Liana TIMBOȘ

Linear Algebra Analytic and Differential Geometry

THEORY AND SEMINAR PROBLEMS

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Preface

This book is addressed to engineering students studying Linear Algebra, Analytic and Differential Geometry in the first year of college.

The book is structured in several chapters, each of them starting with a brief presentation of theoretical notions: definitions, properties, theorems, etc., without pretending to have gone into detail in their presentation or to be rigorously proved, many of the theorems being given only as a statement. After the theoretical part follows a section in which problems solved in detail are presented and then a series of problems proposed to be solved.

The purpose of the book is for the readers to be able to go through the mathematical concepts presented as comfortably as possible, often perceived as difficult, both through their succinct presentation and their highlighting through illustrative figures and through the examples and solved problems, and then to have acquired the skills and the working techniques for solving other exercises and problems, but also for their application in engineering-specific study subjects.

Last but not least, the author gratefully acknowledge the support of Prof. Daniela Inoan and Prof. Adela Capătă who have carefully read the manuscript suggesting valuable improvements.

1

Matrices. Determinants. Systems of linear equations

1.1 Determinants

For every square matrix $A = [a_{ij}]_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}} \in \mathcal{M}_n(\mathbb{R})$ one can assign a scalar denoted $\det(A)$ called the determinant of A. In extended form we write

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Definition 1.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$. The **determinant** of A is the scalar defined by the equation*

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn } (\sigma)\, a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot \ldots \cdot a_{n\sigma(n)}.$$

Let $A \in \mathcal{M}_n(\mathbb{R})$ and let k be an integer, $1 \leqslant k \leqslant n$. Consider the rows $i_1 \ldots i_k$ and the columns $j_1 \ldots j_k$ of A. By deleting the other rows and columns we obtain a submatrix of A of order k, whose determinant is called a **minor** of A and is denoted by $M_{i_1\ldots i_k}^{j_1\ldots j_k}$. By deleting the rows $i_1 \ldots i_k$ and the columns $j_1 \ldots j_k$ of A we obtain the **complementary minor** of $M_{i_1\ldots i_k}^{j_1\ldots j_k}$ denoted by $CM_{i_1\ldots i_k}^{j_1\ldots j_k}$.

A method of calculating determinants is called row expansion and column expansion and it is derived from Laplace Theorem.

Theorem 1.2. *Let* $A \in \mathcal{M}_n(\mathbb{R})$*. Then*

(i) $\det(A) = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} CM_i^k$, *- expansion by row i;*

(ii) $\det(A) = \sum_{k=1}^{n} a_{kj}(-1)^{k+j} CM_k^j$, *- expansion by column j.*

Definition 1.3. *A square matrix* $A \in \mathcal{M}_n(\mathbb{R})$ *is called* ***singular*** *if its determinant is* 0*,* $\det(A) = 0$*. If* $\det(A) \neq 0$ *the matrix* A *is called* ***nonsingular****.*

Properties of the determinant.

Let $A, B \in \mathcal{M}_n(\mathbb{R})$ and let $a \in \mathbb{R}$. Then:

(1) $\det(A^\top) = \det(A)$.

(2) A permutation of the rows, (respectively columns) of A multiplies the determinant by the sign of the permutation.

(3) A determinant with two equal rows (or two equal columns) is zero.

(4) The determinant of A is not changed if a multiple of one row (or column) is added to another row (or column).

(5) $\det(A^{-1}) = \frac{1}{\det(A)}$.

(6) $\det(AB) = \det(A)\det(B)$.

(7) $\det(aA) = a^n \det(A)$.

(8) If A is a triangular matrix, i.e. $a_{ij} = 0$ whenever $i > j$ ($a_{ij} = 0$ whenever $i < j$), then its determinant equals the product of the diagonal entries, that is $\det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn} = \prod_{i=1}^{n} a_{ii}$.

1.2 Rank of a matrix

Rank. Elementary transformations.

A natural number r is called the **rank** of the matrix $A \in \mathcal{M}_{m,n}(\mathbb{R})$ if

1. There exists a square submatrix $M \in \mathcal{M}_r(\mathbb{R})$ of A which is nonsingular (that is $\det(M) \neq 0$).

2. If $p > r$, for every submatrix $N \in \mathcal{M}_p(\mathbb{R})$ of A one has $\det(N) = 0$.

We denote $\text{rank}\,(A) = r$.

Definition 1.4. *The following operations are called elementary row transformations on the matrix $A \in \mathcal{M}_{m,n}(\mathbb{R})$:*

1. *Interchanging of any two rows.*

2. *Multiplication of a row by any non-zero number.*

3. *The addition of one row to another row.*

Similarly one can define the elementary column transformations.

We use elementary transformation in order to compute the rank.

Namely, given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{R})$ we transform it by an appropriate succession of elementary transformations into a matrix B called the echelon form of the initial matrix, such that:

- the diagonal entries of B are either 0 or 1, all the 1's preceding all the 0's on the diagonal.
- all the other entries of B are 0.

Since the rank is invariant under elementary transformations, we have $\operatorname{rank}(A) = \operatorname{rank}(B)$, but it is clear that the rank of B is equal to the number of 1's on the diagonal.

Matrix Invertion

For a square matrix $A \in \mathcal{M}_n(\mathbb{R})$, the matrix $B \in \mathcal{M}_n(\mathbb{R})$ that satisfies

$$AB = I_n \text{ and } BA = I_n$$

(if it exists) is called the **inverse** of A and is denoted by $B = A^{-1}$. Not all square matrices admit an inverse (are invertible). An invertible square matrix is called *nonsingular* and a square matrix with no inverse is called *singular matrix.*

Theorem 1.5. *If a square matrix is reduced to the identity matrix by a sequence of elementary row operations, the same sequence of elementary row transformations performed on the identity matrix produces the inverse of the given matrix.*

1.3 Systems of linear equations

Recall that a system of m linear equations in n unknowns can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases}$$

Here $x_1, x_2, \dots, x_n$ are the unknowns, $a_{11}, a_{12}, \dots, a_{mn}$ are the coefficients of the system, and $b_1, b_2, \dots, b_m$ are the constant terms.

A systems of linear equations may be written as $Ax = b$, with $A = (a_{ij})_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}} \in \mathcal{M}_{m,n}(\mathbb{R})$, $x \in \mathcal{M}_{n,1}(\mathbb{R})$ and $b \in \mathcal{M}_{m,1}(\mathbb{R})$.

The matrix A is called the *coefficient matrix*, while the matrix $[A|b] \in \mathcal{M}_{m,n+1}(\mathbb{R})$,

$$[A|b]_{ij} = \begin{cases} a_{ij} \text{ if } j \neq n+1 \\ b_i \text{ if } j = n+1 \end{cases}$$

is called the *augmented matrix* of the system.

We say that $x_1, x_2, ..., x_n$ is a **solution** of a linear system if $x_1, x_2, ..., x_n$ satisfy each equation of the system. A linear system is **consistent** if it has a solution, and **inconsistent** otherwise. According to the Rouché-Capelli theorem, a system of linear equations is:

- **inconsistent** if $rank(\overline{A}) > rank(A)$, which means that the system has no solution.
- **consistent** if $rank(\overline{A}) = rank(A)$, which means that the system must have at least one solution.
  - The solution is **unique** if and only if $\operatorname{rank} A = n$, where n is the number of variable.
  - The system has **infinitely many solutions** if $\operatorname{rank} A < n$. In this case the general solution has k free parameters where $k = n - \operatorname{rank} A$.

In row reduction, the linear system is represented as an augmented matrix $[A|b]$. This matrix is then modified using elementary row operations until it reaches reduced row echelon form. Because these operations are reversible, the augmented matrix produced always represents a linear system that is equivalent to the original. In this way one can easily read the solutions.

A homogeneous system is equivalent to a matrix equation of the form

$$Ax = O,$$

where $O \in \mathcal{M}_{m,1}$ is the matrix having all the entries zeros. Obviously a homogeneous system is consistent, having the trivial solution $x_1 = x_2 = \cdots = x_n = 0$.

It can be easily realized that a homogeneous linear system has a non-trivial solution if and only if the number of leading coefficients in echelon form is less than the number of unknowns, in other words, the coefficient matrix is singular.

1.4 Solved Problems

Problem 1.1. Compute the following determinant $D = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 1 \\ -1 & 4 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{vmatrix}$.

Solution: We will apply the expansion by a row/column. In order to do that, is more efficient to use properties of determinants to obtain on a row or column as many of zero's we can. So, we choose a_{11} as leading coefficient and we transform the elements of the first column in 0.

$$D = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 1 \\ -1 & 4 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{vmatrix} \overset{-2R_1+R_2, R_1+R_3}{=} \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -5 \\ 0 & 6 & -2 & 3 \\ 0 & 2 & 1 & 1 \end{vmatrix} =$$

$$= 1\cdot(-1)^{1+1}\begin{vmatrix} -5 & 5 & -5 \\ 6 & -2 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 5\begin{vmatrix} -1 & 1 & -1 \\ 6 & -2 & 3 \\ 2 & 1 & 1 \end{vmatrix} \overset{C_2+C_1, C_2+C_3}{=} 5\begin{vmatrix} 0 & 1 & 0 \\ 4 & -2 & 1 \\ 3 & 1 & 2 \end{vmatrix} =$$

$$= 5\cdot 1\cdot(-1)^{1+2}\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = -5(8-3) = -25.$$

Problem 1.2. Solve the equation $\begin{vmatrix} -2-a & -1 & 1 \\ 5 & -1-a & 4 \\ 5 & 1 & 2-a \end{vmatrix} = 0.$

Solution: Of course, we can apply triangle or Sarrus rule, but, is much easier if we apply properties of determinants so that we have the decomposition of the determinant into factors.

$$\begin{vmatrix} -2-a & -1 & 1 \\ 5 & -1-a & 4 \\ 5 & 1 & 2-a \end{vmatrix} = 0 \overset{C_2+C_3}{\Longleftrightarrow} \begin{vmatrix} -2-a & -1 & 0 \\ 5 & -1-a & 3-a \\ 5 & 1 & 3-a \end{vmatrix} = 0 \Longleftrightarrow$$

$$(3-a)\begin{vmatrix} -2-a & -1 & 0 \\ 5 & -1-a & 1 \\ 5 & 1 & 1 \end{vmatrix} = 0 \overset{-R_2+R_3}{\Longleftrightarrow} (3-a)\begin{vmatrix} -2-a & -1 & 0 \\ 5 & -1-a & 1 \\ 0 & 2+a & 0 \end{vmatrix} = 0 \Longleftrightarrow$$

$$(3-a)(2+a)(-1)^{3+2}\begin{vmatrix} -2-a & 0 \\ 5 & 1 \end{vmatrix} = 0 \Longleftrightarrow -(3-a)(2+a)(-2-a) = 0 \Longrightarrow$$

$a \in \{-2, 3\}$.

Problem 1.3. Compute the rank of the following matrices using Gauss-Jordan elimination method.

$$A = \begin{bmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{bmatrix}.$$

Solution: We will apply row transformation so that the last matrix fulfill the two conditions:

C.1 The elements below the diagonal are all zero.

C.2 All non-zero elements on the diagonal are in front of the zeroes.

$$\begin{bmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \overset{-R_1+R_3, \frac{1}{2}R_1}{\simeq} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 & -1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \overset{-R_2+R_3, -R_2+R_4}{\simeq}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of A is $\text{rank}\,(A) = 3$ (we count the non-zero rows in the last form, after we check the conditions **C.1** and **C.2**).

For the matrix B we will apply the same procedure.

$$\begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\simeq} \begin{bmatrix} 1 & 3 & 4 & -2 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & 3 & -1 \\ 4 & -3 & 1 & 1 \end{bmatrix} \overset{-3R_1+R_2, -2R_1+R_3, -4R_1+R_4}{\simeq}$$
$$\begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -10 & -10 & 6 \\ 0 & -5 & -5 & 3 \\ 0 & -15 & -15 & 9 \end{bmatrix} \overset{-\frac{1}{2}R_2+R_3, -3R_2+2R_4}{\simeq} \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & -10 & -10 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The conditions **C.1** and **C.2** are fulfilled so, $\text{rank}\,(B) = 2$.

