 10^{-4}\mathrm{m}^2)\ 0.3\ \mathrm{T}{\cdot}1 \implies \tau = 4.5\ \times 10^{-4}\ \mathrm{N \cdot m}.$$

Problem 114

A circular coil of radius 4 cm and 200 turns carries a current of 1.2 A. A magnetic field of 0.8 T is applied perpendicular to the plane of the coil, so that it points to the same direction as the magnetic dipole $\vec{\mu}$ of the coil (Figure 79-a). How much work is required to turn the coil by 180° (Figure 79-b)?

Solution

The mechanical work required to rotate the magnetic dipole equals the difference between the energies that correspond to the two states of the dipole, antiparallel (b) and parallel (a) to the external magnetic field:

$$\mathrm{W} = \mathrm{E}_2 - \mathrm{E}_1 = -\vec{\mu_2}{\cdot}\vec{\mathrm{B}}_{ext} - (-\vec{\mu_1}{\cdot}\vec{\mathrm{B}}_{ext});$$

$$\mathrm{W} = 2\ \mu\ \mathrm{B}_{ext} = 2\ \mathrm{N}{\cdot}I{\cdot}\mathrm{A}\ \mathrm{B}_{ext}.$$

Where, in absolute values:

$$\mu_1 = \mu_2 = \mu.$$

$$\mathrm{W} = 2{\cdot}200{\cdot}1.2{\cdot}\pi{\cdot}4^2 \times 10^{-4}{\cdot}0.8\ (J) \implies$$

$$\mathrm{W} \cong 1.92\ \mathrm{J}.$$

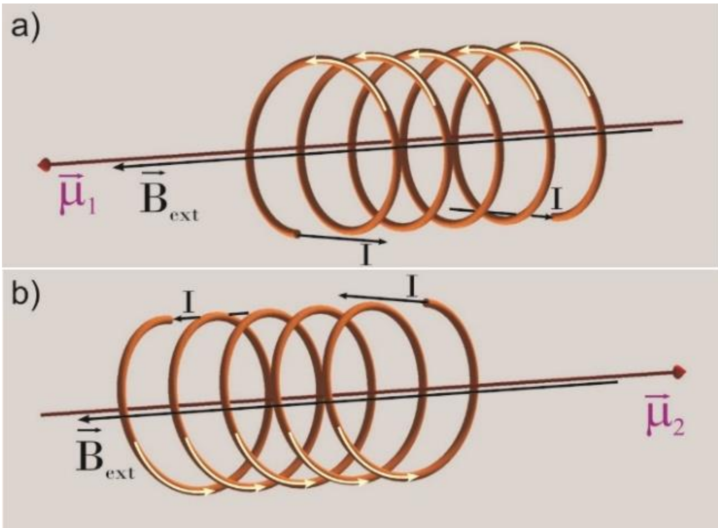


Fig. 79. A coil of current of magnetic dipole $\vec{\mu}$ in an external magnetic field $\vec{\mathrm{B}}_{ext}$.

Problem 115

Use Ampère's law to deduce the magnetic induction of the magnetic field inside and outside a long straight cylinder of radius **R** carrying a current of intensity **I** uniformly distributed over the cross section of the conductor.

Solution

If r > R (for points outside the conductor):

$$\oint_{\Gamma_e} \vec{B}\cdot d\vec{\ell} = \mu_0\, I \Leftrightarrow B\, 2\pi r = \mu_0\, I.$$

Therefore, the magnetic induction outside the conductor that carries the current of intensity **I**, at distance **r** from the axis of the conductor, becomes:

$$B(r) = \mu_0 \frac{I}{2\pi r}.$$

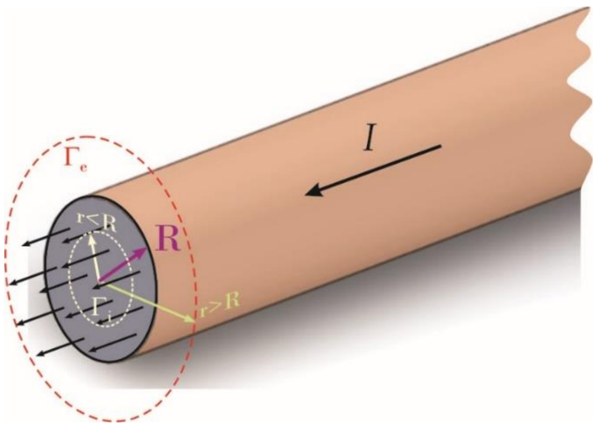


Fig. 80. The magnetic field inside and around a straight conductor carrying current of intensity I.

If r < R (for points situated inside the metallic conductor):

$$\oint_{\Gamma_i} \vec{B}\cdot d\vec{\ell} = \mu_0 \iint_{\Sigma_\Gamma} \vec{j}\cdot d\vec{S}.$$

Where: Σ_Γ = the cross section area (circle of radius **r**) that the current is transmitted through, and the curve Γ is a the closed contour around the cross surface Σ_{Γ}. (Figure 80).

The left side member of the equation can be written:

$$\oint_{\Gamma} \vec{B}\cdot d\vec{\ell} = B\cdot 2\pi r.$$

The integral in the right side member of the equation can also be written:

$$\iint_{\Sigma_\Gamma} \vec{j}\cdot d\vec{S} = \int_0^r \frac{I}{\pi R^2}\, 2\pi r' dr' = \frac{2I}{R^2}\int_0^r r' dr' = \frac{I r^2}{R^2}.$$

Therefore:

$$\oint_{\Gamma} \vec{B}\cdot d\vec{\ell} = \mu_0 \iint_{\Sigma_\Gamma} \vec{j}\cdot d\vec{S} \qquad \Leftrightarrow \qquad B\cdot 2\pi r = \mu_0 \frac{I\cdot r^2}{R^2}.$$

And thus, the magnetic induction inside a metallic conductor carrying the current of intensity **I** is:

$$B(r) = \frac{\mu_0 I\cdot r}{2\pi R^2}\ .$$

Problem 116

What is the magnitude of the magnetic induction **B** of the magnetic field inside a long solenoid that has **n** turns per unit length and carries a current of intensity $\mathbf{I}$?

Solution

Let us consider the contour $(\Gamma) = (abcd)$ in Figure 81, drawn around a 'segment' of length L = ad of the solenoid. Then one can write:

$$\oint_{\Gamma} \vec{B} \cdot d\vec{\ell} = \mu_0 n \cdot L \cdot I \iff$$

$$B \cdot L = \mu_0 N \cdot I.$$

And, therefore, the magnetic induction B in the solenoid volume is:

$$B = \mu_0 \frac{N \cdot I}{L}.$$

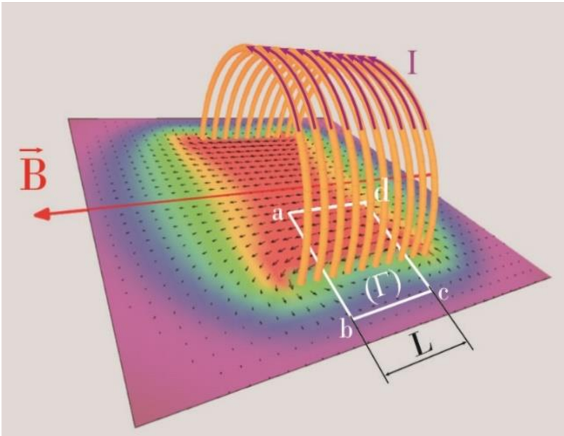


Fig. 81. The magnetic field map inside a solenoid.

Problem 117

What is the energy stored in an air core solenoid of length 10 cm and diameter 1.2 cm, if it has in total N = 200 turns and it carries a current of intensity $\mathbf{I}$ = 1.2 A?

Solution

The energy stored in the magnetic field: $W_B = \frac{1}{2} L \cdot I^2$.

Where **L** is the inductance of the solenoid: $L = \mu_0 \cdot \mu_r \frac{N^2 R^2}{\ell}$.

Then the energy stored in the magnetic field of the solenoid is deduced as (for air $\boldsymbol{\mu_r} = \mathbf{1}$):

$$W_B = \frac{1}{2} \mu_0 \frac{N^2 \cdot R^2}{\ell} I^2 \implies W_B \cong 4.1 \times 10^{-4} \text{ J}.$$

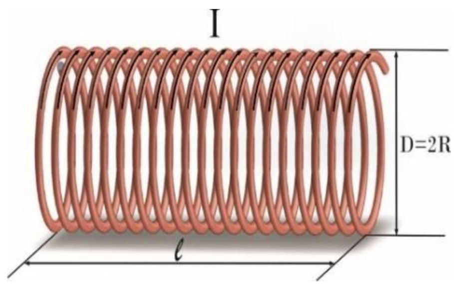


Fig. 82. An air core solenoid of radius R and length l.

Problem 118

A long straight wire is carrying the current of intensity $\mathbf{I}$.
a) Calculate the magnetic field induction **B** at the centre O when the conductor is bent 90° in a circular arc of radius $\mathbf{R_o}$ (Figure 83 - a)?

b) What is the resulting magnetic field induction at point O in the centre of the semicircle of radius **R₀** if the current carrying wire is bent at 180^0, as shown in Figure 83 - b?

Solution

a) The resultant magnetic induction is composed of the magnetic inductions of the magnetic field generated by the parts labelled (1), (2), and (3) in Figure 83-a:

$$\vec{B}_0 = \vec{B}_1 + \vec{B}_2 + \vec{B}_3.$$

All three vectors point into the plane of the figure, and thus: (resulting from the law of Biot-Savart applied to the tree segments of current indicated in the Figure 83):

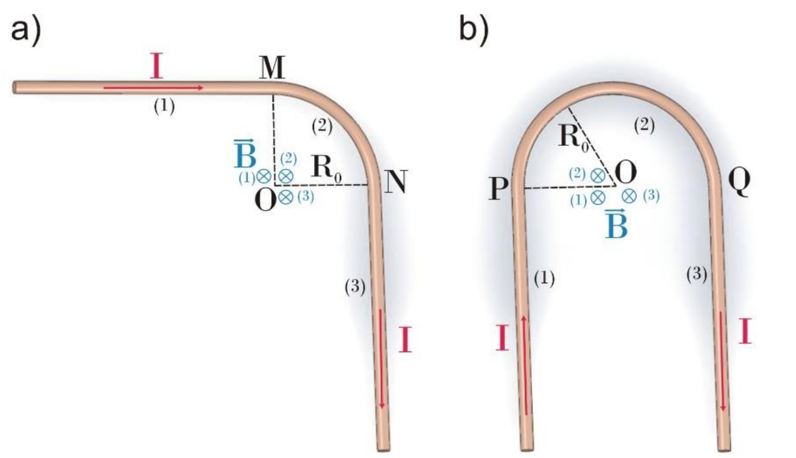


Fig. 83. A current carrying wire being bent at a) 90°, and b) 180°.

$$B_0 = \frac{1}{2}\mu_0 \frac{I}{2\pi\, R_0} + \frac{1}{4}\mu_0 \frac{I}{2R_0} + \frac{1}{2}\,\mu_0 \frac{I}{2\pi R_0} \Longrightarrow B_0 \approx 0.28\ \mu_0 \frac{I}{R_0}.$$

b) In Figure 83-b, the total magnetic field produced at point O is composed by the superposition of the magnetic fields generated by the segment of conductors labelled (1), (2), and (3):

$$\vec{B}_0 = \vec{B}_1 + \vec{B}_2 + \vec{B}_3.$$

All the three vectors are directed into the plane of the figure, and therefore:

$$B_O = \frac{1}{2}\,\mu_0\,\frac{I}{2\pi R_0} + \frac{1}{2}\,\mu_0 \frac{I}{2R_0} + \frac{1}{2}\,\mu_0\,\frac{I}{2\pi R_0}.$$

$$B_O = \mu_0 \frac{I}{2\pi R_0} + \frac{1}{2}\mu_0 \frac{I}{2R_0} = \mu_0 \frac{I}{2R_0}\left(\frac{1}{\pi} + \frac{1}{2}\right) \Rightarrow$$

$$B_0 \cong 0.41\mu_0 \frac{I}{R_0}.$$

Problem 119

A cyclotron accelerator has the oscillator frequency of 12 x 10^6 Hz, and the so called ‘dees’ (Ds, being two halves of a disk, shaped in the form of letter ‘D’) have a radius R = 53.34 cm.

a) Determine the magnetic induction necessary for accelerating the deuterium nuclei, carrying positive charges equal to that of an electron. Take in consideration that the mass of a deuterium nucleus is m = 3.3 x 10^{-27} kg.

b) What is the energy of the deuterium nuclei?

c) What is the period of rotation of the accelerated particles?

Solution

a) $\frac{mv^2}{R} = q \cdot v \cdot B \quad \Rightarrow \quad B = \frac{mv}{R \cdot q} = m\frac{2\pi R}{T}\frac{1}{R \cdot q} = \frac{2\pi\nu \cdot m}{q}$.