Problem 1.4. Find the inverses of the matrix A by using the Gauss-Jordan elimination method, $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -1 & 3 & -1 \\ -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 1 \end{bmatrix}$.

Solution: We will apply row transformation to the matrix A and to I_4 so that the matrix A is transformed into identity matrix I_4. The matrix in which I_4 changes after the same succession of row transformations will be the inverse of A, i.e. A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_1+R_3,-2R_1+R_4}{\simeq}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -3 & -2 & 0 & 0 & 1 \end{array}\right] \overset{R_2+R_4}{\simeq}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -4 & -2 & 1 & 0 & 1 \end{array}\right] \overset{\frac{1}{2}R_4,R_3\leftrightarrow R_4,-R_2}{\simeq}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \overset{2R_4+R_3,-R_4+R_2,-2R_4+R_1}{\simeq}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \overset{3R_3+R_2}{\simeq}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 2 & \frac{1}{2} & 5 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \overset{R_2+R_1}{\simeq} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & 3 & \frac{3}{2} \\ 0 & 1 & 0 & 0 & 2 & \frac{1}{2} & 5 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right]$$

We obtain in the left hand side the identity matrix I_4, so $A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 3 & \frac{3}{2} \\ 2 & \frac{1}{2} & 5 & \frac{3}{2} \\ 1 & \frac{1}{2} & 2 & \frac{1}{2} \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

Problem 1.5. Solve the following systems of linear equations by using Gauss-Jordan elimination method.

$$(S_1) \begin{cases} x_1 + 2x_2 + 4x_3 = 4 \\ 5x_1 + x_2 + 2x_3 = -7 \\ 3x_1 - x_2 + x_3 = -6 \end{cases}$$

$$(S_2) \begin{cases} x - y + z + 2t = 1 \\ -2x + 2y - 3z + 3t = 2 \\ x - y + 2z + 5t = -1 \\ -x + y - 3z + 2t = 4 \end{cases}$$

$$(S_3) \begin{cases} x_1 - 2x_2 + 3x_3 + 4x_4 = 0 \\ -x_1 + x_2 - x_3 - 2x_4 = 0 \\ x_2 - 2x_3 - 2x_4 = 0 \\ x_1 - 3x_2 + 5x_3 + 6x_4 = 0 \end{cases}$$

Solution:

(S_1): We write the system (S_1) using the matrix form and we will transform it using elementary row transformation so we can read easier the rank of the system matrix and the rank of the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 5 & 1 & 2 & -7 \\ 3 & -1 & 1 & -6 \end{array}\right] \overset{-5R_1+R_2,-3R_1+R_3}{\simeq} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & -9 & -18 & -27 \\ 0 & -7 & -11 & -18 \end{array}\right] \overset{-\frac{1}{9}R_2}{\simeq}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -7 & -11 & -18 \end{array}\right] \overset{7R_2+R_3}{\simeq} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{array}\right] \overset{-\frac{1}{3}R_3,-2R_3+R_2,-4R_3+R_1}{\simeq}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \overset{-2R_2+R_1}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\operatorname{rank}(A) = \operatorname{rank}(\overline{A}) = 3$, we have 3 unknowns, so the system have a unique solution and we can read it from the last form, which is $x_1 = -2$, $x_2 = 1$, $x_3 = 1$.

$$(S_2): \left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 1 \\ -2 & 2 & -3 & 3 & 2 \\ 1 & -1 & 2 & 5 & -1 \\ -1 & 1 & -3 & 2 & 4 \end{array}\right] \overset{2R_1+R_2,-R_1+R_3,R_1+R_4}{\simeq}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 7 & 4 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & -2 & 4 & 5 \end{array}\right] \overset{R_2+R_3,-2R_2+R_4}{\simeq}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 7 & 4 \\ 0 & 0 & 0 & 10 & 2 \\ 0 & 0 & 0 & -10 & -3 \end{array}\right] \overset{R_3+R_4}{\simeq} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 7 & 4 \\ 0 & 0 & 0 & 10 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

We can observe that $\operatorname{rank}(A) = 3$, $\operatorname{rank}(\overline{A}) = 4$, so the system is inconsistent.

Remark: if we rewrite the system from the last matrix form, we will get

$$(S_2)\begin{cases} x - y + z + 2t = 1 \\ -3z + 7t = 4 \\ 10t = 2 \\ 0 = -1 \end{cases}$$

and is obviously that the last equation is false, so it doesn't exists $x, y, z, t \in \mathbb{R}$ such that the equations of (S_2) are fulfilled.

(S_3): This system is an homogenous one, so it will have at least the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$. We verify the rank of this matrix to determine if the system has another solutions.

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ -1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -2 & -2 & 0 \\ 1 & -3 & 5 & 6 & 0 \end{array}\right] \overset{R_1+R_2, -R_1+R_4}{\simeq}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & -1 & 2 & 2 & 0 \\ 0 & 1 & -2 & -2 & 0 \\ 0 & -1 & 2 & 2 & 0 \end{array}\right] \overset{R_2+R_3, -R_2+R_4}{\simeq} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & -1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$\operatorname{rank}(A) = \operatorname{rank}(\overline{A}) = 2$, but we have 4 unknowns. So, 2 of them will become free variables, for example $x_3 = \alpha$, $x_4 = \beta$ (the rank is 2 because the minor formed from the coefficients of x_1 and x_2 and the first two equations is not zero, $\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$, so, x_1 and x_2 remain unknowns).

We rewrite the system from the last matrix form and we will get:

$$(S_3)\begin{cases} x_1 - 2x_2 = -3\alpha - 4\beta \\ -x_2 = -2\alpha - 2\beta \end{cases}.$$

We can calculate $x_1 = \alpha$ and $x_2 = 2\alpha + 2\beta$. So, the general solution of the system (S_3) is $S_3 = \{(\alpha, 2\alpha + 2\beta, \alpha, \beta) | \alpha, \beta \in \mathbb{R}\}$.

1.5 Problems

Problem 1.6. Compute the following determinants.

$$D_1 = \begin{vmatrix} 4-x & -5 & 2 \\ 5 & -7-x & 3 \\ 6 & -9 & 4-x \end{vmatrix}, \ D_2 = \begin{vmatrix} 1-x & -1 & -1 \\ -3 & -4-x & -3 \\ 4 & 7 & 6-x \end{vmatrix},$$

$$D_3 = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix}, \ D_4 = \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & -1 & 3 & -1 \\ -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 1 \end{vmatrix}.$$

Problem 1.7. Compute the rank of the following matrices by using the Gauss-Jordan elimination method.

$$A = \begin{bmatrix} 2 & -3 & 0 & 4 \\ 1 & -1 & 5 & 2 \\ 5 & -7 & 5 & 10 \end{bmatrix}, \ B = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 1 & 4 & -1 \\ 0 & -1 & 8 & 5 \\ 2 & -2 & 4 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 1 & 0 & -1 \\ -1 & 2 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & -4 & 2 & 0 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 5 & -8 & -5 & -12 \\ 3 & -7 & 8 & 9 & 13 \end{bmatrix}.$$

Problem 1.8. Solve the following systems of linear equations by using Gauss-Jordan elimination method.

$$(S_1)\begin{cases} x+2y+4z-3t=0\\ 3x+5y+6z-4t=0\\ 3x+8y+24z-19t=0\\ 4x+5y-2z+3t=0\end{cases};\qquad (S_2)\begin{cases} x+y-z-t=0\\ -x-y+2z+t=0\\ x+y+z-t=0\end{cases};$$

$$(S_3)\begin{cases} x_1+x_2+x_3-2x_4-3x_5=0\\ -x_1+3x_2-x_3-2x_4-x_5=0\\ -x_1+x_2-x_3+x_5=0\\ x_1+x_2-x_4-x_5=0\end{cases};\quad (S_4)\begin{cases} x+2y-3z=0\\ x-y+2z-t=0\\ -2x-y+z+t=0\\ -x-8y+13z-2t=0\end{cases}.$$

Problem 1.9. Find the inverses of the following matrices by using the Gauss-Jordan elimination method.

$$A=\begin{bmatrix} 2 & 2 & 3\\ 1 & -1 & 0\\ -1 & 2 & 1\end{bmatrix},\ B=\begin{bmatrix} 1 & 2 & 3\\ 0 & 1 & 2\\ 2 & 2 & 1\end{bmatrix},\ C=\begin{bmatrix} 1 & 1 & 0 & 1\\ 1 & 0 & 0 & -1\\ 1 & -1 & 0 & -1\\ 0 & 0 & -1 & 1\end{bmatrix},$$

$$D=\begin{bmatrix} 1 & -1 & 0 & 2\\ 0 & -1 & 3 & -1\\ -1 & 1 & 0 & -1\\ 2 & -1 & -1 & 1\end{bmatrix}.$$

2

Vector spaces

2.1 Definitions and Properties

Definition 2.1. *A* ***vector space*** V *over a field* $\mathbb{F}$ *(or* $\mathbb{F}$ *vector space) is a set* V *with an addition* $+$ *(internal composition law) such that* $(V, +)$ *is an abelian group and a scalar multiplication* $\cdot : \mathbb{F} \times V \to V, (\alpha, v) \to \alpha \cdot v = \alpha v$*, satisfying the following properties:*

(1) $\alpha(v + w) = \alpha v + \alpha w,\ \forall \alpha \in \mathbb{F}, \forall v, w \in V$

(2) $(\alpha + \beta)v = \alpha v + \beta v, \forall \alpha, \beta \in \mathbb{F}, \forall v \in V$

(3) $\alpha(\beta v) = (\alpha\beta)v$

(4) $1 \cdot v = v, \forall v \in V$

- The elements of V are called **vectors**.
- The elements of $\mathbb{F}$ are called **scalars**.
- The scalar multiplication depends upon $\mathbb{F}$.
- If $\mathbb{F} = \mathbb{R}$ we have the vector space over $\mathbb{R}$ or the real vector space.

- If $\mathbb{F} = \mathbb{C}$ we have the vector space over $\mathbb{C}$ or the complex vector space.

In what follows we will consider $\mathbb{F} = \mathbb{R}$, so we will deal with the real vector space.

Remark. From the definition of a vector space V over $\mathbb{R}$ the following rules for calculus are easily deduced:

- $\alpha \cdot 0_V = 0_V$
- $0_{\mathbb{R}} \cdot v = 0_V$
- $\alpha \cdot v = 0_V \Rightarrow \alpha = 0_{\mathbb{R}}$ or $v = 0_V$.

Definition 2.2. *Let V be a vector space over $\mathbb{R}$. A subset $U \subset V$ is called **subspace** of V over $\mathbb{R}$ if it is stable with respect to the composition laws, that is,*

1. $v + u \in U,\ \forall v, u \in U$
2. $\alpha v \in U,\ \forall \alpha \in \mathbb{R}, v \in U$

and the induced operations verify the properties from the definition of a vector space over $\mathbb{R}$.

Propozition 2.3. *Let V be a $\mathbb{R}$ vector space and $U \subset V$ a nonempty subset. U is a **vector subspace** of V over $\mathbb{R}$ iff the following conditions are met:*

- $v - u \in U, \forall v, u \in U$
- $\alpha v \in U, \forall \alpha \in \mathbb{R}, \forall v \in U$

Propozition 2.4. *Let V be a $\mathbb{R}$ vector space and $U \subset V$ a nonempty subset. U is a vector subspace of V over $\mathbb{R}$ iff*

$$\alpha v + \beta u \in U,\ \ \forall \alpha, \beta \in \mathbb{R},\ \ \forall u, v \in U.$$

Remark 2.5. *$0_{\mathbb{R}}$ is a scalar, so $\forall u \in U$ we have that $0_{\mathbb{R}} \cdot u = 0_V \in U$. Therefore each vector subspace of V has at least one element, namely 0_V.*

Propozition 2.6. *Let V be a vector space and $U, W \subset V$ two vector subspaces. The sets*

$$U \cap W = \{v | v \in U \text{ and } v \in W \}$$

and

$$U + W = \{u + w | u \in U, w \in W\}$$

are subspaces of V.

- The subspace $U \cap W$ is called the intersection vector subspace.
- The subspace $U + W$ is called the sum vector subspace.

Definition 2.7. *Let V be a vector space and $U_1, U_2 \subset V$ subspaces. The sum $U_1 + U_2$ is called* ***direct sum*** *and is denoted by $U_1 \bigoplus U_2$, if every $u \in U_1 + U_2$ can be written uniquely as $u = u_1 + u_2$ where $u_1 \in U_1, u_2 \in U_2$.*

Propozition 2.8. *Let V be a vector space and $U, W \subset V$ be subspaces. The sum $U + W$ is a direct sum iff $U \cap W = \{0_V\}$.*

Definition 2.9. *The sum $\alpha_1 v_1 + \alpha_2 v_2 \ldots \alpha_n v_n$ is called a* ***linear combination*** *of $v_1, \ldots, v_n \in V$, V is $\mathbb{R}$ vector space with scalars $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}$.*

Definition 2.10. *A nonempty set $L = \{v_1, \ldots, v_n\} \subset V$ is called a* ***linearly independent set of vectors*** *if*

$$\alpha_1 v_1 + \ldots \alpha_n v_n = 0_V \Longrightarrow \alpha_i = 0$$

for all $i = \overline{1, n}$, $\alpha_i \in \mathbb{R}$.

A nonempty set of vectors which is not linearly independent is called linearly dependent.

Definition 2.11. *Let V be an $\mathbb{R}$ vector space. A nonempty set $S \subset V$ is called* ***system of generators*** *for V if for every $v \in V$ there exists a finite subset $\{v_1, \ldots, v_n\} \subset V$ and the scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.*

Propozition 2.12. *Let V be a vector space over $\mathbb{R}$ and $U \subset V$ nonempty, $U = \{v_1, v_2, \ldots v_n\}$. The set*

$$\langle U \rangle = \left\{ \sum_{i=1}^{n} \alpha_i v_i : \ \alpha_i \in \mathbb{R} \text{ and } v_i \in U, \forall i = \overline{1,n},\ n \in \mathbb{N} \right\}$$

is a vector subspace over $\mathbb{R}$ of V.

The set $\langle U \rangle$ is the subspace generated by U and is also denoted by

$$\langle U \rangle = \operatorname{span}\{v_1, v_2, \ldots v_n\} = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n | \alpha_i \in \mathbb{R},\ i = \overline{1,n}\}.$$

Definition 2.13. *A subset $B \subset V$ is called* ***basis*** *of V if it is both a system of generators and linearly independent. In this case every vector $v \in V$ can be uniquely written as a linear combination of vectors from B.*

Some important theorems regarding the notion of basis are enumerated in the sequel.

If V is a finitely generated $\mathbb{R}$ vector space and S a finite system of generators of V then:

- Every vector space $V \neq 0$ has a basis.
- From every finite system of generators S, $S \neq \{0\}$ one can extract a basis.
- Every linearly independent set $L \subset S$ can be completed to a basis of V.
- Every basis of V is finite and has the same number of elements.

Definition 2.14. *Let $V \neq \{0\}$ be an $\mathbb{R}$ vector space finitely generated. The number of elements in a basis of V is called **the dimension** of V, is denoted by $\mathbf{dim}_{\mathbb{R}}V$, and it does not depend on the choice of the basis.*

For $V = \{0\}$, $dim_{\mathbb{R}}V = 0$.

Corolary 2.15. *Let V be a vector space over $\mathbb{R}$ of finite dimension, $dim_{\mathbb{R}}V = n$.*

1. *Any linearly independent system of n vectors is a basis. Any system of m vectors, $m > n$ is linearly dependent.*

2. *Any system of generators of V which consists of n vectors is a basis. Any system of m vectors, $m < n$ is not a system of generators.*

Remark 2.16. The dimension of a finite dimensional vector space is equal to any of the following:

- The number of the vectors in a basis.
- The minimal number of vectors in a system of generators.
- The maximal number of vectors in a linearly independent system.

Theorem 2.17. *If U and W are two subspaces of a finite dimensional vector space V, then*

$$dim(U+W) = dim U + dim W - dim(U \cap W) \ .$$

Remark 2.18. *For $V = \mathbb{R}^n$ the vector space over $\mathbb{R}$, a vector $x \in \mathbb{R}^n$ has the form $x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$, or $x = (x_1, x_2, \dots, x_n)$. We will use both of the notations in what follows, as is convenient.*

The internal operation + is defined by

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \dots \\ x_n + y_n \end{bmatrix},$$

where $y = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$.

The scalar multiplication (the external operation) is defined by

$$\alpha x = \alpha \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \dots \\ \alpha x_n \end{bmatrix}, \alpha \in \mathbb{R}.$$

The dimension of $\mathbb{R}^n$ *is* $dim \mathbb{R}^n = n$*, and the canonical basis of* $\mathbb{R}^n$ *is*

$$B_e = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \end{bmatrix} \right\}.$$

e_i *is such that has a* 1 *on the i'th position and* 0 *in the rest.*

2.2 Local computations

Let $\mathbb{R}^n$ be the $\mathbb{R}$ vector space, with the basis $B = \{e_1, \ldots, e_n\}$. Any vector $v \in \mathbb{R}^n$ can be uniquely represented as

$$v = \sum_{i=1}^{n} a_i e_i = a_1 e_1 + \cdots + a_n e_n.$$

The scalars $a_1, \ldots, a_n$ are called the coordinates of the vector v in the basis B.

If we have another basis $B^{'} = \{e_1^{'}, \ldots, e_n^{'}\}$, the coordinates of the same vector in the new basis change.

We have

$$v = a_1 e_1 + \cdots + a_n e_n = b_1 e_1^{'} + \cdots + b_n e_n^{'}.$$

$v_B = \begin{bmatrix} a_1 \\ a_2 \\ \ldots \\ a_n \end{bmatrix}$ is the representation of the vector v in the basis B and

$v_{B'} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$ is the representation of the vector v in the new basis $B^{'}$.

If we consider the representation of the vectors $e_1^{'}, \ldots, e_n^{'}$ with respect to the basis B, we have:

$$\begin{aligned} e_1^{'} &= a_{11} e_1 + \cdots + a_{1n} e_n \\ &\ldots \\ e_n^{'} &= a_{n1} e_1 + \cdots + a_{nn} e_n \end{aligned}$$

Let $A = [a_{ij}]_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}}$ be the matrix formed by the coefficients in the above equations.

The columns of this matrix are given by the coordinates of the vectors of the new basis $B^{'}$ with respect to the old basis B.

The matrix A^t - i.e. the transpose of the matrix A - is called **the transition matrix from B to $B^{'}$** and is denoted by $P^{B',B}$.

Remarks

- The transition matrix from a new basis $B_v = \{v_1, v_2, \ldots, v_n\}$ to the canonical basis $B_e = \{e_1, e_2, \ldots, e_n\}$ is the matrix having as columns the components of the vectors $v_1, v_2, \ldots, v_n$.

- If we consider the change of the basis from $B^{'}$ to B with the matrix $P^{B,B^{'}}$ and the change of the basis from $B^{''}$ to $B^{'}$ with the matrix $P^{B^{'},B^{''}}$, the change of the basis from $B^{''}$ to B is:

$$P^{B,B^{''}} = P^{B,B^{'}} \cdot P^{B^{'},B^{''}}.$$

- If $B^{''} = B$ one has

$$P^{B,B^{'}} P^{B^{'},B} = I_n \Longleftrightarrow (P^{B^{'},B})^{-1} = P^{B,B^{'}}.$$

- The relation between the representation of the same vector in two bases, B and $B^{'}$ is:

$$v_B = P^{B,B^{'}} \cdot v_{B'}.$$

2.3 Solved Problems

Problem 2.1. Determine which of the following sets are vector subspaces of $\mathbb{R}^3$ over $\mathbb{R}$:

- $S_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 - 5x_2 + 4x_3 = 0\}$;

- $S_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 - 5x_2 + 4x_3 = 1\}$;
- $S_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \frac{x_1+4}{-2} = \frac{x_2-3}{3} = \frac{x_3+1}{5}\}$;
- $S_4 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \frac{x_1}{-2} = \frac{x_2}{3} = \frac{x_3}{5}\}$;
- $S_5 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | |x_2| = x_1 + x_3\}$;
- $S_6 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 - 3x_2 + 2x_3^2 = 0\}$.

Solution: Using the Remark 2.5 we can easily observe that $(0, 0, 0)$ is not an element for S_2 and S_3, so these two sets are not vector subspaces of $\mathbb{R}^3$ over $\mathbb{R}$.

It can be proven that only the sets having the conditions given as an homogeneous system of linear equations are vector subspaces of $\mathbb{R}^n$. So, the sets S_5 and S_6 are not vector subspaces of $\mathbb{R}^3$. In what follows we prove that S_5 and S_6 are not vector subspaces of $\mathbb{R}^3$.

Because in the definition 2.2 it is use $\forall$, we can prove that S_5 and S_6 are not vector subspaces of $\mathbb{R}^3$ by just choosing some examples that doesn't satisfy at least one of the conditions in the definition.

For S_5 let's choose $u = (1, 4, 3) \in S_5$ and $v = (1, -4, 3) \in S_5$, but we can see that $u + v = (2, 0, 6) \notin S_5$ (because $|0| \neq 2 + 6$).

For S_6 we can choose $u = (1, 1, 1) \in S_5$ and $v = (1, 1, -1) \in S_6$, but it is clear that $u + v = (2, 2, 0) \notin S_6$ (because $2 - 3 \cdot 2 + 2 \cdot 0^2 \neq 0$).

Let's prove now that S_1 and S_4 are vector subspaces of $\mathbb{R}^3$ over $\mathbb{R}$.

S_1: We will use the Proposition 2.4. So, let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in S_1$, so we have $x_1 - 5x_2 + 4x_3 = 0$ and $y_1 - 5y_2 + 4y_3 = 0$. That means that

$\alpha(x_1 - 5x_2 + 4x_3) = 0$ and $\beta(y_1 - 5y_2 + 4y_3) = 0$ for all $\alpha, \beta \in \mathbb{R} \Longleftrightarrow$

$\alpha(x_1 - 5x_2 + 4x_3) + \beta(y_1 - 5y_2 + 4y_3) = 0$ for all $\alpha, \beta \in \mathbb{R} \Longleftrightarrow$

$\alpha x_1 - 5\alpha x_2 + 4\alpha x_3 + \beta y_1 - 5\beta y_2 + 4\beta y_3 = 0$ for all $\alpha, \beta \in \mathbb{R} \Longleftrightarrow$

$\alpha x_1 + \beta y_1 - 5(\alpha x_2 + \beta y_2) + 4(\alpha x_3 + \beta y_3) = 0$ for all $\alpha, \beta \in \mathbb{R} \Longleftrightarrow$

$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) \in S_1$, so S_1 is a vector subspace of $\mathbb{R}^3$ over $\mathbb{R}$.

S_4: We will rewrite the condition in the definition of S_4, i.e. $\frac{x_1}{-2} = \frac{x_2}{3} = \frac{x_3}{5}$ equivalently $\begin{cases} 3x_1 + 2x_2 = 0 \\ 5x_2 - 3x_3 = 0 \end{cases}$.

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in S_4$, so we have $\begin{cases} 3x_1 + 2x_2 = 0 \\ 5x_2 - 3x_3 = 0 \end{cases}$

and $\begin{cases} 3y_1 + 2y_2 = 0 \\ 5y_2 - 3y_3 = 0 \end{cases} \Longleftrightarrow \begin{cases} 3x_1 + 2x_2 + 3y_1 + 2y_2 = 0 \\ 5x_2 - 3x_3 + 5y_2 - 3y_3 = 0 \end{cases} \Longleftrightarrow$

$\begin{cases} 3(x_1 + y_1) + 2(x_2 + y_2) = 0 \\ 5(x_2 + y_2) - 3(x_3 + y_3) = 0 \end{cases} \Longleftrightarrow x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in S_4.$

Let $x = (x_1, x_2, x_3)$ and $\alpha \in \mathbb{R}$.