The electric charge **q** is equal to that of an electron, and thus the magnetic induction results as:

$$B = \frac{2\pi\nu \cdot m}{e} = \frac{2\pi \cdot 12 \times 10^6 \cdot 3.3 \times 10^{-27}}{1.6 \times 10^{-19}} = \frac{2\pi \cdot 12 \cdot 3.3}{1.6} 10^{-2} \Longrightarrow B = 1.555\ \text{T}.$$

b) The kinetic energy $\mathbf{K}$ of the accelerated deuterium nucleus:

$$K = \frac{m\,v^2}{2} = \frac{m\,q^2B^2R^2}{2\,m^2} = \frac{q^2B^2R^2}{2\,m} \Longrightarrow$$
$$K = 2.67 \times 10^{-2}\ \text{J} = 16.7\ \text{MeV}\,.$$

Where: $1\ \text{MeV} = 10^6\ \text{eV} = 10^6 \times 1.6 \times 10^{-19}\ \text{J} = 1.6 \times 10^{-13}\ \text{J}.$

c) The period of rotation is then:

$$T = \frac{1}{\nu} \Longrightarrow T = 0.833\ \times 10^{-7}\text{s} = 0.0833\ \mu\text{s}.$$

9.3 Electromagnetic fields and waves

Problem 120

An electric voltage given by $V_0 = V_m \sin \omega t$ is applied between the ends of a transmission line, without electrical resistance.

What is the equation of the electric voltage at point P, a distance $d = 1.5\,\lambda$ from the electric oscillator if the frequency of the alternative voltage is: $\nu = \frac{\omega}{2\pi} = 3\text{ GHz} = 3 \times 10^9\text{ s}^{-1}$?

Solution

The wave equation:

$$V(x,t) = V_m\ \sin(\omega t - k{\cdot}x).$$

Where: the distance $x = 1.5\,\lambda$ and $k = \frac{2\pi}{\lambda}$ is the wave number. Then the wave equation, at point P can be rewritten as:

$$V(P) = \ V_m\ \sin(\omega t - \frac{2\pi}{\lambda}{\cdot}1.5\,\lambda)\,;$$

$$V(P) = \ V_m\ \sin(\omega t - 3\pi) = \ -V_m\ \sin(\omega t).$$

Problem 121

A parallel plates capacitor (with circular plates, of radius **R**, separated at distance **d**) has air between its plates and is being charged up (Figure 84).

a) What is the magnetic field induced at r < R, and outside the capacitor, for r > R ?
b) Calculate the magnetic induction for r = R, if dE/dt = 10^{12} $(\mathrm{V/m{\cdot}s})$. Consider a radius R = 5 cm of the capacitor plates.
c) Show that while it is being charged the Poynting vector points everywhere radially into the volume between the capacitor plates.
d) Show that the rate at which energy flows into this volume is equal to the rate at which the electrostatic energy increases.
e) What is the displacement current in the dielectric of the capacitor?

Solution

a) The law of Maxwell-Ampère for the magneto-electric induction phenomenon states that:

$$\oint_\Gamma \vec{B}{\cdot}d\vec{\ell} = \mu_0\, I_c + \varepsilon_0\, \mu_0\, \frac{d\Phi_E}{dt},$$

Where: $\mathbf{I_c}$ is the conduction current – this is null between the capacitor plates, separated by an insulating medium(here, vacuum); $\boldsymbol{\varepsilon_0}$ is the electric permittivity of free space; $\boldsymbol{\mu_0}$ is the magnetic permeability of free space; Φ_E is the electric flux.

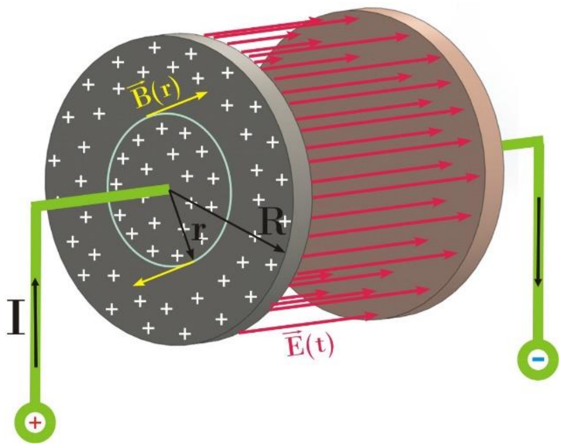


Fig. 84 A parallel plates capacitor is being charged.

<u>For r < R</u> (inside the volume of the capacitor) - see Figure 84:

$$\oint_{\Gamma} \vec{B}\cdot d\vec{\ell} = \varepsilon_0\mu_0 \frac{d}{dt}(E\cdot\pi\cdot r^2)$$

$$B\cdot 2\pi r = \varepsilon_0\mu_0\cdot\pi r^2 \frac{dE}{dt}.$$

Then the magnetic induction of the field inside the capacitor, while it is being charged up is:

$$B(r) = \frac{\varepsilon_0\mu_0\cdot r}{2}\cdot\frac{dE}{dt}.$$

<u>For r > R</u> (points situated outside the capacitor):

$$\oint_{\Gamma} \vec{B}\cdot d\vec{\ell} \Leftrightarrow B\cdot 2\pi r = \varepsilon_0\mu_0\cdot\pi R^2 \frac{dE}{dt}.$$

Then the magnetic induction of the magnetic field outside the capacitor volume becomes:

$$B(r) = \frac{\varepsilon_0\mu_0\cdot R^2}{2r}\frac{dE}{dt}.$$

At the frontier of the capacitor volume, where r=R, the magnetic induction is: $B(R) = \frac{\varepsilon_0\mu_0\cdot R}{2}\frac{dE}{dt}$.

b) The magnetic induction at r = R = 5 cm is given by:

$$B(R) = \frac{8.856\times 10^{-12}\cdot 4\pi\times 10^{-7}\cdot 5\times 10^{-2}}{2}10^{12}(T) = \frac{8.856\cdot 4\pi\cdot 5}{2}\times 10^{-9}\ (T),$$

$$B(R) \cong 0.278\ \mu T.$$

Note that this seems to be a small value, but then note that the magnetic induction of the terrestrial magnetic field at the Earth surface, at the equator is approximately:

$$B_{Eq} = 3.05\ \times 10^{-5}\ T = 30.5\times 10^{-6}\ T = 30.5\ \mu T.$$

c) The Poynting vector is given by the equation:

$$\vec{\wp} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}\vec{E}\times\vec{B}.$$

As shown in Figure 85, during the charging up of the capacitor, owing to the electric field between the parallel plates, E(t), and to the magnetic field thus generated, the

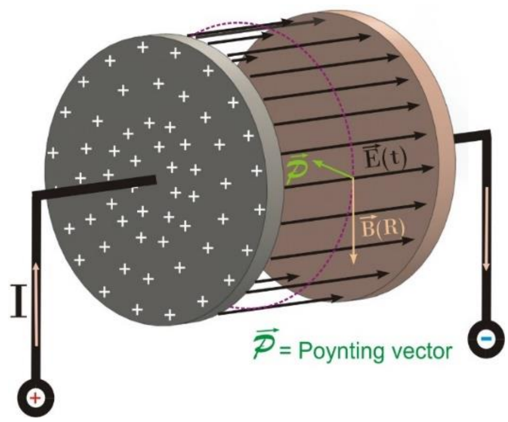


Fig. 85. The Poynting vector during the charging up of a parallel plates capacitor.

Poynting vector is directed towards the volume limited between the plates of the capacitor.

d) The rate dW/dt at which electromagnetic energy enters the volume between the capacitor plate (i.e., the power), is given by the equation:

$$P = \frac{dW}{dt} = \iint_S \vec{\wp} \cdot d\vec{S} = \iint_S \frac{E \cdot B}{\mu_0}\, d(2\pi R \cdot \ell)\,;$$

In the above equations **S** represent the lateral surface of the cylinder volume between the capacitor plates (Figure 86).

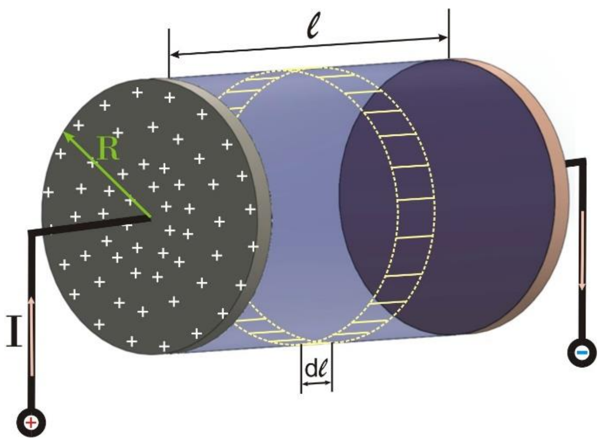


Fig. 86. Electromagnetic energy is entering the volume between the plates of the capacitor.

Thus, the power of the electromagnetic field, or the rate at which energy enters the volume between the plates becomes:

$$P = \frac{E \cdot B}{\mu_0} 2\pi R \int_0^{\ell} d\ell = \frac{E \cdot B}{\mu_0} 2\pi R \cdot \ell.$$

Considering that:

$$V = \pi R^2 \cdot \ell \qquad \textit{and} \qquad B(R) = \frac{\mu_0 \varepsilon_0 \cdot R}{2} \frac{dE}{dt},$$

the power transported by the electromagnetic field becomes:

$$P = \frac{E}{\mu_0} \frac{\mu_0 \varepsilon_0 R}{2} \frac{dE}{dt} 2\pi R \cdot \ell\,;$$

$$P = \pi R^2 \cdot \ell \frac{d}{dt}\left(\frac{\varepsilon_0 \cdot E^2}{2}\right) = V \frac{dw_e}{dt} = \frac{d(V \cdot w_e)}{dt} \Longrightarrow P = \frac{dW_e}{dt} = \frac{dW}{dt}.$$

e) The displacement current is given by the equation:

$$I_d = \varepsilon_0 \frac{d\Phi_e}{dt} = \varepsilon_0 \cdot \pi R^2 \frac{dE}{dt} \Leftrightarrow I_d = 8.856 \times 10^{-12} \frac{F}{m} \cdot 3.14 \cdot 25 \times 10^{-4} m^2 \times 10^{12} \frac{V}{m} \cdot s\,,$$

$$I_d \cong 69.5 \text{ mA}.$$

Problem 122

A long coaxial cable (radii **a** and **b**, in Figure. 87) carries a steady current of intensity **I**. Find the energy stored in the magnetic field of the cable, along a length **L**.

Solution

At radius **r** (a < r < b in Figure 87), one can write that: $\oint_\Gamma \vec{B} \cdot d\vec{\ell} = \mu_0 I \Rightarrow$

$$B = \mu_0 \frac{I}{2\pi \cdot r}.$$

For points outside the cable:

$$\vec{B}_{I_+} = -\vec{B}_{I_-} ,$$

The total magnetic induction is:

$$\vec{B} = 0 .$$

For points inside the coaxial cable, the density of magnetic energy is:

$$w_m = \frac{1}{2\mu_0} B^2 = \frac{1}{2\mu_0} \frac{\mu_0^2 \cdot I^2}{4\pi^2 r^2} \Longrightarrow$$

$$w_m = \mu_0 \frac{I^2}{8\pi^2 r^2}.$$

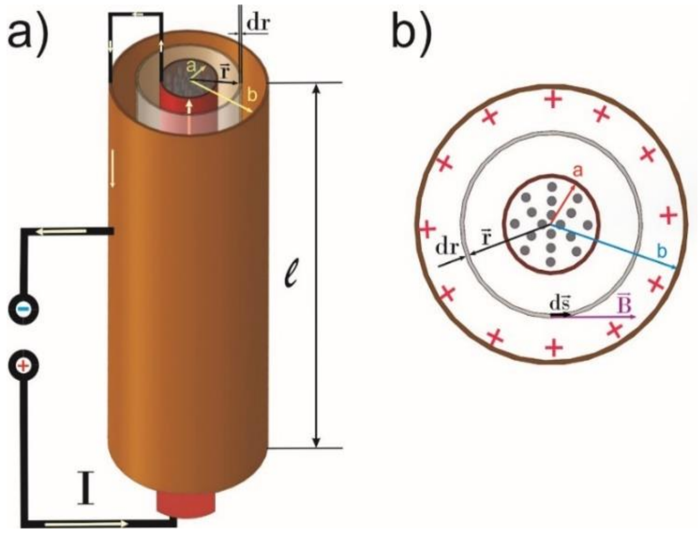


*Fig. 87. Energy stored in a coaxial cable (radii **a, b**)*

The element of volume can be written: $dV = 2\pi r \cdot dr \cdot \ell$.

Therefore, the energy **dW** stored in the element of volume $2\pi r \cdot \ell \cdot dr$, becomes:

$$dW = w_m \cdot dV$$

$$dW = \mu_0 \frac{I^2}{8\pi^2 \cdot r^2} 2\pi r \cdot \ell \cdot dr .$$

Thus, the total energy stored in the magnetic field in the coaxial cable is found by integrating dW between **a** and **b**:

$$W = \int dW = \int_a^b \frac{\mu_0 I^2}{4\pi} \frac{\ell}{r} dr \Longrightarrow W = \mu_0 \frac{I^2 \cdot \ell}{4\pi} \ln\left(\frac{a}{b}\right).$$

Problem 123

It has been hypothesized that birds might use the emf (electromotive force, i.e. the induced electromagnetic voltage) between their wing tips by the terrestrial magnetic field, as a help for their '*navigation*' during migration. Calculate:

a) The emf induced between the wing tips of a bird with a wingspread of 1.5 m, flying at 20 m/s in a region where the vertical component of the Earth magnetic field is B_{Earth} = 2 x 10^{-5} T.

b) The emf induced between the tips of the wings of a Boeing 747 with a sing span of 60 m, flying at a speed of 900 km/h in the same region of the terrestrial magnetic field.