$\begin{cases} 3x_1 + 2x_2 = 0 \\ 5x_2 - 3x_3 = 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha(3x_1 + 2x_2) = 0 \\ \alpha(5x_2 - 3x_3) = 0 \end{cases} \Longleftrightarrow \begin{cases} 3\alpha x_1 + 2\alpha x_2 = 0 \\ 5\alpha x_2 - 3\alpha x_3 = 0 \end{cases} \Longleftrightarrow$

$\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3) \in S_4$.

We just proved both $x + y \in S_4$ and $\alpha x \in S_4$, so S_4 is a vector subspace of $\mathbb{R}^3$ over $\mathbb{R}$.

Problem 2.2. Prove that $\mathfrak{B} = \left\{ v_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Determine the coordinates of the vector $v = \begin{bmatrix} 4 \\ 4 \\ -9 \end{bmatrix}$ in the basis $\mathfrak{B}$.

Solution: $\mathfrak{B}$ is a basis for $\mathbb{R}^3$ if the cardinal of $\mathfrak{B}$ is 3, which is obvious, and if the vectors v_1, v_2, v_3 are linearly independent.

v_1, v_2, v_3 are linearly independent if $av_1 + bv_2 + cv_3 = 0_{\mathbb{R}^3} \Longleftrightarrow a = b = c = 0$.

$av_1 + bv_2 + cv_3 = 0_{\mathbb{R}^3} \Longleftrightarrow$

$$a \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow$$

$$\begin{bmatrix} 2a \\ a \\ -3a \end{bmatrix} + \begin{bmatrix} 3b \\ 2b \\ -5b \end{bmatrix} + \begin{bmatrix} c \\ -c \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow$$

$$\begin{bmatrix} 2a + 3b + c \\ a + 2b - c \\ -3a - 5b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow$$

$$\begin{cases} 2a + 3b + c = 0 \\ a + 2b - c = 0 \\ -3a - 5b + c = 0 \end{cases}.$$

The vectors are linearly independent if $a = b = c = 0$ is the unique solution of this system of linear equations. So, we will compute the rank of the system matrix. In this case is faster if we just determine the determinant of the matrix, since is obviously that the rank is at least 2.

$$D = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -3 & -5 & 1 \end{vmatrix} \overset{R_2+R_3, R_2+R_1}{=} \begin{vmatrix} 3 & 5 & 0 \\ 1 & 2 & -1 \\ -2 & -3 & 0 \end{vmatrix} = (-1)(-1)^{2+3} \begin{vmatrix} 3 & 5 \\ -2 & -3 \end{vmatrix} =$$

$-9 + 10 = 1 \neq 0$.

The rank $= 3$ equals the number of the unknowns and we have that the system has a unique solution $a = b = c = 0$ and we can conclude that the vectors are linearly independent therefore they form a basis in $\mathbb{R}^3$.

The vector $v = \begin{bmatrix} 4 \\ 4 \\ -9 \end{bmatrix}$ is given in the canonical basis of $\mathbb{R}^3$, i.e.

$$B_e = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The coordinates of v with respect to the basis $\mathfrak{B}$ are $v_{\mathfrak{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

$$\begin{bmatrix} 4 \\ 4 \\ -9 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Longleftrightarrow$$

$$\begin{bmatrix} 4 \\ 4 \\ -9 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 \\ \alpha_1 \\ -3\alpha_1 \end{bmatrix} + \begin{bmatrix} 3\alpha_2 \\ 2\alpha_2 \\ -5\alpha_2 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ -\alpha_3 \\ \alpha_3 \end{bmatrix} \Longleftrightarrow$$

$$\begin{bmatrix} 4 \\ 4 \\ -9 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 + \alpha_3 \\ \alpha_1 + 2\alpha_2 - \alpha_3 \\ -3\alpha_1 - 5\alpha_2 + \alpha_3 \end{bmatrix} \Longleftrightarrow$$

$$\begin{cases} 2\alpha_1 + 3\alpha_2 + \alpha_3 = 4 \\ \alpha_1 + 2\alpha_2 - \alpha_3 = 4 \\ -3\alpha_1 - 5\alpha_2 + \alpha_3 = -9 \end{cases}.$$

We can solve it using Gauss-Jordan elimination method.

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 1 & 2 & -1 & 4 \\ -3 & -5 & 1 & -9 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 3 & 1 & 4 \\ -3 & -5 & 1 & -9 \end{array}\right] \overset{R_2-2R_1, 3R_1+R_3}{\simeq}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -1 & 3 & -4 \\ 0 & 1 & -2 & 3 \end{array}\right] \overset{R_2+R_3}{\simeq} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right] \Longleftrightarrow$$

$$\begin{cases} \alpha_1 + 2\alpha_2 - \alpha_3 = 4 \\ -\alpha_2 + 3\alpha_3 = -4 \\ \alpha_3 = -1 \end{cases}$$

We calculate $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -1$, so $v_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Problem 2.3. Find the dimension and a basis for the subspace $U \subseteq \mathbb{R}^4$

$$U = \operatorname{span}\left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, u_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix} \right\}.$$

Solution: By definition of U, the vectors form a system of generators of U. We need to determine if they are linearly independent or, if not, the maximum number of linearly independent vectors from this set. We compute a linear combination of u_1, u_2, u_3, u_4 to determine this.

$$au_1 + bu_2 + cu_3 + du_4 = 0_{\mathbb{R}^4} \Longleftrightarrow$$

$$a\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow$$

$$\begin{bmatrix} a \\ 0 \\ 0 \\ -a \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix} + \begin{bmatrix} 2c \\ c \\ c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ 2d \\ 3d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow$$

$$\begin{bmatrix} a+b+2c \\ b+c+d \\ b+c+2d \\ -a+b+3d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{cases} a+b+2c=0 \\ b+c+d=0 \\ b+c+2d=0 \\ -a+b+3d=0 \end{cases}.$$

The vectors are linearly independent if $a=b=c=d=0$ is the unique solution of this system of linear equations. So, we will compute the rank of the system matrix.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ -1 & 1 & 0 & 3 \end{bmatrix} \overset{R_1+R_4}{\simeq} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 3 \end{bmatrix} \overset{-R_2+R_3, 2R_2-R_4}{\simeq} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \overset{R_3+R_4}{\simeq}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can observe that rank $=3$. That means that we have more than one solution ($\text{rank}\,(A)=3<$ the number of the unknowns $=4$), so the vectors are not linearly independent. Because the rank is 3, that means that there are 3 linearly independent vectors from the list. We will choose 3 of the vectors such that the rank of the matrix formed by their components is 3. For instance, we can choose u_1, u_2, u_4. $\mathfrak{B}_U = \{u_1, u_2, u_4\}$ and $\dim(U) = 3$.

Problem 2.4. Find a basis in the real space of solutions of the following systems of linear equations:

$$(S_1)\begin{cases} x+y-z+t=0 \\ x-y+2z-t=0 \\ 2x+y-z-t=0 \\ x+2y-3z=0 \end{cases}$$

$$(S_2)\begin{cases} x-y+z-t=0 \\ x+y+z-2t=0 \\ x-3y+z=0. \end{cases}$$

Solution: We will apply Gauss-Jordan elimination method for finding the general solution of the system (S_1).

$$(S_1):\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 2 & 1 & -1 & -1 & 0 \\ 1 & 2 & -3 & 0 & 0 \end{array}\right] \overset{R_2-R_1, 2R_1-R_3, R_4-R_1}{\simeq}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right] \overset{R_2+2R_3, R_2+2R_4}{\simeq}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & -4 & 0 \end{array}\right] \overset{R_3+R_4}{\simeq} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The $rank(A) = \text{rank}\left(\overline{A}\right) = 3$ and we have 4 unknowns, so we have 1 free variable, $t=\alpha$.

The system is equivalent to

$$\begin{cases} x + y - z = -\alpha \\ -2y + 3z = 2\alpha \\ z = -4\alpha. \end{cases}$$

We can calculate $-2y = -3z + 2\alpha = 12\alpha + 2\alpha \Rightarrow y = -7\alpha$, and $x = -y + z - \alpha = 7\alpha - 4\alpha - \alpha = 2\alpha$.

The general solution is

$$S_1 = \left\{ \begin{bmatrix} 2\alpha \\ -7\alpha \\ -4\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{R} \right\} = \left\{ \alpha \begin{bmatrix} 2 \\ -7 \\ -4 \\ 1 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 2 \\ -7 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

So, a basis for S_1 is $B_{S_1} = \left\{ \begin{bmatrix} 2 \\ -7 \\ -4 \\ 1 \end{bmatrix} \right\}$ and the dimension is $\dim(S_1) = 1$.

Now we determine the general solution for the second system.

$$(S_2) : \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 1 & 1 & 1 & -2 & 0 \\ 1 & -3 & 1 & 0 & 0 \end{array}\right] \overset{R_1 - R_2, R_1 - R_3}{\simeq} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right] \overset{R_2 + R_3}{\simeq} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The $rank(A) = 2$ and we have 4 unknowns, so we have 2 free variables, $x = \alpha$ and $y = \beta$. The system is equivalent to $\begin{cases} z - t = -\alpha + \beta \\ t = 2\beta \end{cases}$, therefore $z = -\alpha + \beta + t = -\alpha + 3\beta$.

The general solution is

$$S_2 = \left\{ \begin{bmatrix} \alpha \\ \beta \\ -\alpha + 3\beta \\ 2\beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} \alpha \\ 0 \\ -\alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \\ 3\beta \\ 2\beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

A basis for (S_2) is $B_{S_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$ and the dimension is $\dim(S_2) = 2$.

Problem 2.5. Determine a basis and the dimension for each of the vector subspaces $U + V$ and $U \cap V$, if

$$V = \{(x, y, z, t) \in \mathbb{R}^4 \mid x - 2y - z + t = 0\}$$

and

$$U = \text{span} \left\{ u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}, u_4 = \begin{bmatrix} 3 \\ 4 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

Solution:

$U + V = \{u + v | u \in U \text{ and } v \in V\}$.

If $u \in U$ and a basis of U is $B_U = \{u_1, \ldots, u_k\}$ then $u = \alpha_1 u_1 + \ldots \alpha_k u_k$, $\alpha_i \in \mathbb{R}$, $i \in \{1, \ldots k\}$.

If $v \in V$ and a basis of V is $B_V = \{v_1, \ldots, v_l\}$ then $v = \beta_1 v_1 + \ldots \beta_l v_l$, $\beta_i \in \mathbb{R}$, $i \in \{1, \ldots l\}$.

Hence,

$$\begin{aligned} U + V &= \{u + v | u \in U \text{ and } v \in V\} \\ &= \{\alpha_1 u_1 + \ldots \alpha_k u_k + \beta_1 v_1 + \ldots \beta_l v_l | \alpha_i \in \mathbb{R}, i = \overline{1,k},\ \beta_j \in \mathbb{R}, j = \overline{1,l}\} \\ &= \text{span}\{u_1, \ldots, u_k, v_1, \ldots, v_l\}. \end{aligned}$$

In what follows we will determine a basis in each of the subspaces U and V.

The dimension of the vector subspace U equals the rank of the matrix having as columns the vectors u_1, u_2, u_3, u_4:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 3 \\ 1 & 0 & -2 & -1 \end{bmatrix} \underset{\sim}{2R_1 - R_2, R_1 - R_3, R_1 - R_4} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 3 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} \underset{\sim}{R_2 - R_3}$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 4 \end{bmatrix} \underset{\sim}{2R_3 - R_4} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the matrix is 3, so $\dim U = 3$. A basis for U is

$$B_U = \left\{ u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

For the vector subspace V we will write the general solution of the equation $x - 2y - z + t = 0$. The rank is obviously 1, we have 4 unknowns, therefore the general solution has 3 free unknowns, $y = \alpha$, $z = \beta$ and $t = \gamma$ and we calculate $x = 2\alpha + \beta - \gamma$. The general solution of the equation is

$$V = \left\{ \begin{bmatrix} 2\alpha + \beta - \gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} 2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \beta \\ 0 \\ \beta \\ 0 \end{bmatrix} + \begin{bmatrix} -\gamma \\ 0 \\ 0 \\ \gamma \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R} \right\}$$
$$= \left\{ \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

$$V = \text{span}\left\{ v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for V is $B_V = \{v_1, v_2, v_3\}$.

$U + V = \text{span}\{u_1, u_2, u_3, v_1, v_2, v_3\}$.

The dimension of $U + V$ equals the rank of the matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 1 & -1 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix} \overset{2R_1-R_2, R_1-R_3, R_1-R_4}{\simeq} \begin{bmatrix} 1 & 0 & 2 & 2 & 1 & -1 \\ 0 & -1 & 3 & 3 & 2 & -2 \\ 0 & -1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 4 & 2 & 1 & -2 \end{bmatrix} \overset{R_2-R_3}{\simeq}$$

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 1 & -1 \\ 0 & -1 & 3 & 3 & 2 & -2 \\ 0 & 0 & 2 & 1 & 2 & -1 \\ 0 & 0 & 4 & 2 & 1 & -2 \end{bmatrix} \overset{2R_3-R_4}{\simeq} \begin{bmatrix} 1 & 0 & 2 & 2 & 1 & -1 \\ 0 & -1 & 3 & 3 & 2 & -2 \\ 0 & 0 & 2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

The rank is 4, hence $\dim(U+V) = 4$. A basis for $U+V$ is $B_{U+V} = \{u_1, u_2, u_3, v_2\}$.

The dimension of the subspace $S \cap V$ is

$\dim U \cap V = \dim U + \dim V - \dim(U+V) = 3 + 3 - 4 = 2.$

If $v \in U \cap V$ then v can be uniquely written as a linear combination of vectors from both B_U and B_V. Therefor, for $v \in U \cap V \Rightarrow v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$, which leads us to the system:

$$\begin{cases} \alpha_1 + 2\alpha_3 = 2\beta_1 + \beta_2 - \beta_3 \\ 2\alpha_2 + \alpha_2 + \alpha_3 = \beta_1 \\ \alpha_1 + \alpha_2 + \alpha_3 = \beta_2 \\ \alpha_1 - 2\alpha_3 = \beta_3 \end{cases} \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & -1 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array}\right] \simeq$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & -1 \\ 0 & -1 & 3 & 3 & 2 & -2 \\ 0 & 0 & 2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{array}\right] \text{ (see the rank of } U+V).$$

The rank of the matrix is 4, and because the system has 6 unknowns, two of them will be free unknowns, $\beta_1 = a$ and $\beta_3 = b$.

The system is equivalent to:

$$\begin{cases} \alpha_1 + 2\alpha_3 - \beta_2 = 2a - b \\ -\alpha_2 + 3\alpha_3 - 2\beta_2 = 3a - 2b \\ 2\alpha_3 - 2\beta_2 = a - b \\ 0 = \beta_2 \end{cases}.$$

Because we only need one of the writings $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$ or $v = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$ is easy to choose the second one where $\beta_1 = a$, $\beta_2 = 0$ and $\beta_3 = b$, so

$$v = av_1 + 0v_2 + bv_3 = a\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow v \in \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Therefore, a basis for $U \cap V$ is $B_{U\cap V} = \left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Problem 2.6. Prove that $B = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Determine the transition matrix from the canonical basis to B. Determine the coordinates of the vector $v = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ given in the canonical basis in the new basis B.

Solution: The rank of the matrix having the coordinates of v_1, v_2, v_3 as columns is 3 since the determinant

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} \overset{-C_3+C_2}{=} \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -1 \neq 0.$$

Hence, the vectors are linearly independent therefore they form a basis in $\mathbb{R}^3$.

The vector $v = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ is given in the canonical basis of $\mathbb{R}^3$,

$$e = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The coordinates of v with respect to the basis B are $v_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $v = av_1 + bv_2 + cv_3$.

$$\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \iff$$

$$\begin{cases} 6 = a + b \\ 3 = a + b + c \\ 8 = a + 2b + 2c \end{cases}.$$

We can solve it using Gauss-Jordan elimination method, or, we can use the transition matrix from the new basis B to the canonical basis, $P^{e,B} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ to determine the coordinates of v with respect to the new basis B.

As a matrix multiplication, the previous system can be written as

$$\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Longrightarrow v_e = P^{e,B} \cdot v_B \Longrightarrow v_B = \left(P^{e,B}\right)^{-1} \cdot v_e, \text{ where}$$

$$P^{e,B} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

We need to compute the inverse of the matrix $P^{e,B}$ which is the matrix $P^{B,e}$ i.e. the transition matrix from the canonical basis e to the new basis B.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \overset{R_1-R_2, R_1-R_3}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & -2 & 1 & 0 & -1 \end{array}\right] \overset{R_2 \leftrightarrow R_3}{\simeq}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right] \overset{2R_3-R_2}{\simeq}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right] \overset{-R_2+R_1, -R_3}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right].$$

So, $P^{B,e} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

We can now determine the coordinates of v in the new basis B:

$$v_B = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -3 \end{bmatrix}.$$

Remark. *Using the transition matrix we can now have the coordinates of any vector v from $\mathbb{R}^3$ in the new basis B by just multiplying by v the matrix $P^{B,e}$.*

Problem 2.7. In the space $\mathbb{R}^3$ we consider the basis $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

and $B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \right\}$. Determine the transition matrix from B to B'. Determine the coordinates of the vector $v = \begin{bmatrix} -1 \\ -4 \\ 4 \end{bmatrix}$ in the basis B'. Which are the coordinates of the vector v in the basis B?

Solution: We can write the transition matrix from both B and B' to the canonical basis, i.e.

$$P^{e,B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ P^{e,B'} = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 0 \\ -1 & -3 & 3 \end{bmatrix}.$$

We know that $P^{B'B} = P^{B',B''} \cdot P^{B'',B}$, and we use as B'' the canonical basis e. Therefore, $P^{B'B} = P^{B',e} \cdot P^{e,B} = (P^{e,B'})^{-1} \cdot P^{e,B}$.

We calculate the inverse of the matrix $P^{e,B'}$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -1 & -3 & 3 & 0 & 0 & 1 \end{array}\right] \overset{R_1-R_2, R_1+R_3}{\simeq}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \overset{2R_3+R_2, 2R_3+R_1}{\simeq}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \overset{-3R_2+R_1}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 3 & -4 \\ 0 & 1 & 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

So, the transition matrix from B to B' is

$$P^{B',B} = P^{B',e} \cdot P^{e,B} = \begin{bmatrix} -6 & 3 & -4 \\ 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -3 & -7 \\ 3 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix}.$$

We can now determine the coordinates of v in the basis B':

$$v_{B'} = P^{B',e} \cdot v_e = \begin{bmatrix} -6 & 3 & -4 \\ 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -22 \\ 9 \\ 3 \end{bmatrix}.$$

We can determine the coordinates of v with respect to B using $v_B = P^{B,e} \cdot v_e$ or $v_B = P^{B,B'} \cdot v_{B'}$. Is easier to compute $P^{B,e} = (P^{e,B})^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, so

$$v_B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 4 \end{bmatrix}.$$

2.4 Problems

Problem 2.8. Prove that $(V, \oplus)$ is a vector space over $(\mathbb{R}, +, \cdot)$, where $V = \mathbb{R}_+^*$, the internal composition law (internal operation) is $x \oplus y = x \cdot y$, and the scalar multiplication (external operation) is $\alpha \star x = x^\alpha$, $\alpha \in \mathbb{R}$, $x \in V$.

Problem 2.9. Determine which of the following sets are vector subspaces of $\mathbb{R}^3$ over $\mathbb{R}$:

- $A_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 0\}$;
- $A_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 1\}$;
- $A_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + 2x_2 - 3x_3 = 0\}$;
- $A_4 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \frac{x_1+1}{3} = \frac{x_2}{-2} = \frac{x_3-2}{2}\}$;

- $A_5 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \frac{x_1}{3} = \frac{x_2}{-2} = \frac{x_3}{2}\}$;
- $A_6 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | |x_1| = x_2 - x_3\}$;
- $A_7 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 - x_2 + x_3 = 0\}$;
- $A_8 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 - x_2 + x_3 = 0; x_1^2 + 2x_2 + 4x_3 = 0\}$;
- $A_9 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 - x_2 + x_3 = 2; x_1^2 + 2x_2 + 4x_3 = -1\}$.

Problem 2.10. Determine if the following sets are linearly independent sets of vectors:

a) $S_1 = \{v_1 = 1 + X, v_2 = 2 + X, v_3 = 2 - 3X\}$, $S_1 \subset \mathbb{R}[X]$;

b) $S_2 = \left\{ v_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$,
$S_2 \subset \mathcal{M}_2(\mathbb{R})$;

c) $S_3 = \left\{ v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ -11 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -4 \\ 10 \end{bmatrix} \right\}$, $S_3 \subset \mathbb{R}^3$;

d) $S_4 = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$, $S_4 \subset \mathbb{R}^4$.

Problem 2.11. Consider V a vector space over $\mathbb{R}$, and $v_1, v_2, v_3 \in V$ are linearly independent vectors. Prove that the vectors $w_1 = v_1 - v_2 - v_3$, $w_2 = -v_1 - 2v_2 + 3v_3$ and $w_3 = -v_1 + v_2 - v_3$ are linearly independent.

Problem 2.12. Prove that the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ and $v_4 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix}$ are linearly dependent. Write the vector v_4 as a linear combination of the vectors v_1, v_2, v_3.

Problem 2.13. Find the coordinates of the vector v in the basis B (prove that B is a basis of $\mathbb{R}^3$, respectively $\mathbb{R}^4$) if:

a) $v = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$;

b) $v = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$;

c) $v = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$;

d) $v = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -4 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$;

e) $v = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 0 \\ -2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Problem 2.14. Find the dimension and a basis for the subspaces generated by the following sets of vectors:

a) $U_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 4 \\ -1 \end{bmatrix} \right\}$, $U_1 \subset \mathbb{R}^4$;

b) $U_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$, $U_2 \subset \mathbb{R}^4$;

c) $U_3 = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$, $U_3 \subset \mathbb{R}^4$;

d) $U_4 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$, $U_4 \subset \mathbb{R}^5$;

e) $U_5 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} \right\}, U_5 \subset \mathbb{R}^4;$

f) $U_6 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}, U_6 \subset \mathbb{R}^3;$

g) $U_7 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, U_7 \subset \mathbb{R}^3;$

h) $U_8 = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -5 \end{bmatrix}, \begin{bmatrix} 10 \\ 3 \end{bmatrix} \right\}, U_8 \subset \mathbb{R}^2.$

Problem 2.15. Find a basis in the real space of solutions of the following systems:

$$(S_1) \quad \begin{cases} x + 2y + z + -t = 0 \\ x + y + z = 0. \end{cases}$$

$$(S_2) \quad \begin{cases} x + y - z - 2t = 0 \\ x - y + 2t = 0 \\ y + -z + t = 0. \end{cases}$$

$$(S_3) \quad \begin{cases} x - y + z - t = 0 \\ 2x + 3y + 3z - t = 0 \\ 3x + 2y + 4z - 2t = 0 \\ x + 4y + 2z = 0. \end{cases}$$

$$(S_4)\quad \begin{cases} x - y - z + t = 0 \\ x + y - 2z + t = 0 \\ x - 5y - z + t = 0. \end{cases}$$

$$(S_5)\quad \begin{cases} x + y - t = 0 \\ x + z - 2t = 0 \\ y - z + t = 0. \end{cases}$$

$$(S_6)\quad \begin{cases} x - y + 2z - 4t = 0 \\ -x + 2y + z = 0 \\ x + t = 0 \\ y - z = 0. \end{cases}$$