Solution

a) The induced *emf* (the electromotive force, actually a voltage) is:

$$\mathcal{E} = \ell \cdot B \cdot v = 1.5\ \mathrm{m} \cdot 2 \times 10^{-5}\mathrm{T} \cdot 20\frac{\mathrm{m}}{\mathrm{s}} \Rightarrow \mathcal{E} = 0.6\ \mathrm{mV}.$$

Note that the cellular voltages at the biological cells are of about 70 mV, therefore, an induced voltage of 0.6 mV is hard to be detected by the bird.

b) For a Boeing plane, the *emf* induced during the flight is:

$$\mathcal{E} = \ell \cdot B \cdot v = 60\ \mathrm{m} \cdot 2 \times 10^{-5}\mathrm{T} \cdot 250\frac{\mathrm{m}}{\mathrm{s}} = 300\ \mathrm{mV}.$$

Note that this is a measurable voltage, and so the electromagnetic induction effect is appreciable in this case.

Problem 124

A current of intensity **I** is set along a resistor (electrical resistance **R**, resistivity ρ, length **L**, cross section area **S**). Represent the Poynting vector and show that the rate at which energy enters the resistor equals the rate at which it generates Joule heat (Figure 88).

Solution

The Poynting vector is given by:

$$\vec{\wp} = \vec{E} \times \vec{H} = \frac{\vec{E} \times \vec{B}}{\mu_0}.$$

On the other hand, the power, or the rate at which the energy is transmitted across a surface of area **S** equals the flux of the Poynting vector across that surface **S**:

$$P = \frac{dW}{dt} = \iint_S \vec{\wp} \cdot \vec{S}.$$

Therefore, by substituting the Poynting vector, the power becomes:

$$P = \iint_S \wp\ dS = \iint_S E\,\frac{B}{\mu_0}\,dS = \int_0^L \frac{E \cdot B}{\mu_0} 2\pi R \cdot d\ell\,,$$

$$P = \frac{E \cdot B \cdot 2\pi R \cdot L}{\mu_0}.$$

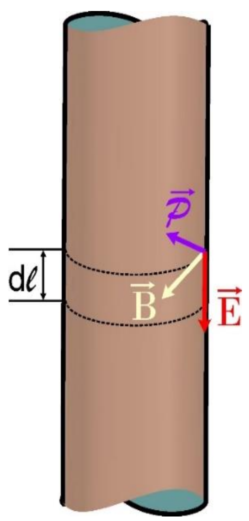


Fig. 88. The Poynting vector around a current (I) through the resistor (R).

After substituting $E = \frac{U}{L}$ and rearranging the terms:

$$P = \frac{2\pi R}{\mu_0} E \cdot B \cdot L = \frac{2\pi R}{\mu_0}\left(\frac{U}{L}\right)\left(\mu_0 \frac{I}{2\pi R}\right) L\,.$$

Simplifying the terms results in: $P = U \cdot I = (I \cdot R) \cdot I = I^2 \cdot R.$

Therefore, the source of the joule power (I^2R), or the heat released when the current of intensity **I** travels along the resistor of resistance **R** is the rate at which electromagnetic energy is carried along the resistor **R**.

Problem 125

A variable magnetic field fills the space within a cylindrical region of radius R = 10 cm.

a) Based on Maxwell-Faraday law, find the electric field intensity E(r) at r < R (inside the volume) and r > R (outside the volume).

b) Represent a graph showing the dependence of E(r) versus **r**.

Solution

a) At points inside the volume (r < R) where the magnetic field is variable, Maxwell-Faraday law:

$$\oint_{\Gamma} \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_m}{dt}\,.$$

Then: $E \cdot 2\pi r = -(\pi r^2)\frac{dB}{dt} \quad \Rightarrow \quad E(r) = -\frac{r}{2}\frac{dB}{dt}.$

b) At points outside the volume (r > R) where the magnetic field is variable, Maxwell-Faraday law is written as follows:

$$E \cdot 2\pi r = -\pi R^2 \frac{dB}{dt} \quad \Leftrightarrow \quad E(r) = -\frac{R^2}{2r}\frac{dB}{dt}.$$

The graph showing the dependence of E(r) versus **r** is shown in red, in Figure 89-(b). For values of the radius r < R the intensity of the electric field increases linearly, directly proportional to **r**. For values r > R the electric field intensity decreases with **1/r**.

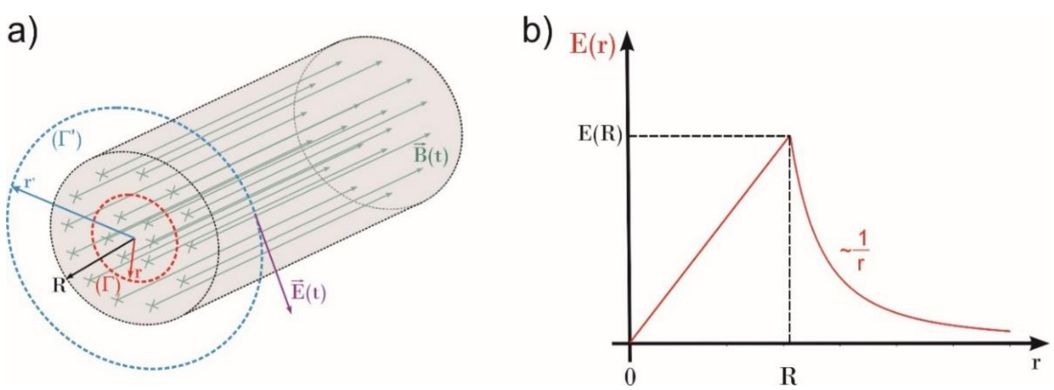


Fig. 89. Variable magnetic field in a cylindrical region and the electric field thus generated

Problem 126

An observer is at distance **r** from a point source of electromagnetic radiation, of power $\mathbf{P_0}$. Assuming that the source is monochromatic, it irradiates uniformly in all the directions, and at long distance the wave is planar, determine what is the amplitude of the electric (E_m) and that of the magnetic field (B_m) in the electromagnetic wave?

Solution

The Poynting vector $(\vec{\wp})$ describes the transport of energy and has the equation:

$$\vec{\wp} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \vec{E} \times \vec{H}.$$

The average Poynting vector represents the power emitted across the unit surface (the source emits uniformly along all the directions in space, therefore its energy is carried on a spherical surface of radius R centred on the point source):

$$\bar{\wp} = \frac{P_0}{4\pi \cdot R^2}.$$

The average Poynting vector can be written:

$$\bar{\wp} = \overline{\left(\frac{1}{\mu_0} E \cdot B\right)} = \frac{1}{\mu_0}\,\overline{E \cdot B} = \frac{1}{\mu_0} \overline{E\frac{E}{c}} = \frac{1}{\mu_0 \cdot c}\,\overline{E^2}.$$

As $E(t) = E_m \sin(\omega t)$, the average becomes:

$$\overline{E(t)^2} = \overline{[E_m \sin(\omega t)]^2} = \frac{1}{2} E_m^2.$$

It follows that the average Poynting vector is: $\bar{\wp} = \frac{1}{2\mu_0 c} E_m^2$.

Then the power of the electromagnetic wave is: $P_0 = \bar{\wp} \cdot 4\pi \cdot r^2 = \frac{2\pi \cdot r^2}{\mu_0 c} E_m^2$

Then the maximum electric field results as: $E_m = \sqrt{\frac{\mu_0\, c}{2\pi r^2} P_0} = \frac{1}{r}\sqrt{\frac{P_0 \cdot \mu_0 c}{2\pi}}$.

The maximum magnetic field results as: $B_m = \frac{1}{r \cdot c}\sqrt{\frac{P_0 \cdot \mu_0 c}{2\pi}} = \frac{1}{r}\sqrt{\frac{P_0 \cdot \mu_0}{2\pi \cdot c}}$.

Problem 127

Calculate the electric and magnetic field amplitudes for the electromagnetic wave in Problem 128, at distance r = 1 m, from the point source with the radiating power P_0 = 1 kW.

Solution

The amplitude of the electric field component: $E_m = \frac{1}{1}\sqrt{\frac{10^3 \cdot 4\pi\times10^{-7}\cdot 3\times10^8}{2\pi}} \Longrightarrow E_m \cong 245\,\frac{V}{m}$.

The magnetic field induction: $B_m = \frac{E_m}{c} \cong \frac{245}{3\times10^8}\,T \Longrightarrow B_m \cong 8.2\times10^{-7}\,T$.

Problem 128

The amplitude of the electric field component of a planar electromagnetic wave is $E_m = 10^{-4}\ V/m$. Determine:

a) The amplitude of the magnetic field component of the electromagnetic wave, B_m =?

b) The intensity of the electromagnetic wave, or else the average of Poynting vector $\vec{\wp}$ =?

Solution

a) One knows that: $E_m = c \cdot B_m$, or else:

$$B_m = \frac{1}{c}E_m = \frac{10^{-4}\,\frac{V}{m}}{3\times10^8\,\frac{m}{s}} \Rightarrow B_m = 3.3(3)\times10^{-13}\,T.$$

b) The intensity of a wave is the amount of energy carried by the wave as it propagates, during the unit time across the unit surface, or the power transferred across the unit surface:

$$I = \frac{\text{energy}}{\Delta t \cdot A} = \frac{dE}{dt \cdot dA} = \frac{dP}{dA} = \bar{\wp} = \text{average of the Poynting vector.}$$

Where the Poynting vector of an electromagnetic wave is, at any moment and position, the vector product (or 'cross product') between the electric and magnetic intensities at that moment and at that position, in the electromagnetic wave: $\vec{\wp} = \vec{E}\times\vec{H}$.

The average of the Poynting vector over a period: $\bar{\wp} = \frac{1}{2\mu_0 c}\,E_m^2$. (see Problem 126)

Therefore, the wave intensity becomes:

$$I = \bar{\wp} = \frac{1}{2\cdot 4\pi\times10^{-7}\cdot 3\times10^8}(10^{-4})^2\,\frac{W}{m^2} \Longrightarrow$$

$$I = 1.33\times10^{-11}\,\frac{W}{m^2}.$$

Problem 129

A LASER beam of 100 W power and 5000 Å wavelength has the cross-section area of 10 mm^2.

a) Calculate the intensity of the LASER beam, I =?

b) Write the equations of the electric field $\vec{E}(x,t)$ and magnetic field $\vec{B}(x,t)$ components of the LASER beam.

Solution

a) The electromagnetic wave intensity is defined as the power transmitted across the unit area:

$$\mathrm{I} = \frac{\mathrm{P}}{\mathrm{A}} = \frac{100\ \mathrm{W}}{10 \times 10^{-6}\mathrm{m}^2} \Longrightarrow$$

$$\mathrm{I} = 10^7\frac{\mathrm{W}}{\mathrm{m}^2}.$$

b) The absolute value of the Poynting vector equals the wave intensity:

$$|\vec{\wp}| = \mathrm{I}.$$

Where: $\vec{\wp} = \vec{\mathrm{E}} \times \vec{\mathrm{H}}$, and its absolute value then results as: $|\vec{\wp}| = \mathrm{E_{eff}}\,\mathrm{H_{eff}}$. Given that the electric and magnetic field are linked by the equation: $\mathrm{E} = \mathrm{c{\cdot}B}$, or for the intensity of the magnetic field: $\mathrm{H} = \frac{\mathrm{B}}{\mu_0} = \frac{\mathrm{E}}{c{\cdot}\mu_0}$, that holds for their instant [B(x, t) and E(x, t)], maximum [B_{max} and E_{max}], and effective [B_{eff} and E_{eff}] values, respectively, the relationship linking the intensity with the electric and magnetic components of the wave can therefore be written:

$$\mathrm{I} = \frac{\mathrm{E_{eff}^2}}{\mathrm{c}{\cdot}\mu_0}.$$

It follows then that the effective value of the electric field in the LASER beam is:

$$\mathrm{E_{eff}} = \sqrt{\mu_0{\cdot}\mathrm{c}{\cdot}I} = \sqrt{4\pi \times 10^{-7}{\cdot}3 \times 10^8{\cdot}10^7} \Longrightarrow$$

$$\mathrm{E_{eff}} \cong 6.14 \times 10^4\,\frac{\mathrm{V}}{\mathrm{m}}.$$

And the effective value of the magnetic induction of the magnetic field component is:

$$\mathrm{B_{eff}} = \frac{\mathrm{E_{eff}}}{\mathrm{c}} \cong 2.047 \times 10^{-4}\ \mathrm{T}.$$

Then the equations of the electric and magnetic field components of the LASER beam result as:

$$\begin{cases} \mathrm{E(x,t)} = \mathrm{E_{max}}\ \sin(\omega\mathrm{t} - \mathrm{kx})\ \left(\frac{\mathrm{V}}{\mathrm{m}}\right) \\ \mathrm{B(x,t)} = \mathrm{B_{max}}\ \sin(\omega\mathrm{t} - \mathrm{kx})\ (\mathrm{T}). \end{cases}$$

Where the amplitudes of the two waves are deduced as follows:

$$\mathrm{E_{max}} = \sqrt{2}\,\mathrm{E_{eff}} \cong 8.68 \times 10^4\,\frac{\mathrm{V}}{\mathrm{m}} \Rightarrow \mathrm{E_{max}} = 86.8\,\frac{\mathrm{kV}}{\mathrm{m}}.$$

$$\mathrm{B_{max}} = \sqrt{2}\ \mathrm{B_{eff}} \cong 2.89 \times 10^{-4}\,\mathrm{T} \Rightarrow \mathrm{B_{max}} = 0.289\,\mathrm{mT}.$$

Problem 130

A point like source radiates electromagnetic waves (emw) at a rate $P_0 = 10^3$ W. What are the maximum values (i.e. the amplitudes) of the electric (**E_m**) and magnetic field (**B_m**) components of this electromagnetic wave at distance r = 1 m from the source?