Problem 2.16. Find the dimension and a basis of the union (sum) $S + V$ and intersection $S \cap V$ of the linear subspaces S and V if:

a) $S = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$

$V = span \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\};$

b) $S = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}, V = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\};$

c) $S = span\left\{\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-2\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\\-1\end{bmatrix}\right\}$,

$V = span\left\{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\\0\end{bmatrix}\right\}$;

d) $S = span\left\{\begin{bmatrix}1\\2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}\right\}$,

$V = \{(x, y, z, t) \in \mathbb{R}^4 | x + y + 2z - t = 0\}$;

e) $S = \{(x, y, z) \in \mathbb{R}^3 | -x + y + z = 0\}$,

$V = \{(x, y, z) \in \mathbb{R}^3 | 2x - y - 2z = 0\}$;

f) $S = \{(x, y, z) \in \mathbb{R}^3 | x - 2y + 5z = 0\}$,

$V = \{(x, y, z) \in \mathbb{R}^3 | -3x + y + z = 0\}$;

g) $S = \{(x, y, z, t) \in \mathbb{R}^4 | x - y + 2z - t = 0; x + 2y - z + t = 0; x + 5y - 4z + 3t = 0\}$,

$V = span\left\{\begin{bmatrix}0\\1\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\-3\\-3\\6\end{bmatrix}, \begin{bmatrix}-1\\2\\0\\3\end{bmatrix}\right\}$;

h) $$S = span\left\{\begin{bmatrix}1\\1\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\0\\-2\end{bmatrix},\begin{bmatrix}2\\1\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\-4\end{bmatrix}\right\},$$

$$V = span\left\{\begin{bmatrix}3\\1\\2\\-4\end{bmatrix},\begin{bmatrix}-1\\0\\0\\-2\end{bmatrix},\begin{bmatrix}-5\\-1\\-2\\0\end{bmatrix}\right\}.$$

Problem 2.17. Prove that each of the two sets of vectors is a basis in $\mathbb{R}^3$ and find the relationship between the coordinates of one and the same vector in the two bases: $B = \left\{a_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}, a_2 = \begin{bmatrix}2\\3\\3\end{bmatrix}, a_3 = \begin{bmatrix}3\\1\\7\end{bmatrix}\right\}$

and

$$B' = \left\{b_1 = \begin{bmatrix}3\\4\\1\end{bmatrix}, b_2 = \begin{bmatrix}5\\1\\2\end{bmatrix}, b_3 = \begin{bmatrix}1\\-6\\1\end{bmatrix}\right\}.$$

Problem 2.18. Prove that each of the two sets of vectors is a basis in the space of polynomials of degree $\leqslant 3$ with real coefficients and find the transition matrix between the two bases:

$$B = \{e_1 = 1, e_2 = X, e_3 = X^2, e_4 = X^3\}$$

and

$$B' = \{e_1' = 1 + X, e_2' = 1 - X^2, e_3' = X^2 + X, e_4' = X^3 - X^2\}.$$

Problem 2.19. In the space $\mathbb{R}^3$ we consider the bases

$$B = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and

$$B' = \left\{ \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \right\}.$$

Determine the transition matrix from B to B'. Determine the coordinates of the vector $v = \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix}$ in the basis B'. Which are the coordinates of the vector v in the basis B?

Problem 2.20. Prove that each of the two sets of vectors is a basis in the space $R_2[X]$ and find the transition matrix between the two bases if

$$B = \{X^2, X + X^2, 1 + X + X^2\}$$

and

$$B' = \{2 + 3X^2, 4 + X - X^2, 2 + 5X + 3X^2\}.$$

Determine the transition matrix from B to B'. Find the coordinates of the polynomial $4 - 4X - X^2$ in both of the basis B and B'.

3

Inner product spaces

3.1 Definitions and Properties

Definition 3.1. *An* **inner product** *on a vector space* V *over the field* $\mathbb{F}$ *is a function (bilinear form)* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ *with the properties:*

1. $\langle v, v \rangle \geqslant 0$ *and* $\langle v, v \rangle = 0$ *iff* $v = 0$. - ***positivity and definiteness***

2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$, *for all* $u, v, w \in V$. - ***additivity in the first slot***

3. $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ *for all* $\alpha \in \mathbb{F}$ *and* $v, w \in V$. - ***homogeneity in the first slot***

4. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ *for all* $v, w \in V$. - ***conjugate symmetry***.

An **inner product space** *is a pair* $(V, \langle \cdot, \cdot \rangle)$, *where* V *is vector space and* $\langle \cdot, \cdot \rangle$ *is an inner product on* V.

Properties. From the definition one can easily deduce the following properties of an inner product:

1. $\langle v, 0\rangle = \langle 0, v\rangle = 0$,

2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$,

3. $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$,

for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$.

The most important example of an inner product space is $\mathbb{R}^n$.

Definition 3.2. *Let* $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ *and* $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$. *The* **the Euclidean inner product** *of v and w is defined by*

$$\langle v, w\rangle = v_1 w_1 + \cdots + v_n w_n.$$

When $\mathbb{R}^n$ is referred to as an inner product space, one should assume that the inner product is the Euclidean one, unless explicitly stated otherwise.

Norm and distances

Definition 3.3. *Let V be a vector space over $\mathbb{R}$. A function*

$$\| \cdot \| : V \to \mathbb{R}$$

is called ***a norm*** *on V if:*

1. $\|v\| \geqslant 0,\ v \in V,\ \|v\| = 0 \Leftrightarrow v = 0_V$

2. $\|\alpha v\| = |\alpha| \cdot \|v\|,\ \forall \alpha \in \mathbb{R},\ \forall v \in V$

3. $\|u + v\| \leqslant \|u\| + \|v\|,\ \forall u, v \in V.$

A **normed space** *is a pair* $(V, \|\cdot\|)$, *where* V *is a vector space and* $\|\cdot\|$ *is a norm on* V.

Example 3.4. *Let* $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. *On the real linear space* $\mathbb{R}^n$ *one can define a norm in several ways.*

1. *The* ***Euclidian norm***

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\langle x, x\rangle}.$$

2. *The* $p-$***norm,*** *for any* $p \in \mathbb{R}$, $p \geqslant 1$

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

3. *The* ***maximum norm***

$$\|x\|_{max} = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Definition 3.5. *Let* X *be a nonempty set. A function* $d : X \times X \to \mathbb{R}$ *satisfying the following properties:*

- $d(x, y) \geqslant 0, \forall x, y \in X$ *and* $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x), \forall x, y \in X$
- $d(x, y) \leqslant d(x, z) + d(z, y), \forall x, y, z \in X$

is called a **metric** *or* **distance** *on* X.

A set X *with a metric defined on it is called a* **metric space**.

Example 3.6. *Let* $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$. *On* $\mathbb{R}^n$ *are can be defined the following distances:*

1. *The* ***euclidian distance*** *is defined as*

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \|x - y\|_2.$$

2. *The* ***Minkowski distance*** *or* ***Manhattan distance*** *is defined as*

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|.$$

3. *The* ***Chebyshev distance*** *is defined as*

$$d_{max}(x, y) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|.$$

Definition 3.7. *Two vectors* $u, v \in V$ *are said to be* **orthogonal** *and we denote* $u \perp v$ *if* $\langle u, v \rangle = 0$.

In a real inner product space we can define *the angle of two vectors* as

$$\widehat{(v, w)} = \arccos \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}$$

We have

$$v \perp w \Leftrightarrow \langle v, w \rangle = 0 \Leftrightarrow \widehat{(v, w)} = \frac{\pi}{2}.$$

Theorem 3.8. ***(Parallelogram law)*** *Let* V *be an inner product space and* $u, v \in V$. *Then*

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Proof.

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + \langle u,v\rangle + \langle v,u\rangle + \|v\|^2\end{aligned} \tag{3.1}$$

$$\begin{aligned}\|u-v\|^2 &= \langle u-v, u-v\rangle \\ &= \langle u, u-v\rangle - \langle v, u-v\rangle \\ &= \langle u,u\rangle - \langle u,v\rangle - (\langle v,u\rangle - \langle v,v\rangle) \\ &= \|u\|^2 - \langle u,v\rangle - \langle v,u\rangle + \|v\|^2\end{aligned} \tag{3.2}$$

By adding (3.1) and (3.2) we obtain $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$. □

Theorem 3.9. (Pythagorean Theorem) *Let V be an inner product space, and $u, v \in V$ orthogonal vectors. Then*

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof.

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + \langle u,v\rangle + \langle v,u\rangle + \|v\|^2.\end{aligned}$$

Since $u \perp v$ then $\langle u, v\rangle = 0$ so, we have $\|u+v\|^2 = \|u\|^2 + \|v\|^2$. □

3.2 Orthonormal Bases

Definition 3.10. *Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and let I be an arbitrary index set. A family of vectors $A = \{e_i \in V | i \in I\}$ is called an* orthogonal *family, if $\langle e_i, e_j\rangle = 0$ for every $i, j \in I$, $i \neq j$. The family A is called* **orthonormal** *if it is orthogonal and $\|e_i\| = 1$ for every $i \in I$.*

One of the reason that one studies orthonormal families is that in such special bases the computations are much more simple.

Propozition 3.11. *If $\{e_1, e_2, \ldots, e_m\}$ is an orthonormal family of vectors in V, then*

$$\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_m e_m\|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_m|^2$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$.

Proof. From Pythagorean Theorem we have

$$\begin{aligned}\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_m e_m\|^2 &= \|\alpha_1 e_1\|^2 + \|\alpha_2 e_2\|^2 + \cdots + \|\alpha_m e_m\|^2 \\ &= |\alpha_1|^2\|e_1\|^2 + |\alpha_2|^2\|e_2\|^2 + \cdots + |\alpha_m|^2\|e_m\|^2 \\ &= |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_m|^2.\end{aligned}$$

□

Corolary 3.12. *Every orthonormal list of vectors is linearly independent.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal family. Then, by definitions, $v_i \perp v_j$ for all $i \neq j$, $i, j = \overline{1,n}$, and $\|e_i\| = 1, \forall i = \overline{1,n}$.

The vectors $v_1, v_2, \ldots, v_n$ are linearly independent if

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0_V \Longleftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

$\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|^2 = \|0_V\|^2 = 0 \Longleftrightarrow |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_m|^2 = 0$. Since $|\alpha_i|^2 \geqslant 0 \, \forall i = \overline{1,n}$, we have that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ which implies that the vectors $v_1, v_2, \ldots, v_n$ are linearly independent. □

Theorem 3.13. ***(Gram-Schmidt)*** *If $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set of vectors in V, then there exists an orthonormal set of vectors $\{e_1, \ldots, e_k\}$ in V, such that*

$$span\{v_1, v_2, \ldots, v_k\} = span\{e_1, e_2 \ldots, e_k\}$$

$k \in \{1, 2, \ldots, m\}$.

Proof. We will determine first an orthogonal set of vectors $u_1, u_2, \ldots, u_k$.

Let $u_1 = v_1$. We will determine u_2 as a linear combination of v_2 and u_1, $u_2 = v_2 + \alpha_1 u_1$ such that $u_1 \perp u_2$.

We have $\langle u_1, u_2\rangle = 0 \Longleftrightarrow \langle v_2 + \alpha u_1, u_1\rangle = 0 \Longleftrightarrow \langle v_2, u_1\rangle + \alpha\langle u_1, u_1\rangle = 0 \Longrightarrow \alpha = -\frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle}$.

So, $u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1$ and we have that $u_1 \perp u_2$.

Next we will write u_3 as a linear combination of v_3, u_1 and u_2 such that $u_3 \perp u_1$ and $u_3 \perp u_2$, that is $u_3 = v_3 + \alpha_1 u_1 + \alpha_2 u_2$ and $\langle u_3, u_1\rangle = 0$, $\langle u_3, u_2\rangle = 0$.

$$\begin{aligned}
\langle u_3, u_1\rangle = 0 &\Longleftrightarrow \langle v_3 + \alpha_1 u_1 + \alpha_2 u_2, u_1\rangle = 0 \\
&\Longleftrightarrow \langle v_3, u_1\rangle + \alpha_1\langle u_1, u_1\rangle + \alpha_2\langle u_2, u_1\rangle = 0 \\
&\Longleftrightarrow \langle v_3, u_1\rangle + \alpha_1\langle u_1, u_1\rangle = 0 \\
&\Longleftrightarrow \alpha_1 = -\frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle}.
\end{aligned}$$

$$\begin{aligned}\langle u_3, u_2\rangle = 0 &\Longleftrightarrow \langle v_3 + \alpha_1 u_1 + \alpha_2 u_2, u_2\rangle = 0\\ &\Longleftrightarrow \langle v_3, u_2\rangle + \alpha_1\langle u_1, u_2\rangle + \alpha_2\langle u_2, u_2\rangle = 0\\ &\Longleftrightarrow \langle v_3, u_2\rangle + \alpha_2\langle u_2, u_2\rangle = 0\\ &\Longleftrightarrow \alpha_2 = -\frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle}.\end{aligned}$$

So, $u_3 = v_3 - \frac{\langle v_3,u_1\rangle}{\langle u_1,u_1\rangle}u_1 - \frac{\langle v_3,u_2\rangle}{\langle u_2,u_2\rangle}u_2$, and $u_3 \perp u_1$, $u_3 \perp u_2$.

By induction, we will have that

$u_k = v_k - \frac{\langle v_k,u_1\rangle}{\langle u_1,u_1\rangle}u_1 - \frac{\langle v_k,u_2\rangle}{\langle u_2,u_2\rangle}u_2 - \cdots - \frac{\langle v_k,u_{k-1}\rangle}{\langle u_{k-1},u_{k-1}\rangle}u_{k-1}$.

Because the set $\{u_1, u_2, \ldots, u_k\}$ is orthogonal, then the set $\{e_1 = \frac{u_1}{\|u_1\|}, e_2 = \frac{u_2}{\|u_2\|} \ldots, e_k = \frac{u_k}{\|u_k\|}\}$ is orthonormal. □

We can summarise the Gram-Schmidt process for the orthogonalization of the vectors $v_1, v_2, \ldots, v_n$ in the following:

- $u_1 = v_1$;
- $u_2 = v_2 - \frac{\langle v_2,u_1\rangle}{\langle u_1,u_1\rangle}u_1$;
- $u_3 = v_3 - \frac{\langle v_3,u_1\rangle}{\langle u_1,u_1\rangle}u_1 - \frac{\langle v_3,u_2\rangle}{\langle u_2,u_2\rangle}u_2$;

  ...
- $u_n = v_n - \frac{\langle v_n,u_1\rangle}{\langle u_1,u_1\rangle}u_1 - \frac{\langle v_n,u_2\rangle}{\langle u_2,u_2\rangle}u_2 - \cdots - \frac{\langle v_n,u_{n-1}\rangle}{\langle u_{n-1},u_{n-1}\rangle}u_{n-1}$.

Corolary 3.14. *Every finitely dimensional inner product space has an orhtonormal basis.*

Orthogonal complement

Let $U \subseteq V$ be a subset of an inner product space V. The *orthogonal complement* of U, denoted by $U^\perp$ is the set of all vectors in V which are orthogonal to every vector in U i.e.:

$$U^{\perp} = \{v \in V | \langle v, u\rangle = 0, \ \forall u \in U\}.$$

Theorem 3.15. *If U is a subspace of V, then*

$$V = U \oplus U^{\perp}.$$

3.3 Solved Problems

Problem 3.1. Let $\mathbb{R}^4$ be the inner product space with the canonical inner product. Apply the Gram-Schmidt orthogonalization method to construct orthogonal basis for the subspace

$$V = span\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -5 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 2 \\ 8 \\ -7 \end{bmatrix} \right\}.$$

Solution: We will apply Gram-Schmidt orthogonalization method on v_1, v_2, v_3.

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}.$$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1$$

$$= \begin{bmatrix} 1 \\ 1 \\ -5 \\ 3 \end{bmatrix} - \frac{1+2-10-3}{1+4+4+1} \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -5 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -3 \\ 2 \end{bmatrix}.$$

$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle}u_2$$
$$= \begin{bmatrix} 3 \\ 2 \\ 8 \\ -7 \end{bmatrix} - \frac{3+4+16+7}{1+4+4+1}\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} - \frac{6+6-24-14}{4+9+9+4}\begin{bmatrix} 2 \\ 3 \\ -3 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ 2 \\ 8 \\ -7 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -2 \end{bmatrix}.$$

We can easily verify that $u_1 \perp u_2$, $u_1 \perp u_3$ and $u_2 \perp u_3$ by computing $\langle u_1, u_2\rangle = 0$, $\langle u_1, u_3\rangle = 0$ and $\langle u_2, u_3\rangle = 0$.

So, an orthogonal basis for V is
$$B_V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \\ -2 \end{bmatrix} \right\}.$$

Problem 3.2. Let S be the solutions of the system

$$(S): \begin{cases} x + y + z - t = 0 \\ x + 2y + 3z + w = 0 \\ x - y - 3z - 3t - 2w = 0. \end{cases}$$

Find an orthonormal basis in S.

Solution: We will determine first a basis from the general solution of the system (S).

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & -3 & -3 & -2 & 0 \end{array}\right] \overset{L_1-L_2, L_1-L_3}{\simeq} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 2 & 4 & 2 & 2 & 0 \end{array}\right] \overset{2L_2+L_3}{\simeq} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$\operatorname{rank}(A) = 2$ and there are 5 unknowns, so $z = \alpha$, $t = \beta$ and $w = \gamma$ are free variables. We can determine x and y solving the system $\begin{cases} x + y = -\alpha + \beta \\ -y = 2\alpha + \beta + \gamma \end{cases}$.

Hence, $x = \alpha + 2\beta + \gamma$ and $y = -2\alpha - \beta - \gamma$.

$$S = \left\{ \begin{bmatrix} \alpha + 2\beta + \gamma \\ -2\alpha - \beta - \gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ s_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, s_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$B_S = \{s_1, s_2, s_3\}$ is a basis for S.

In order to obtain an orthogonal basis, we will apply Gram-Schmidt algorithm for s_1, s_2, s_3.

$$u_1 = s_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u_2 = s_2 - \frac{\langle s_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$u_3' = s_1 - \frac{\langle s_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle s_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -1 \\ 1 \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}.$$

Now we can choose $u_3 = 3u_3' = \begin{bmatrix} -1 \\ -3 \\ 3 \\ -1 \\ -2 \end{bmatrix}$.

We can easily verify that $u_1 \perp u_2$, $u_1 \perp u_3$ and $u_2 \perp u_3$ by computing $\langle u_1, u_2 \rangle = 0$, $\langle u_1, u_3 \rangle = 0$ and $\langle u_2, u_3 \rangle = 0$.

For an orthonormal basis we divide each of u_1, u_2, u_3 by its norm, i.e.

$$n_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix},$$

$$n_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix},$$

$$n_3 = \frac{u_3}{\|u_3\|} = \frac{1}{2\sqrt{6}} \begin{bmatrix} -1 \\ -3 \\ 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\sqrt{6}} \\ -\frac{3}{2\sqrt{6}} \\ \frac{3}{2\sqrt{6}} \\ -\frac{1}{2\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}.$$

Therefore, an orthonormal basis for S is $B_n = \{n_1, n_2, n_3\}$.

Problem 3.3. Let $S = \{(x, y, z, t) \in \mathbb{R}^4 | 2x - y - z + 3t = 0\}$ be a vector subspace of $\mathbb{R}^4$. Determine bases in S and in the orthogonal complement $S^{\perp}$.

Solution: The general solution of the equation $2x - y - z + 3t = 0$ can be written if one choose $x = \alpha$, $z = \beta$ and $t = \gamma$ and we calculate $y = 2\alpha - \beta + 3\gamma$.

$$\begin{aligned} S &= \left\{ \begin{bmatrix} \alpha \\ 2\alpha - \beta + 3\gamma \\ \beta \\ \gamma \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R} \right\} \\ &= \left\{ \alpha \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ s_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, s_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

In order to obtain an orthogonal basis, we will apply Gram-Schmidt algorithm on s_1, s_2 and s_3.

$$\begin{aligned} u_1 &= s_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \\ u_2 &= s_1 - \frac{\langle s_1, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 \end{aligned}$$

$$u_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_3' = s_3 - \frac{\langle s_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle s_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$$= \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} - \frac{-3}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Now we can choose $u_3 = 2u_3' = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$.

We can easily verify that $u_1 \perp u_2$, $u_1 \perp u_3$ and $u_2 \perp u_3$ by computing $\langle u_1, u_2 \rangle = 0$, $\langle u_1, u_3 \rangle = 0$ and $\langle u_2, u_3 \rangle = 0$.

For an orthonormal basis we divide each of u_1, u_2, u_3 by its norm, i.e.

$$n_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix},$$

$$n_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}$$

$$n_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{bmatrix}.$$

Therefore, an orthonormal basis for S is $B_n = \{n_1, n_2, n_3\}$.

We have that $\dim S = 3$ so, $\dim S^{\perp} = 4 - 3 = 1$.

$$\begin{aligned} S^{\perp} &= \{v \in \mathbb{R}^4 | \langle v, s\rangle = 0, \forall s \in S\} \\ &= \{v \in \mathbb{R}^4 | \langle v, s_1\rangle = 0, \langle v, s_2\rangle = 0, \langle v, s_3\rangle = 0\} \\ &= \{v = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 + 2x_2 = 0, -x_2 + x_3 = 0, 3x_2 + x_4 = 0\}. \end{aligned}$$

The matrix system has the rank 3, so one of the unknowns become free variable, $x_2 = \alpha$, and we can determine $x_1 = -2\alpha$, $x_3 = \alpha$ and $x_4 = -3\alpha$.

$$S^{\perp} = \left\{ \begin{bmatrix} -2\alpha \\ \alpha \\ \alpha \\ -3\alpha \end{bmatrix} \mid \alpha \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ -3 \end{bmatrix} \right\}.$$

An orthogonal basis for $B_S^{\perp} = \left\{ s_4 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -3 \end{bmatrix} \right\}$, an orthonormal basis is

$$B_{S^{\perp}} = \{n_4\}, \text{ where } n_4 = \frac{s_4}{\|s_4\|} = \frac{1}{\sqrt{15}} \begin{bmatrix} -2 \\ 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{3}{\sqrt{15}} \end{bmatrix}.$$

Problem 3.4. Verify that the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal and complete them to form orthogonal basis of $\mathbb{R}^4$.

Solution: Because $\langle v_1, v_2 \rangle = 1 \cdot 1 + 0 \cdot 2 + 2 \cdot 0 + (-1) \cdot 1 = 0$ we can conclude that v_1 and v_2 are orthogonal.

Because $\dim R^4 = 4$ we will choose another two vectors v_3 and v_4 such that each of v_3 and v_4 is orthogonal on both v_1 and v_2.

Let $v = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in \mathbb{R}^4$ such that

$$\begin{cases} v \perp v_1 \\ v \perp v_2 \end{cases} \iff \begin{cases} \langle v, v_1 \rangle = 0 \\ \langle v, v_2 \rangle = 0 \end{cases} \iff \begin{cases} x + 2z - t = 0 \\ x + 2y + t = 0 \end{cases}.$$

The general solution of the system if we denote $y = \alpha$ and $z = \beta$ is

$$\begin{aligned} S &= \left\{ \begin{bmatrix} -\alpha - \beta \\ \alpha \\ \beta \\ \beta - \alpha \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} \\ &= \left\{ \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} \end{aligned}$$

$$= \text{span}\left\{ u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Because $\langle u_1, u_2 \rangle = 0 \Longrightarrow u_1 \perp u_2$, and we can choose $v_3 = u_1$ and $v_4 = u_2$.

We can conclude that $\{v_1, v_2, v_3, v_4\}$ is an orthogonal basis of $\mathbb{R}^4$.