Solution

The Poynting vector:

$$\vec{\wp} = \vec{E} \times \vec{H}.$$

The average value of the Poynting vector:

$$\bar{\wp} = E_{ef} \cdot H_{ef} = E_{ef} \frac{E_{ef}}{\mu_0 \cdot c} = \frac{1}{\mu_0 \cdot c} \frac{E_m^2}{2}.$$

The absolute value of the Poynting vector is also equal to the energy transmitted by the wave across the unit surface during the unit time, or the power transmitted across the unit surface. Considering that the point like source emits isotropic and uniformly in space, this power is spread over a spherical surface of radius r = 1 m. The Poynting vector average is also given by (see Problem 126):

$$\bar{\wp} = \frac{P_0}{4\pi r^2}.$$

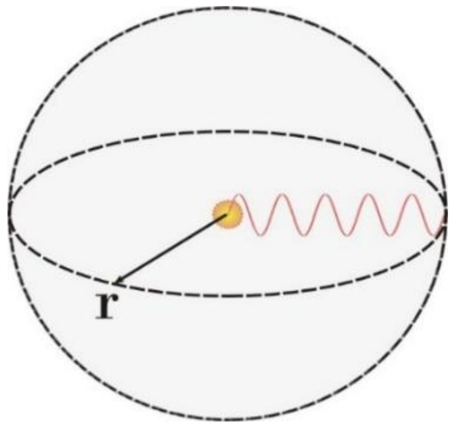


Fig. 90. A point like source emits isotropic electromagnetic radiation, uniformly in space.

Therefore, one can write that:

$$\frac{P_0}{4\pi r^2} = \frac{1}{\mu_0 \cdot c} \frac{E_m^2}{2}.$$

Then the amplitudes of the electric and magnetic components of the electromagnetic wave are:

$$E_m = \sqrt{2\mu_0 c \frac{P_0}{4\pi r^2}} = \sqrt{\frac{2 \cdot 4\pi \times 10^{-7} \cdot 3 \times 10^8 \cdot 10^3}{4\pi \cdot 1^2}} \Rightarrow$$

$$E_m \cong 244.95 \frac{V}{m}.$$

$$B_m = \frac{E_M}{c} = \frac{244.95}{3 \times 10^8} \Rightarrow$$

$$B_m \cong 8.16 \times 10^{-7} T = 0.8\ \mu T.$$

Problem 131

Sunlight hits the superior terrestrial atmosphere with an intensity of 2 $\mathrm{cal/(cm^2 \cdot min)}$ (also called the *solar constant*) after travelling approximately 150 million km from the Sun. Calculate the amplitudes of the electric and magnetic components of this emw (electromagnetic wave).

Solution

The solar constant can be converted in SI units:

$$2\ \mathrm{cal \cdot cm^{-2} \cdot min^{-1}} \cong 1.34\ \frac{kW}{m^2},$$

where we converted the calorie unit of energy (heat) in Joules, using that 1 cal ≅ 4.85 J. The *intensity* of the solar radiation represents the *energy propagated during the unit time across the unit surface*, or else, it equals the average value of the Poynting vector of the radiation:

$$\bar{\wp} \cong 1340 \frac{\mathrm{W}}{\mathrm{m}^2}.$$

The average value of the Poynting vector for the solar radiation when it impacts the Earth atmosphere can be written as a function of the amplitude of the electric field component (see Problem 126):

$$\bar{\wp} = \frac{1}{\mu_0 c} \frac{E_m^2}{2}.$$

Then, the maximum value of the electric field intensity results as:

$$E_m = \sqrt{2\mu_0 c \cdot \bar{\wp}} = \sqrt{2 \cdot 4\pi \times 10^{-7} \cdot 3 \times 10^8 \cdot 1340} \cong 1.005 \frac{\mathrm{kV}}{\mathrm{m}}.$$

It follows then that the induction of the magnetic component of the solar radiation at the impact on the atmosphere is:

$$B_m = \frac{E_m}{c} = \frac{1005}{3 \times 10^8} \Longrightarrow B_m \cong 3.35\ \mu\mathrm{T}.$$

Problem 132

At the Earth surface, the sun rays have an intensity of $1.34\ \mathrm{kW/m^2}$ (***Note*** *that the intensity of the Solar radiation at the Earth surface is smaller than the Solar constant due to the absorption of the solar radiation through the atmosphere*). What pressure does this electromagnetic radiation exert on a black surface?

Solution

The pressure exerted by the emw on the black surface equals the ratio between the average value of the Poynting vector and the speed of light in vacuum:

$$p = \frac{\bar{\wp}}{c} = \frac{1000\ \mathrm{W/m^2}}{3 \times 10^8\ \mathrm{m/s}} \cong 3.3(3) \times 10^{-6} \frac{\mathrm{N}}{\mathrm{m}^2} = 3.3(3) \times 10^{-6} \cdot 0.9869 \times 10^{-5} \mathrm{atm} \Rightarrow$$

$$p \cong 3.29 \times 10^{-11} \mathrm{atm}.$$

Problem 133

A LASER beam has the cross section of 10 mm^2, a power of 100 W and the wavelength of 500 nm.

a) What is the intensity of this LASER beam?

b) What are the amplitudes of the electric and magnetic components of this emw?

c) What are the equations of **E(t)** and of **B(t)**?

Solution

a) The intensity of the LASER radiation is equal to the average value of the Poynting vector:

$$I = \bar{\wp} = \frac{P}{S} = \frac{100\ \mathrm{W}}{10 \times 10^{-6}\mathrm{m}^2} = 10^7\ \frac{\mathrm{W}}{\mathrm{m}^2}.$$

b) The average of the Poynting vector:

$$\bar{\wp} = E_{ef}\, H_{ef} = \frac{E_m \cdot H_m}{2} = \frac{E_m \cdot B_m}{2\mu_0} = \frac{{E_M}^2}{2\mu_0 \cdot c}.$$

Then the maximum of the electric component of the LASER wave is:

$$E_m = \sqrt{2\mu_0 c\, \bar{\wp}} = \sqrt{2 \cdot 4\pi \times 10^{-7} \cdot 3 \times 10^8 \times 10^7} \Longrightarrow E_m \cong 8.68 \times 10^4\, \frac{\mathrm{V}}{\mathrm{m}} = 86.8\ \frac{\mathrm{kV}}{\mathrm{m}}.$$

The amplitude of the magnetic induction of the magnetic component of the LASER radiation:

$$B_m = \frac{E_m}{c} = \frac{8.68 \times 10^4}{3 \times 10^8} \Longrightarrow B_m \cong 2.04\,\sqrt{2} \times 10^{-4}\mathrm{T} = 2.89 \times 10^{-4}\mathrm{T} = 0.27889\ \mathrm{mT}.$$

c) The instant values of the electric intensity and of the magnetic induction of the LASER beam:

The electric component: $E(t) = 86.8 \sin(\omega t - kx) \left(\frac{\mathrm{V}}{\mathrm{m}}\right)$.

The magnetic component: $B(t) = 0.27889 \sin(\omega t - kx)(\mathrm{mT})$.

Problem 134

The visual sensation is produced by the electric component of the electromagnetic wave in the visible spectrum, therefore just the electric component will be considered when describing visual phenomena. Thus, the electric field of an electromagnetic wave, at time **t**, at distance **r** from the electromagnetic waves source is: $E(r, t) = E_m \sin(\omega\, t - k\, r)$, where: $k = \frac{2\pi}{\lambda}$ is the wave number. In empty space (free space, vacuum), the amplitude of the electric field of an electromagnetic wave (emw) is E_m = 0.05 V/m. Determine:

a) What is the amplitude of the magnetic field component of the emw in free space, $\mathbf{B_m}$ = ?

b) What is the amplitude of the electric field of the electromagnetic wave in a medium with the relative electric permittivity ε_r = 81 and relative magnetic permeability μ_r = 1, if the magnetic field amplitude remains the same ($B = B_m$), $\mathbf{E_m}$ = ?

Solution

a) The magnetic field amplitude in the electromagnetic wave in free space (of electric permittivity ε_0 and magnetic permeability μ_0):

$$B_m = \frac{E_m}{c} = \frac{0.05}{3 \times 10^8}\ (\mathrm{T}) \Longrightarrow B_m = 1.66(6) \times 10^{-10}\,\mathrm{T}.$$

b) In a medium of dielectric constant ε_r and relative magnetic permeability μ_r :

$$E = v \cdot B.$$

$$E = \frac{c}{\sqrt{\varepsilon_r\,\mu_r}} B = \frac{3 \times 10^8}{\sqrt{\varepsilon_r\,\mu_r}}\, 1.66(6) \times 10^{-10} \left(\frac{V}{m}\right) = \frac{3 \times 10^8}{\sqrt{81}}\, 1.66(6) \times 10^{-10} \left(\frac{V}{m}\right).$$

$$E = 0.55(5) \times 10^{-2}\frac{V}{m} = 0.0056\,\frac{V}{m} = 5.5(5)\,\frac{mV}{m}.$$

Problem 135

Consider Young's experiment of interference for two coherent sources of light, S_1 and S_2, placed at distance **d** from each other. Deduce the fringe spacing between two neighbouring maxima of interference for the light waves of wavelength λ, on a screen placed at distance **D** from the sources S_1 and S_2 (Figure 91).

Solution

The path difference, Δr for the waves travelling from S_1 (r_1) and S_2 (r_2), respectively, to the plan of the screen, where they interfere, say at point M:

$$\Delta r = d \cdot \sin(\theta).$$

For small angles θ, we have that: $\sin(\theta) \approx \tan(\theta)$.

In Figure 91, one can observe that:

$$\tan(\theta) = \frac{OM}{D} = \frac{x}{D}.$$

Therefore, the first equation becomes:

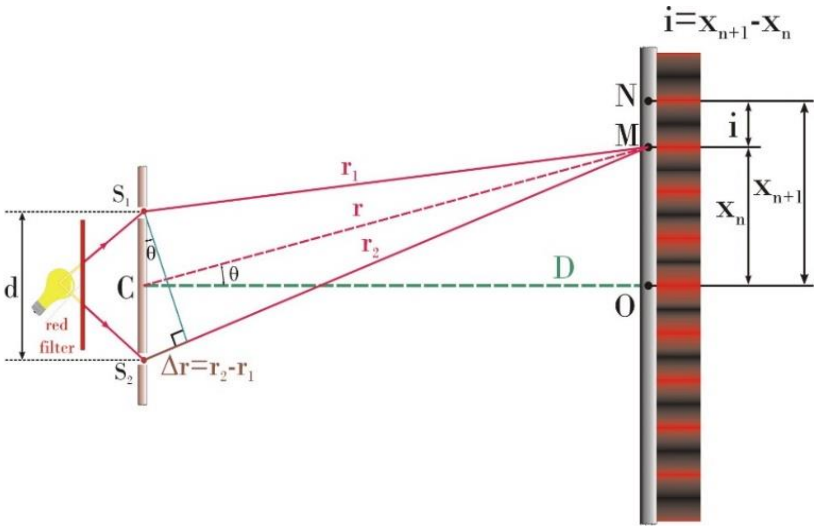


Fig. 91. Young's experiment: interference of coherent light waves of wavelength λ. The red filter selects monochromatic waves from the whole spectrum emitted by the source of light.

$$\Delta r = d \cdot \sin(\theta) \approx d\frac{x}{D}.$$

If the path difference $\Delta r = n \cdot \lambda$ (an integer number of wavelengths), then at point M (distance x_n from the centre O), there will be observed a maximum of interference. A neighbouring maximum, of the order (n+1), will be observed at a point N (at distance x_{n+1} from the centre O).

The equation for the path difference corresponding to maxima: $d\frac{x_n}{D} = n \cdot \lambda$.

Then, the positions of the next consecutive maxima can be written as:

$$\begin{cases} x_n = n\dfrac{\lambda\cdot D}{d} \\ x_{n+1} = (n+1)\dfrac{\lambda\cdot D}{d} \end{cases}.$$

And the distance between two consecutive maxima of interference, or the *fringe spacing* (i) can be therefore deduced as:

$$i = x_{n+1} - x_n = \frac{\lambda D}{d}.$$

Problem 136

In a Young experiment of interference with yellow light (λ = 546 nm), on the image screen (at distance D = 0.8 m from the sources) (see Fig. 91), one can observe N = 20 fringe separations (20 maxima of interference alternating with 20 dark minima) along a total distance y = 10.92 mm. Determine what is the distance between the two coherent sources of light, **d** = ?

Solution

The total width of the 20 maxima alternating with 20 dark minima is given by: $y = N\cdot i = N\frac{\lambda\cdot D}{d}$.

It follows that the distance between the two coherent sources of light is:

$$d = \frac{N\cdot\lambda\cdot D}{y} = \frac{20\cdot 546\times10^{-9}\text{m}\cdot 0.8\text{ m}}{10.92\ \times 10^{-3}\text{m}} \Longrightarrow d = 0.8\text{ mm}.$$

Problem 137

Two polarizers have their polarising axes parallel, such that the light transmitted across them both has maximum intensity $\mathbf{I_0}$. What is the angle of rotation for one of the polarisers, to reduce the intensity of the light transmitted across the polarisers to half of its maximum?

Solution

Malus's law can be written as: $I = I_0\cos^2(\theta)$.

Then by substituting the light intensity I with half of its initial value: $I = \frac{1}{2}I_0$, one obtains:

$$\frac{1}{2}I_m = I_m\cos^2(\theta).$$

Therefore, we can deduce that:

$$\cos^2(\theta) = \frac{1}{2} \quad \Rightarrow \cos(\theta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \Rightarrow$$

$$\theta = \arccos\left(\frac{\sqrt{2}}{2}\right) \Longrightarrow \theta = \frac{\pi}{4} = 45^{o}.$$

The result shows that for the light intensity (I_0) to drop to half of its original intensity ($I = I_0/2$), the axes of the two polarizers must be at an angle of 45^{o}.

10. QUANTUM PHYSICS and MATTER

Problem 138

Consider a mass-spring system, of mass m = 1 kg and with the spring force constant k = 20 N/m, which oscillates with an amplitude Ψ_0 = 1 cm.

a) If the energy of this oscillator is quantified, what is the quantum number **n** ?

b) What is the relative change in the oscillator energy when it changes its state from **n** to **(n+1)**?

c) What is the value of its energy on the lowest energy level, i.e. for **n** = 1?

Solution

a) The oscillator energy is given by the equation:

$$E = \frac{1}{2} k\, \Psi_0^2 = \frac{1}{2} 20 \frac{N}{m} \cdot (10^{-2}\ m)^2 \Longrightarrow E = 10^{-3}\, J\,.$$

If it is quantified, then one can write (for its energy in state 'n'):

$$E = E_n = n \cdot h\nu.$$

With Planck's constant $h \cong 6.6\ \times 10^{-34}\, J \cdot s$, and frequency: $\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \Longrightarrow \nu \cong\ 0.71\ Hz.$

Then '**n**', or the quantum number of the energy state that the oscillator is in results as:

$$n = \frac{E_n}{h\nu} = \frac{10^{-3}}{6.6\ \times 10^{-34} \times 0.71} \Longrightarrow n \cong 2.1\ \times 10^{30}\,.$$

This means that the energy calculated for the oscillator corresponds to its quantum state number 2.1 x 10^{30}. For such large numbers **n**, the energy levels are very close and densely packed, that they end up belonging to a continuum of energy values.

b) When the oscillator changes its state from quantum number 'n' to quantum number 'n+1', the change in its quantum number is $\Delta n = 1$, and the change in energy:

$$\Delta E = h\nu \cdot \Delta n = h\nu.$$

Then the relative change in energy, between states 'n+1' and 'n' is:

$$\frac{\Delta E}{E_n} = \frac{h\nu}{n \cdot h\nu} = \frac{1}{n} \Longrightarrow \frac{\Delta E}{E_n} \cong 10^{-31} =\ 10^{-22}\ \times 10^{-9} = 10^{-22}\ ppb.$$

(Where: 1 ppb = parts per billion = 10^{-9}). The lowest energy level corresponds to n = 1 and is really a very small quantity:

$$E_1 = 1 \cdot h\nu\ \cong 6{,}6\ \times 10^{-34} Js\ \times 0.71\ s^{-1} = 4.7\ \times 10^{-34}\, J \cong 2.9 \times 10^{-15} eV.$$

If this corresponds to a kinetic type of energy, when found on this lowest kinetic energy level, the speed of motion of the particle is really slow (although not zero):

$$v = \sqrt{\frac{2\cdot E_1}{m}} = \sqrt{\frac{2\cdot 4.7 \times 10^{-34}}{1}} \left(\frac{m}{s}\right) = \sqrt{9.4} \times 10^{-17}\frac{m}{s} \Longrightarrow v \cong 3.07 \times 10^{-17}\frac{m}{s}.$$

Problem 139

The lifetime of a confined particle in an excited state is Δt = 1 ns. What is the uncertainty in the energy of the particle in that excited state?

Solution

Heisenberg's uncertainty principle can be written as:

$$\Delta E\cdot\Delta t = \frac{\hbar}{2}, \qquad \text{where } \hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}\,\mathrm{J\cdot s} \approx 10^{-34}\,\mathrm{J\cdot s}.$$

Therefore:

$$\Delta E = \frac{\frac{\hbar}{2}}{\Delta t} = \frac{10^{-34}\,\mathrm{J\cdot s}}{10^{-9}\,\mathrm{s}} \Longrightarrow$$

$$\Delta E = 10^{-25}\,\mathrm{J} \approx 0.625 \times 10^{-6}\mathrm{eV} = 625\ \mathrm{meV}.$$

Problem 140

a) Deduce the IS unit of Planck's constant. Use the energy of the radiation of frequency ν.

b) Convert 1 eV of energy in Joules.

Solution

a) The energy of a photon of frequency ν is given by the equation: $E = h\nu$.

Therefore, the Planck constant results:

$$h = \frac{E}{\nu}.$$

And it IS unit then becomes:

$$[h]_{SI} = \frac{[E]_{SI}}{[\nu]_{SI}} = \frac{1\,\mathrm{J}}{1\,\mathrm{Hz}} = 1\,\frac{\mathrm{J}}{\mathrm{s}^{-1}} = 1\,\mathrm{J\cdot s}.$$

b) Converting 1 eV of energy:

$$1\,\mathrm{eV} = 1.6 \times 10^{-19}\mathrm{C}\cdot 1\,\mathrm{V} = 1.6 \times 10^{-19}\,(\mathrm{C\cdot V}) = 1.6 \times 10^{-19}\,\mathrm{J}.$$

Problem 141

Calculate the energy E_γ and the linear momentum **p** of a photon (γ) that has the wavelength λ = 400 nm (violet, or the high frequency end of the visible spectrum of electromagnetic waves).

Solution

The photon energy:

$$E_\gamma = h\nu = h\frac{c}{\lambda} = 6.6 \times 10^{-34}\ \mathrm{J{\cdot}s}\ \frac{3 \times 10^8\ \frac{\mathrm{m}}{\mathrm{s}}}{4 \times 10^{-7}\ \mathrm{m}} = 4.95 \times 10^{-19}\ \mathrm{J} = \frac{4.95 \times 10^{-19}\ \mathrm{J}}{1.6 \times 10^{-19}\ \frac{\mathrm{J}}{\mathrm{eV}}}$$

$$E_\gamma \cong 3.09\ \mathrm{eV}.$$

The photon momentum:

$$p_\gamma = \frac{E_\gamma}{c} = \frac{4.95 \times 10^{-19}\ \mathrm{J}}{3 \times 10^8\ \mathrm{m/s}} \Rightarrow\ p_\gamma = 1.65 \times 10^{-27}\ \mathrm{kg}{\cdot}\frac{\mathrm{m}}{\mathrm{s}}.$$

Problem 142

Calculate the energy (in eV) carried by the following photons:

a) An X ray photon of frequency $\nu = 1.5 \times 10^{20}$ Hz;

b) A photon of UV radiation of wavelength λ = 340 nm;

c) A photon of red light, of wavelength λ = 0.675 µm;

d) A photon of IR radiation (heat) of wavelength λ = 9600 Å;

e) A quantum of microwave radiation of wavelength λ = 2 cm.

Solution

a) $E_X = h\nu = (6.63 \times 10^{-34}\mathrm{J{\cdot}s})\cdot(1.5 \times 10^{20}\ \mathrm{s}^{-1}) \Longrightarrow E_X = 9.945 \times 10^{-14}\mathrm{J} = \frac{9.945 \times 10^{-14}\ \mathrm{J}}{1.6 \times 10^{-19}\ \frac{\mathrm{J}}{\mathrm{eV}}};$

$$E_X \approx 6.216 \times 10^5\ \mathrm{eV} = 621.6\ \mathrm{keV}.$$

b) $E_{UV} = h\nu = h\frac{c}{\lambda} \approx 6.63 \times 10^{-34}\mathrm{J{\cdot}s}\ \frac{3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}}{340 \times 10^{-9}\mathrm{m}} \approx 0.0585 \times 10^{-17}\mathrm{J} \Longrightarrow E_{UV} = 5.85 \times 10^{-19}\mathrm{J}.$

$$E_{UV} = \frac{5.85 \times 10^{-19}\mathrm{J}}{1.6 \times 10^{-19}\mathrm{J/eV}} = 3.656\ \mathrm{eV}.$$

c) $E_{Red} = h\frac{c}{\lambda} = 6.63 \times 10^{-34}\mathrm{Js} \cdot \frac{3 \times \frac{10^8\mathrm{m}}{\mathrm{s}}}{0.675 \times 10^{-6}\mathrm{m}} \Longrightarrow E_{Red} = 29.46(6) \times 10^{-20}\mathrm{J} \approx 18.41(6) \times 10^{-1}\mathrm{eV},$

$$E_{Red} = 1.841(6)\ \mathrm{eV}.$$

d) $E_{IR} = h\frac{c}{\lambda_{IR}} = 6.63\times10^{-34}\text{Js}\cdot\frac{3\times10^{8}\frac{\text{m}}{\text{s}}}{9600\times10^{-10}\text{m}} \Longrightarrow E_{IR} \approx 2.07\times10^{-19}\ \text{J} \approx 1.29\ \text{eV};$

e) $E_{\mu w} = h\frac{c}{\lambda_{\mu w}} = 6.63\times10^{-34}\text{Js}\cdot\frac{3\times10^{8}\frac{\text{m}}{\text{s}}}{2\times10^{-2}\text{m}} \Longrightarrow E_{\mu w} = 9.945\times10^{-24}\text{J} \cong 6.216\times10^{-5}\text{eV}$

$$E_{\mu w} \approx 62\ \mu\text{eV}.$$

Problem 143

Based on the equations used in the previous problem, fill in the table below with the missing parameters, characteristic to each example of electromagnetic waves. The emw in pink cells in the table below (i.e. Extreme UV, X rays and gamma rays) are ionizing radiations: their photons carry enough energy to knock out electrons from atoms and molecules. (***Note*** *that the various range 'limits' are only approximate and some neighbouring ranges may overlap at their minimum or maximum values of wavelengths/frequencies/energies*).

Photon energy (eV)	***Photon wavelength***	***Photon frequency***	***Sources of radiation/Observations***
………………..	…………………..	50 Hz	**The network that distributes electrical energy to consumers**
……… - ……..	1 m – 10 km	……… - ……..	**Radio waves: L, M, S, US (FM)**
……… - ……..	……… - ……..	(1 – 20) GHz	**Microwaves and RADAR (range)**
……… - ……..	1 mm – 10 mm	30 GHz – 300 GHz	**Extremely high radio frequencies – EHF (the millimetre band) (range)**
……… - ……..	……… - ……..	600 MHz – 70 GHz	**5G cellular networks (range)**
……… - ……..	……… - ……..	600 MHz – 900 MHz	• *Low band 5G (as in the 4G band)*
……… - ……..	……… - ……..	1.7 GHz – 4.7 GHz	• *Mid band 5G*
……… - ……..	……… - ……..	24 GHz – 47 GHz	• *High band 5G*
……… - ……..	1 μm – 1 mm	……… - ……..	**Heat / IR radiation (range)**
……… - ……..	(400 – 700) nm	……… - ……..	**Light or visible emw (range)**
1.86	…………………..	3.8×10^{14} Hz	• *Red (e.g. monochromatic)*
………….	410 nm	……………….	• *Violet (e.g. monochromatic)*
……… - ……..	(10 – 400) nm	……… - ……..	**Ultraviolet UV (range)**
……… - ……..	(315 – 400) nm	……… - ……..	• *UVA (range)*
……… - ……..	(280 – 315) nm	……… - ……..	• *UVB (able to modify DNA) (range)*

4.92 – 13.76	(100 – 280) nm	……… - ……..	• *UVC (range). The 13.76 eV photons are able to ionize the H atom, i.e. to remove its electron.*
11.47 – 137.64	(10 - 100) nm	……… - ……..	• *Extreme UV (range)*
……… - ……..	……… - ……..	10^{16} Hz – 10^{19} Hz	**X rays** (emitted by electrons during their transitions between K, L, M and N electronic shells) (range)
……… - ……..	……… - ……..	10^{19} Hz - 10^{21} Hz	**Gamma rays (released during nuclear transitions) (range)**

Problem 144

Consider a hydrogen molecule at room temperature and a photon in the yellow-green range of visible light.

a) Compare the energy of the photon in the visible range (λ = 5000 Å, yellow-green) with the kinetic energy of a hydrogen molecule (H_2) at the room temperature.

b) Calculate the wavelength **λ** of a photon that has the energy equal to that of a hydrogen molecule at room temperature.

c) Calculate the absolute temperature **T** at which the kinetic energy of a H_2 molecule equals the energy carried by a photon of wavelength 5000 Å.

d) At what wavelength is the linear momentum of a photon equal to that of a H_2 molecule at room temperature (***Hint****: use T_{room} = 300 K*)?

Solution

a) The energy E_γ carried by a photon with λ = 5000 Å is:

$$\mathrm{E_\gamma = h\nu = h\frac{c}{\lambda} \Longrightarrow E_\gamma \cong 3.978 \times 10^{-19}\,J \ \cong 2.49\ eV.}$$

The kinetic energy of a H_2 molecule at room temperature (consider $\mathrm{T_{room}} \cong 300$ K):

$$\mathrm{\overline{E}_{H_2} = \frac{3}{2}k_B T_{room} = \frac{3}{2}\cdot 1.38 \times 10^{-23}\ \frac{J}{K}\cdot 300\ K \Longrightarrow \overline{E}_{H_2} \cong 6.21 \times 10^{-21} J \cong 3.88 \times 10^{-2}\ eV = 0.388\ meV.}$$

To compare these two values, we calculate their ratio:

$$\mathrm{\frac{\overline{E}_{H_2}}{E_\gamma} = \frac{0.388\ meV}{2.49\ eV} = 1.559\ \times 10^{-4}.}$$

In other words, the average kinetic energy of a hydrogen molecule at room temperature is much smaller than the amount of energy carried by a photon in the visible yellow-green range.

b) The H_2 molecule kinetic energy at room temperature was calculated above:

$$\mathrm{\overline{E}_{H_2} = 6.21 \times 10^{-21}\ J.}$$

The energy of a photon of wavelength λ is: $\mathrm{E_\lambda = h\frac{c}{\lambda}}$.

Then if $E_\lambda = E_{H_2}$, one can determine the wavelength at which the photon carries the kinetic energy of a H_2 molecule at room temperature:

$$\lambda = \frac{hc}{E_{H_2}} = \frac{6.63 \times 10^{-34} \cdot 3 \times 10^{8}}{6.21 \times 10^{-21}} \cong 32\ \mu m\,.$$

This wavelength belongs to the infrared range of the emw spectrum.

c) Equating the two amounts of energy: $\frac{3}{2}k_B T = h\frac{c}{\lambda}$., the absolute temperature results then:

$$T = \frac{2}{3}\frac{hc}{k_B \lambda} = \frac{2}{3}\frac{6.63 \times 10^{-34} J \cdot s \cdot 3 \times 10^{8}\frac{m}{s}}{1.38 \times 10^{-23}\frac{J}{K} \cdot 5000 \times 10^{-10} m} \Longrightarrow T \cong 19217.4\ (K).$$

d) The kinetic energy-momentum relationship: $\bar{E}_{H_2} = \frac{{p_{H_2}}^2}{2\,m}$,

Then, using the mass of a H_2 molecule: $m = 2 \cdot m_p = 2 \cdot 1.00785 \times 10^{-27}$ kg, its momentum results:

$$p_{H_2} = \sqrt{2m \cdot \bar{E}_{H_2}} = \sqrt{3m \cdot k_B T} = \sqrt{3 \cdot 2 \cdot 1.00785 \times 10^{-27} \cdot 1.38 \times 10^{-23} \cdot 19217.4} \Longrightarrow$$

$$p_{H_2} \cong 40.05 \times 10^{-24} kg\frac{m}{s}.$$

A photon of 5000 Å wavelength has the momentum: $p_\lambda = \frac{h}{\lambda}$.

Therefore, the wavelength results:

$$\lambda = \frac{h}{p_{H_2}} = \frac{6.63 \times 10^{-34}}{40.05 \times 10^{-24}} \Rightarrow \lambda \cong 0.166\ Å.$$

Problem 145

Considering the classical definition of momentum ($p = m \cdot v$), calculate what is the de Broglie wavelength for the following objects and particles:

a) A stone of mass 0.2 kg thrown at speed 3 m/s;
b) A granule of sand of mass 20 µg carried by wind at speed 0.2 m/s;
c) An electron (mass $m_e = 9.1 \times 10^{-31}$ kg) accelerated under U = 20 V;
(***Hint***: *the kinetic energy of the accelerated electron:* $E_k = eU = 20$ *eV; 1 eV =* 1.6×10^{-19} *J).*

Solution

a) The de Broglie wavelength of a stone:

$$\lambda_B = \frac{h}{p} = \frac{h}{mv} = \frac{6.63 \times 10^{-34}\ Js}{0.2\ kg \cdot 3\ m/s} = 11.05 \times 10^{-34}\frac{J}{kg\frac{m}{s^2}} = 11.05 \times 10^{-34}\frac{N\,m}{N}$$

$$\lambda_B = 1.105 \times 10^{-33} m.$$

b) The de Broglie wavelength of a sand granule:

$$\lambda_B = \frac{h}{p} = \frac{h}{mv} = \frac{6.63 \times 10^{-34}\,\mathrm{J\,s}}{20 \times 10^{-6} \times 10^{-3}\,\mathrm{kg}\cdot 0.2\,\frac{m}{s}} = \frac{6.63}{0.4} \times \frac{10^{-34}}{10^{-8}}\,\mathrm{m}$$

$$\lambda_B = 16.575 \times 10^{-26}\ \mathrm{m}.$$

c) The de Broglie wavelength of an electron:

$$\lambda_B = \frac{h}{p} = \frac{h}{m_e v} = \frac{h}{\sqrt{2m_e E_k}} = \frac{h}{\sqrt{2m_e \cdot e \cdot U}} = \frac{6.63 \times 10^{-34}}{\sqrt{2\cdot 9.1 \times 10^{-31}\cdot 1.6 \times 10^{-19}\cdot 20}}\ \mathrm{m}.$$

$$\lambda_B = \frac{6.63 \times 10^{-34}}{\sqrt{5.824 \times 10^{-48}}}\ \mathrm{m} = \frac{6.63 \times 10^{-34}}{2.412 \times 10^{-24}}\ \mathrm{m} \Longrightarrow$$

$$\lambda_B = 2.75 \times 10^{-10}\mathrm{m} = 0.275\ \mathrm{nm}.$$

Problem 146

An electron of mass $m_0 \cong 9.1 \times 10^{-31}$ kg is accelerated at 1 eV, 10^4 eV, and 10^5 eV, respectively.

a) What are the de Broglie wavelengths that correspond to the electron at each accelerating voltage if the relativistic effects are not considered?

b) Calculate the de Broglie wavelengths of the electron corresponding to the above accelerating voltages if the relativistic effects are considered.

Solution

a) If no relativistic effects are considered, the linear momentum of the electron of energy $\mathbf{E_e}$ is:

$$p_e = \sqrt{2m\cdot E_e} = \sqrt{2m\cdot eU}.$$

Then the de Broglie wavelength associated to an electron of momentum $\mathbf{p_e}$ is given by:

$$\lambda_e = \frac{h}{p_e} = \frac{h}{\sqrt{2m\cdot E_e}}.$$

If $E_e = 1\ \mathrm{eV} = 1.6 \times 10^{-19}\ \mathrm{J}$ then ⇒ $\lambda_e = \frac{6.63\times10^{-34}\mathrm{Js}}{\sqrt{2\cdot9.1\times10^{-31}\mathrm{kg}\cdot1.6\times10^{-19}\mathrm{J}}} \cong 12.28$ Å.

If $E_e = 10^4\ \mathrm{eV}$ then ⇒ $\lambda_e \cong 0.123$ Å.

If $E_e = 10^5\mathrm{eV}$ then ⇒ $\lambda_e \cong 0.039$ Å.

b) If the relativistic effects are considered at the accelerating voltage 10^5 V, then the momentum of the electron is found as follows. The total relativistic energy W = 10^5 eV:

$$W^2 = p^2c^2 + m_0^2\cdot c^4.$$

The equation can be solved for momentum: $p^2 = \frac{W^2 - m_0^2\cdot c^4}{c^2} \Rightarrow p = \frac{\sqrt{E^2 - m_0^2\cdot c^4}}{c}$.

It results that: $p \cong 0.5 \times 10^{-22}\,\mathrm{kg}\cdot\frac{m}{s}$.

Therefore, the de Broglie wavelength corresponding to the electron accelerated under 10^5 V becomes equal to:

$$\lambda = \frac{h}{p} = \frac{6.63\cdot 10^{-34}}{0.5\cdot 10^{-22}}\ \mathrm{m} \cong 0.133\ \text{Å}$$

Problem 147

The lifetime of an excited state of an electron in an atom is $\tau = 10^{-8}$ s. Find:

a) The width of the energy level, ΔW = ?

b) The width of the spectral line emitted when the physical quantum system (electron - atom) undergoes a transition to the ground state level of energy.

c) The bandwidth of frequency relative to the photon frequency if its wavelength is 5500 Å?

d) The uncertainty in locating the photon while it is found in this state.

Solution

a) Heisenberg's principle states that:

$$\Delta p\cdot\Delta x \geq \frac{\hbar}{2} \text{ or that: } \Delta W\cdot\tau \geq \frac{\hbar}{2}$$

(***Note**: τ, or the lifetime of the excited energy state is a time interval,* $\tau = \Delta t$). Then the energy width of the excited energy level is:

$$\Delta W = \frac{\hbar}{2\tau} = \frac{6.63\times 10^{-34}}{2\cdot\pi\cdot 2\cdot 10^{-8}} \cong 0.329\times 10^{-7}\ \mathrm{eV}.$$

$$\Delta W = 3.26\ \mu\mathrm{eV}$$

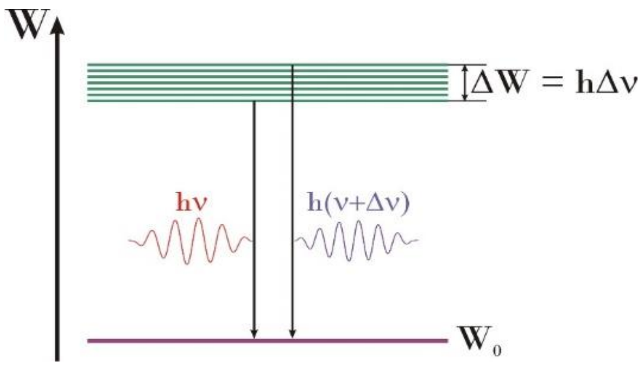


Fig. 92.The energy width of an excited state of an electron in an atom.

b) The frequency width $\Delta\nu$ of the spectral line can be calculated as the ratio between the energy width and Planck's constant:

$$\Delta\nu = \frac{\Delta W}{h} = \frac{5.275\times 10^{-26} J}{6.63\times 10^{-34} J\cdot s} \cong 7.956\times 10^{6}\ \mathrm{Hz}\,.$$

c) If the photon has the wavelength $\lambda = 5500$ Å, this means that the frequency of the radiation (released during the transition from the state in energy W to the state of energy W_0) is found from:

$$\nu = \frac{c}{\lambda} = \frac{3\times 10^{8}\mathrm{m/s}}{5500\ \text{Å}} = \frac{300}{55}\times\frac{10^{6}}{100\times 10^{-10}}\ \frac{\mathrm{m/s}}{\mathrm{m}} = 5.454\times 10^{14}\ \mathrm{s}^{-1}\,.$$

Then, the frequency width of the radiation band, relative to the transition frequency is:

$$\frac{\Delta\nu}{\nu} = \frac{7.956\ \times 10^{6}\ \mathrm{Hz}}{5.454\times 10^{14}\ \mathrm{Hz}} = 1.458\times 10^{-8} = 0.1458\times 10^{-7} \cong 0.15\times 10^{-5}\ \%$$

$$\frac{\Delta\nu}{\nu} = 1.5\ \mathrm{ppm}$$

d) The uncertainty in the electron *position* is the distance travelled by the photon (with the speed of light **c**, if in empty space) during the lifetime of the excited energy level of the electron:

$$\Delta x = c \cdot \Delta t = c \cdot \tau \Longrightarrow$$

$$\Delta x \cong 3 \text{ m}\,.$$

Problem 148

Using Wien's displacement law, calculate the temperature of a glowing heated block of metal, if the maximum intensity of the radiation emitted by this block has the wavelength $\lambda_{max} = 1.8\ \mu m$.

Solution

Wien's law for blackbody radiation:

$$\lambda_{max} T = b$$

(where Wien's constant is b = 2898 µm·K, with: 1 µm = 10^{-6} m and K = 1 Kelvin degree).

Therefore, the temperature of the glowing body is:

$$T = \frac{b}{\lambda_{max}} = \frac{2898\ \times 10^{-6} \text{m} \cdot \text{K}}{1.8 \times 10^{-6} \text{m}} \Longrightarrow T = 1610 \text{ K}\,.$$

Problem 149

The surface of a metal is illuminated with photons of green light (wavelength λ = 555 nm). One photoelectron emitted from the metal under this light has the kinetic energy E_k = 0.04 eV. Determine:

a) What is the energy carried by one of the incident photons, E_λ = ?

b) What is the energy required to remove an electron from the metal surface, E_0 = ?

c) What is the speed of the electron released by the photoelectric effect in this case?

Solution

a) The energy carried by one photon in the incident beam of light:

$$E_\lambda = h\frac{c}{\lambda} = 6.63 \times 10^{-34} \text{ Js} \cdot \frac{3 \times 10^8\ \frac{\text{m}}{\text{s}}}{555 \times 10^{-9} \text{ m}} = 3.58 \times 10^{-19} \text{ J}.$$

$$E_\lambda = 2.24 \text{ eV}$$

b) The energy of one green photon is equal to the sum of the energy required to remove the electron from the metal surface and the kinetic energy of the electron released:

$$E_\lambda = E_0 + E_k.$$

Therefore, the energy necessary to remove one electron from this metal surface is:

$$E_0 = E_\lambda - E_k = 2.24\ \text{eV} - 0.04\text{eV} \Longrightarrow E_0 = 2.2\ \text{eV}.$$

c) The speed of the electron is:

$$v = \sqrt{\frac{2E_k}{m_e}} = \sqrt{\frac{2{\cdot}0.04{\cdot}1.6 \times 10^{-19}\text{J}}{9.1 \times 10^{-31}\text{kg}}};$$

$$v = \sqrt{0.01407 \times 10^{12}}\,\frac{m}{s} \approx \sqrt{1.407} \times 10^5\,\frac{m}{s} \Longrightarrow$$

$$v \approx 1.186 \times 10^5\,\frac{m}{s}.$$

Problem 150

Tungsten has the *work function* (i.e. the minimum photon energy required to extract an electron from the metal surface) equal to 4.6 eV. Suppose that monochromatic UV radiation of frequency $\nu = 1.7{\times}10^{15}$ Hz is falling on the clean tungsten surface. Calculate:

a) The wavelength of the UV radiation $\lambda = ?$

b) The energy carried by one photon in this beam of UV radiation $E_\lambda = ?$

c) The maximum energy of the electrons released by the photoelectric effect $E_k = ?$

d) Will the photoelectric effect be still produced if IR radiation of wavelength λ = 950 nm is used instead of the UV light?

Solution

a) $\lambda = \frac{c}{\nu} = \frac{3{\times}10^8}{1.7{\times}10^{15}}\ \text{m} \Longrightarrow \lambda = 1.76 \times 10^{-7}\text{m} = 0.176\ \mu\text{m}.$

b) $E_\lambda = h\nu = 6.63 \times 10^{-34}{\cdot}1.7 \times 10^{15}\ \text{J} \Longrightarrow E_\lambda = 11.271 \times 10^{-19}\text{J} \approx 7.044\ \text{eV}.$

c) $E_k = E_\lambda - E_0 = 7.044\ \text{eV} - 4.6\ \text{eV} \Longrightarrow E_k = 2.444\ \text{eV}.$

d) The energy of an IR photon:

$$E_{IR} = h\frac{c}{\lambda} = 6.63 \times 10^{-34}\,\frac{3 \times 10^8}{950 \times 10^{-9}} = 2.09 \times 10^{-19}\text{J}\,;$$

$$E_{IR} \approx 1.308\ \text{eV}.$$

As $E_{IR} < E_0$, it means that the IR radiation is not carrying enough energy to produce the photoelectric effect when impacting on the tungsten surface.

Problem 151

The photoelectric threshold for sodium is λ_0 = 542 nm.

a) What is the *work function* for sodium? W_{ex} = ?

b) Calculate the maximum kinetic energy of the electrons released from the sodium metal surface under electromagnetic radiation of wavelength 480 μm, E_k = ?

Solution

a) The *work function* can be deduced from Einstein's equation for energy conservation, when the electron is released from the metal surface without kinetic energy ($E_k = 0$):

$$W_0 = h\frac{c}{\lambda_0} = 6.63 \times 10^{-34}\frac{3 \times 10^8}{542 \times 10^{-9}} = \frac{6.63 \cdot 3}{542} \times 10^{-17}\text{J} = 3.67 \times 10^{-19}\text{J} = \frac{3.67 \times 10^{-19}}{1.6 \times 10^{-19}}\ \text{eV},$$

$$W_0 \approx 2.29\ \text{eV}.$$

b) For an incident radiation of the given wavelength, the law of the photoelectric effect:

$$h\frac{c}{\lambda} = W_o + E_k.$$

Therefore, the maximum kinetic energy of the electrons ejected from the metal surface is:

$$E_k = h\frac{c}{\lambda} - W_0 = 6.63 \times 10^{-34}\frac{3 \times 10^8}{480 \times 10^{-9}} \cdot \frac{1}{1.6 \times 10^{-19}} - 2.29\ (\text{eV})$$

$$E_k = 2.59\ \text{eV} - 2.29\ \text{eV} \Longrightarrow E_k = 0.30\ \text{eV}.$$

Problem 152

A copper plate with the cross section of **a** × **b** (6 mm × 0.5 mm) is submitted to a magnetic field of magnetic induction B = 2 T. An electric current of intensity I = 1.5 A crosses the plate as shown in Figure 93. The Hall voltage thus generated is $\mathbf{U_H}$ = 1.54 μV. Determine:

a) The number of electrons per unit volume in the copper plate, **n** = ?

b) The average drift speed of the electrons $\mathbf{v_D}$ = ?

Solution

a) The Hall electric field generated across the Cu plate has the intensity: $E_H = \frac{U_H}{a}$.

Then the Hall voltage: $U_H = a \cdot E_H$.

The Lorentz force on the electrons as they enter the magnetic field is balanced by the electric static force exerted by the electric field generated by the Hall effect:

$$e \cdot v_D \cdot B = e \cdot E_H \Rightarrow E_H = v_D \cdot B.$$

The current density (the electric current across the unit surface of the conductor) is given by:

$$j = n \cdot e \cdot v_D \quad \Rightarrow$$

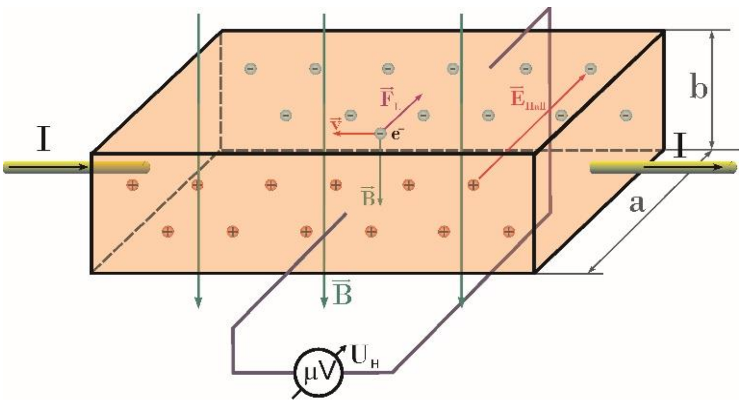


Fig. 93. The Hall effect in a slab of Cu.

$$\frac{\mathrm{I}}{\mathrm{a \cdot b}} = \mathrm{n \cdot e \cdot v_D} \Rightarrow$$

$$\mathrm{n} = \frac{\mathrm{I}}{\mathrm{a \cdot b \cdot e \cdot v_D}} = \frac{\mathrm{I}}{\mathrm{a \cdot b \cdot e \cdot \frac{E_H}{B}}}$$

$$\mathrm{n} = \frac{\mathrm{I \cdot B}}{\mathrm{a \cdot b \cdot e \cdot \frac{U_H}{a}}}.$$

The number of electrons per unit volume then becomes:

$$\mathrm{n} = \frac{\mathrm{I \cdot B}}{\mathrm{b \cdot e \cdot U_H}} = \frac{1.5 \cdot 2}{0.5 \times 10^{-3} \cdot 1.6 \times 10^{-19} \cdot 1.54 \times 10^{-6}} \Longrightarrow \mathrm{n} \approx 2.435 \times 10^{28}\ \mathrm{m^{-3}}.$$

b) From the density of current:

$$\mathrm{j = n \cdot e \cdot v_D},$$

the drift velocity can be deduced as follows:

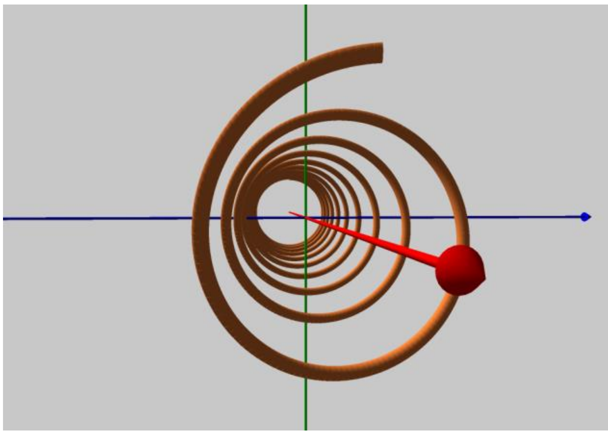
$$\mathrm{v_D} = \frac{\mathrm{j}}{\mathrm{n \cdot e}} = \frac{I}{\mathrm{a \cdot b \cdot n \cdot e}} = \frac{1.5}{6 \times 10^{-3} \cdot 0.5 \times 10^{-3} \cdot 2.435 \times 10^{28} \cdot 1.6 \times 10^{-19}} \left(\frac{\mathrm{m}}{\mathrm{s}}\right).$$

$$\mathrm{v_D} = \frac{1.5}{6 \cdot 0.5 \cdot 2.435 \cdot 1.6 \times 10^{3}} \frac{\mathrm{m}}{\mathrm{s}} \Longrightarrow$$

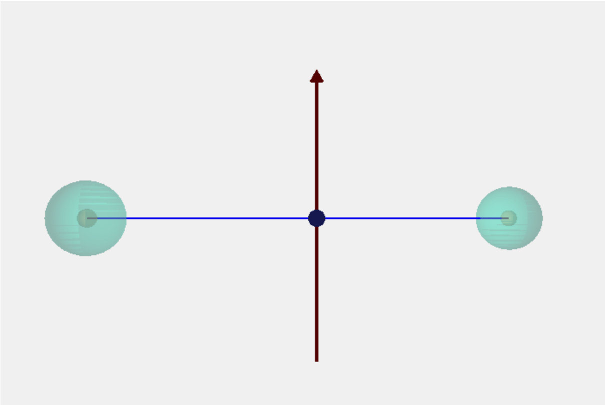
$$\mathrm{v_D} \approx 1.28 \times 10^{-4} \frac{\mathrm{m}}{\mathrm{s}} = 0.128\ \frac{\mathrm{mm}}{\mathrm{s}}.$$

11. ONLINE SIMULATIONS OF THE EXPERIMENTAL SETINGS IN THE PROBLEMS

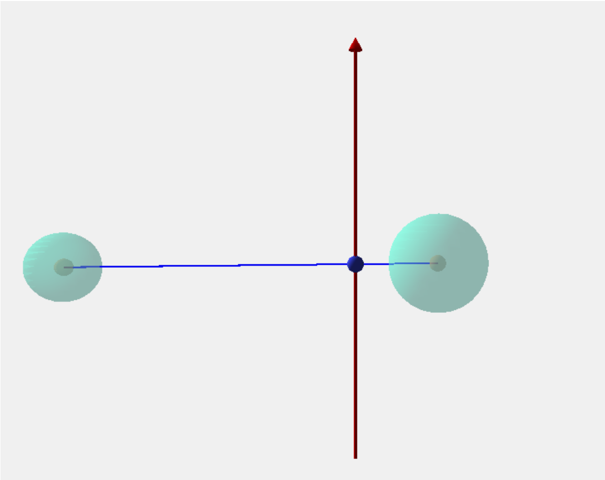
Problem 6. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig06/



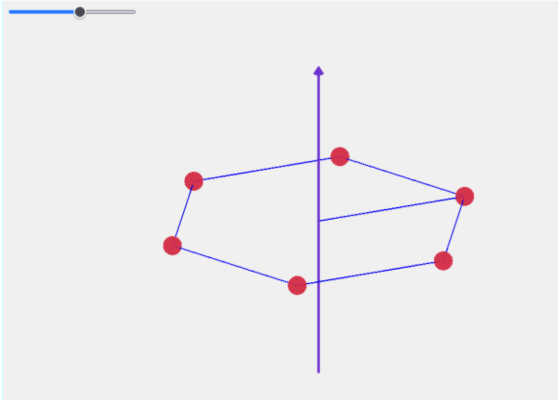
Problem 15. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig15/



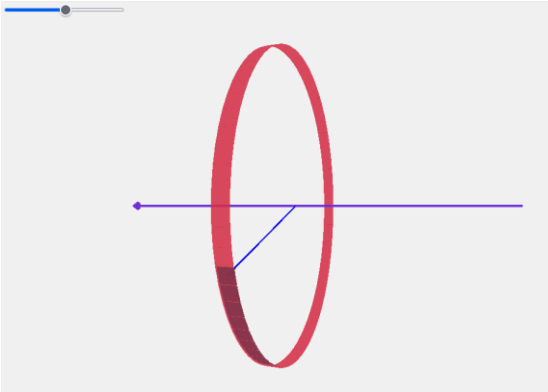
Problem 16. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig16/



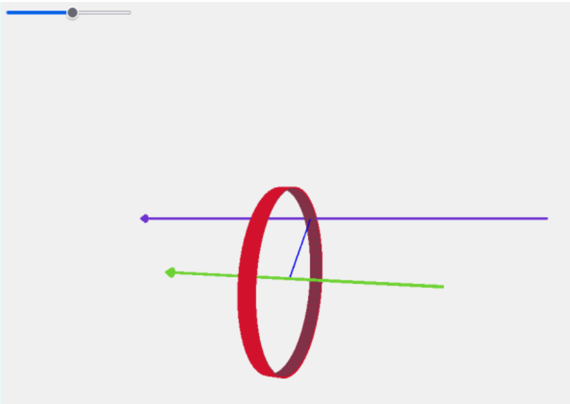
Problem 17a. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig17a/



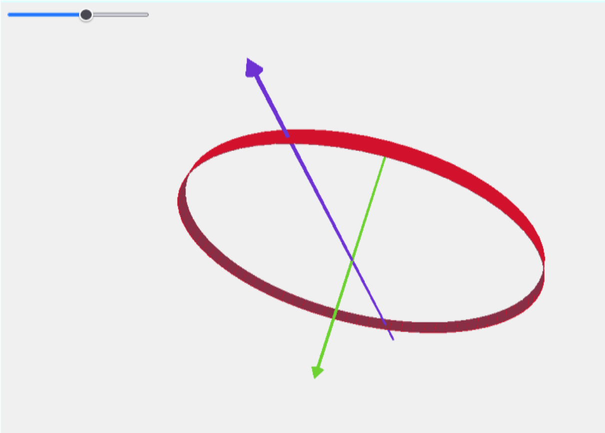
Problem 18a. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig18a/



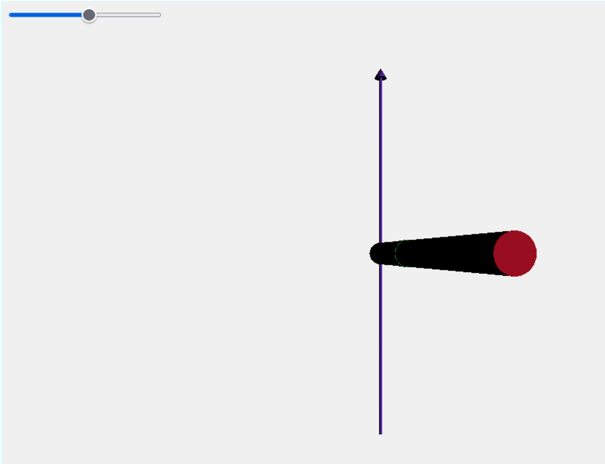
Problem 18b. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig18b/



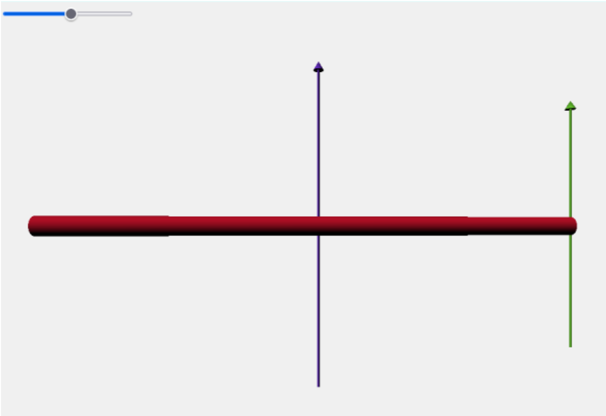
Problem 18c. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig18c/



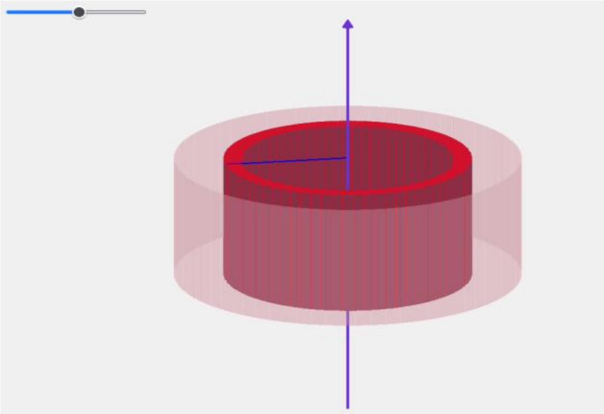
Problem 19a. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig19a/



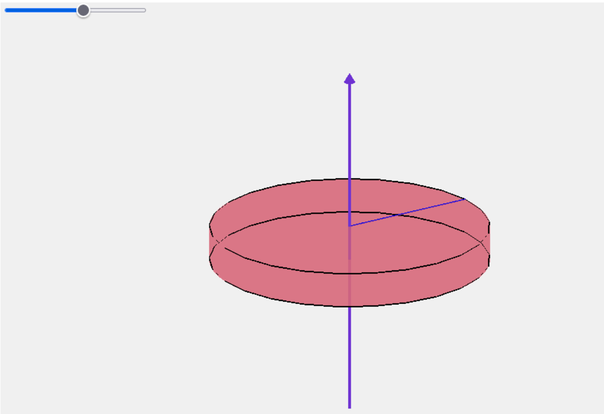
Problem 19b. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig19b/



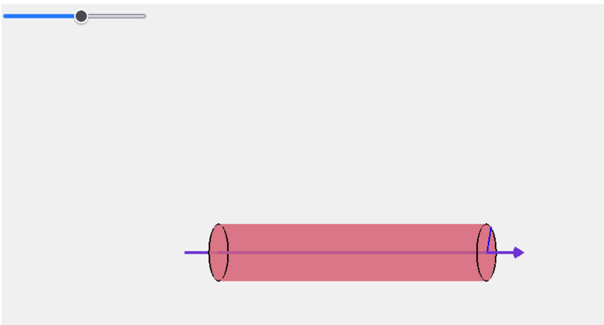
Problem 20a. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig20a/



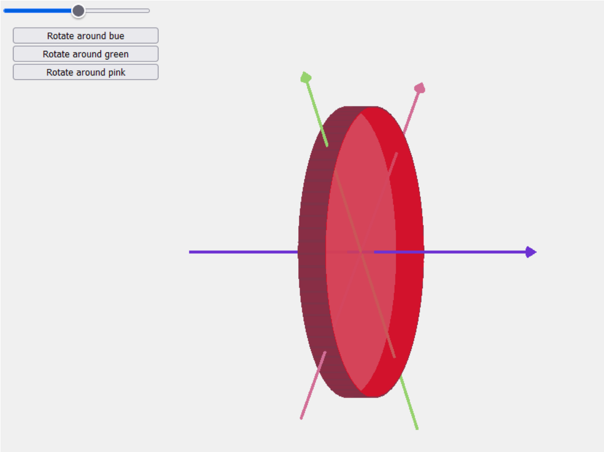
Problem 20b. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig20b/



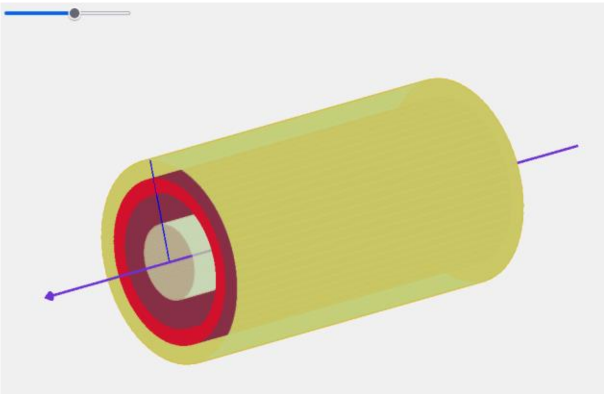
Problem 20c. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig20c/



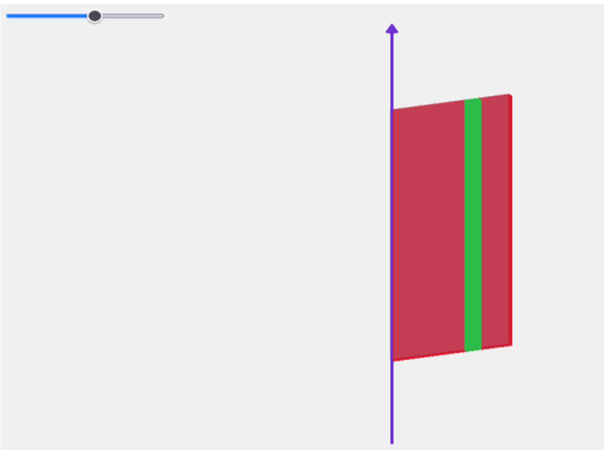
Problem 21. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig21/



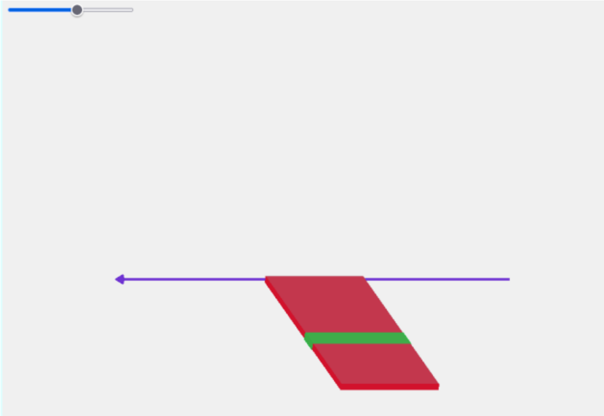
Problem 22. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig22/



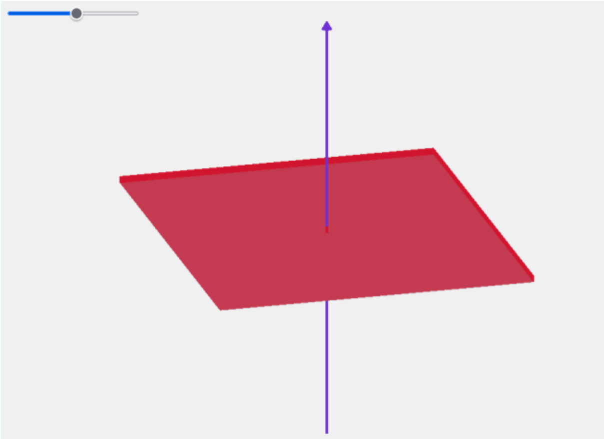
Problem 23a. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig23a/



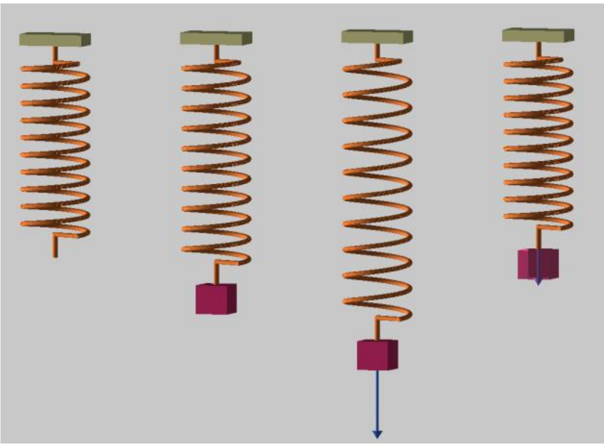
Problem 23b. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig23b/



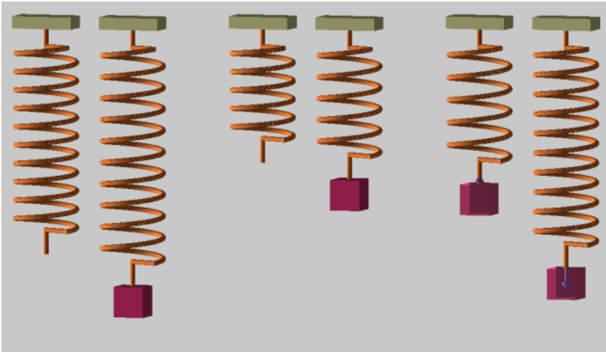
Problem 24. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig24/



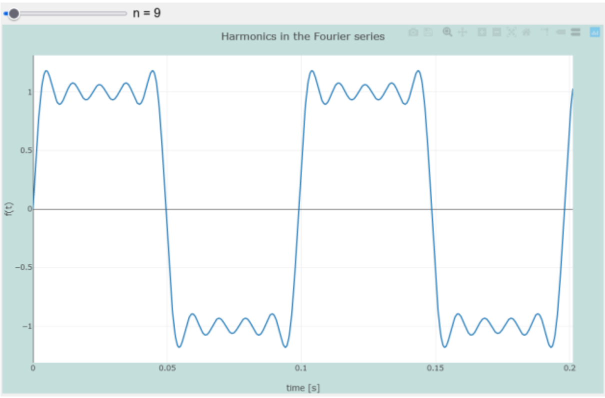
Problem 27. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig27/



Problem 28. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig28/



Problem 32. Link: https://phys.utcluj.ro/resurse/Seminarii/JS-Fig32/



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