Remark. If $\langle u_1, u_2 \rangle \neq 0$ then we need to apply Gram Schmitd orthogonalization method to obtain two orthogonal vectors.

3.4 Problems

Problem 3.5. Let S be the solutions of the system. Find an orthonormal basis in S if:

a) $(S) : \begin{cases} x + y + z - t = 0 \\ x + 2y + 3z = 0 \\ x - y - 3z - 3t = 0; \end{cases}$

b) $(S) : \begin{cases} x + y + t = 0 \\ 2x + y + z = 0 \\ x - y + 2z - 3t = 0; \end{cases}$

c) $(S) : \begin{cases} x + y - z + t = 0 \\ x - y - z + 2t = 0 \\ x + 3y - z = 0. \end{cases}$

Problem 3.6. Let S be the set of solutions of the following systems. Find bases in S and in the orthogonal complement of S, $S^{\perp}$, if:

a) $(S): \begin{cases} x+y+2z=0 \\ 2x+3y+z=0 \\ x+2y-z=0; \end{cases}$

b) $(S): \begin{cases} 2x-y-z+t=0 \\ x+y+3z-t=0; \end{cases}$

c) $(S): \begin{cases} x+y-z+t=0 \\ x+y+3z-t=0 \\ x+y-5z+3t=0. \end{cases}$

Problem 3.7. Verify that the following sets of vectors are orthogonal and complete them to form orthogonal basis of $\mathbb{R}^4$:

a) $v_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$;

b) $v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}$;

c) $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -3 \end{bmatrix}$;

d) $v_1 = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Problem 3.8. Let $\mathbb{R}^4$ be the inner product space with the canonical inner product. Apply the Gram-Schmidt orthogonalizations to construct orthogonal bases for the subspaces spanned by the following lists of vectors:

a) $\begin{bmatrix} 1 \\ 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -8 \\ -7 \end{bmatrix}$;

b) $\begin{bmatrix} -1 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 8 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 3 \\ 8 \end{bmatrix}$;

c) $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -11 \\ 0 \\ 1 \end{bmatrix}$;

d) $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ -2 \\ -5 \end{bmatrix}.$

Problem 3.9. Let S be the set of solutions of the system

$$\begin{cases} x + 2y - z - t = 0 \\ x + y - z + 2w = 0 \\ x - y + 2z - t + w = 0. \end{cases}$$

Find an orthonormal basis in $S^{\perp}$.

Problem 3.10. Let $U = \{(x, y, z, t) \in \mathbb{R}^4 | x - y - 3z + 2t = 0\}$ be a vector subspace of R^4. Find an orthonormal basis for S and for $S^{\perp}$.

4

Vectors

4.1 Space and Plane Coordinates

Coordinates in $\mathbb{R}^2$

In $\mathbb{R}^2$ we consider the Cartesian orthogonal coordinate system xOy. The **cartesian coordinates** or *rectangular coordinates* of the point $M \in \mathbb{R}^2$ is the ordered pair (x_0, y_0). The **polar coordinates** of M are (r, θ) where:

- r is the length of the line segment $[OM]$, $r \geqslant 0$;
- θ is the angle between the positive direction of Ox and OM, $\theta \in [0, 2\pi]$.

  The angle is measured in radians in the counterclockwise direction from the Ox axis to OM.

r and θ can be converted to the Cartesian coordinates x and y by using the trigonometric functions sine and cosine:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

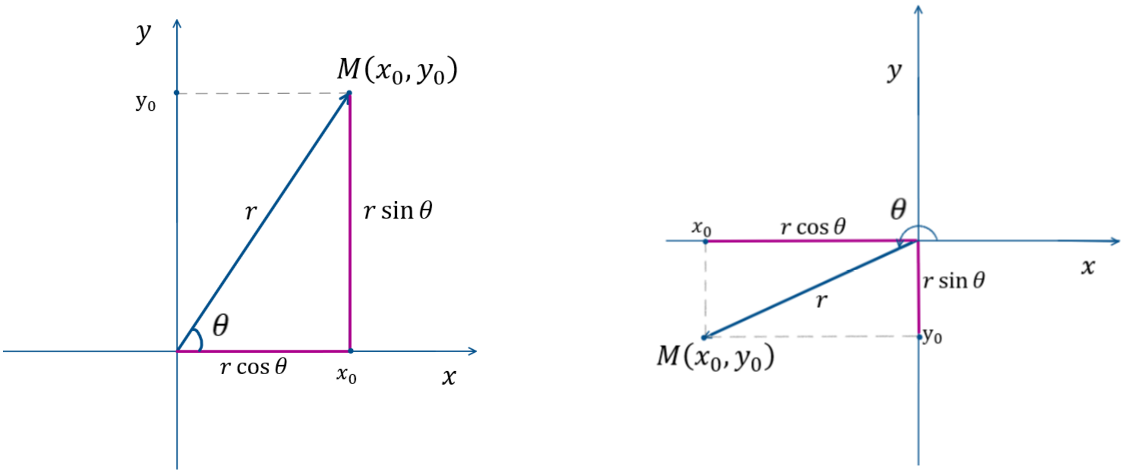


Figure 4.1: The relationship between polar and Cartesian coordinates

Viceversa:

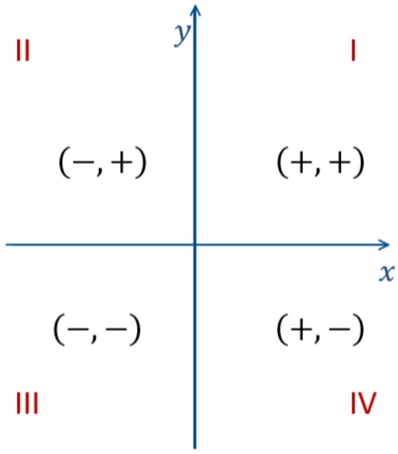
$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \tan\theta = \dfrac{y}{x}, \ x \neq 0. \end{cases}$$

If $x = 0$ and $y > 0$ then $\theta = \dfrac{\pi}{2}$.

If $x = 0$ and $y < 0$ then $\theta = \dfrac{3\pi}{2}$.

Remark 4.1. *When we include negative values, the Ox and Oy axes divide the space up into 4 pieces Quadrants I, II, III and IV, numbered in a counter-clockwise direction.*

- *In Quadrant I both x and y are positive.*
- *In Quadrant II x is negative y is positive.*
- *In Quadrant III x and y are negative.*
- *In Quadrant IV x is positive and y is negative.*

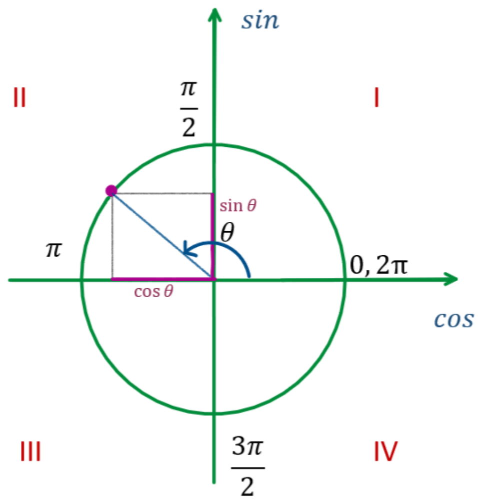


The trigonometric functions can be reduced to the first quadrant using the following relations:

- *If* $\theta \in \left(\frac{\pi}{2}, \pi\right) \Longrightarrow \theta = \pi - x$, *where* $x \in \left(0, \frac{\pi}{2}\right) \Longrightarrow \begin{cases} \sin(\pi - x) = \sin(x) \\ \cos(\pi - x) = -\cos(x) \\ \tan(\pi - x) = -\tan(x). \end{cases}$

- *If* $\theta \in \left(\pi, \frac{3\pi}{2}\right) \Longrightarrow \theta = \pi + x$, *where* $x \in \left(0, \frac{\pi}{2}\right) \Longrightarrow \begin{cases} \sin(\pi + x) = -\sin(x) \\ \cos(\pi + x) = -\cos(x) \\ \tan(\pi + x) = \tan(x). \end{cases}$

- *If* $\theta \in \left(\frac{3\pi}{2}, 2\pi\right) \Longrightarrow \theta = 2\pi - x$, *where* $x \in \left(0, \frac{\pi}{2}\right) \Longrightarrow \begin{cases} \sin(2\pi - x) = -\sin(x) \\ \cos(2\pi - x) = \cos(x) \\ \tan(2\pi - x) = -\tan(x). \end{cases}$



Space Coordinates

In $\mathbb{R}^3$ we consider the Cartesian orthogonal coordinate system $Oxyz$. The **cartesian coordinates** of the point $M \in \mathbb{R}^3$ is the ordered triple of real numbers (x_0, y_0, z_0) and we denote this by $M(x_0, y_0, z_0)$.

The **cylindrical coordinates** of M are (r, θ, z) where:

- r is the length of the line segment $[OM']$, where M' is the projection on xOy of M, $r \geqslant 0$.
- θ is the angle between the positive direction of Ox and OM', $\theta \in [0, 2\pi]$.

We denote the cylindrical coordinates by $M(r, \theta, z)$.

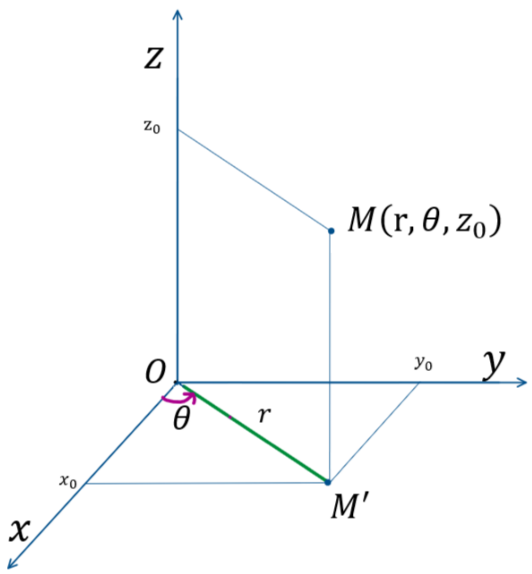


Figure 4.2: The cylindrical coordinates

The relationship between the Cartesian (x, y, z) and cylindrical (r, θ, z) coordinates are:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} ; \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x},\ x \neq 0 \\ z = z. \end{cases} .$$

If $x = 0$ and $y > 0$ then $\theta = \dfrac{\pi}{2}$.

If $x = 0$ and $y < 0$ then $\theta = \dfrac{3\pi}{2}$.

The **spherical coordinates** to locate the point M is space are (ρ, φ, θ) where :

- ρ is the length of the line segment $[OM]$, $\rho \geqslant 0$ *distance*;
- φ is the angle between the positive direction of Oz and OM, $\varphi \in [0, \pi]$; *elevation*;

- θ is the angle between the positive direction of Ox and OM', where M' is the projection on xOy plane of the point M, $\theta \in [0, 2\pi]$, *azimuth*.

We denote the spherical coordinates $M(\rho, \varphi, \theta)$.

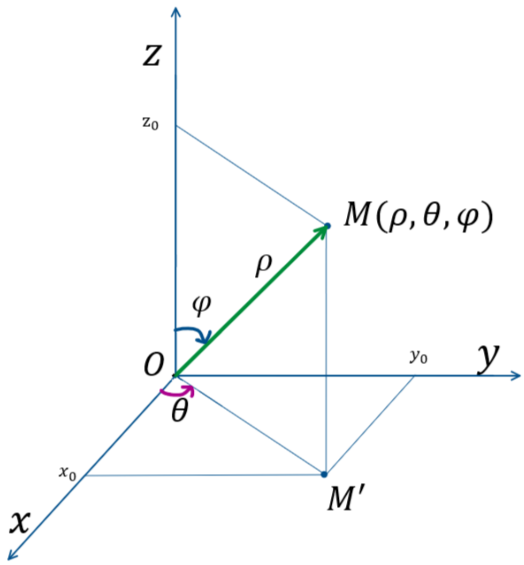


Figure 4.3: The spherical coordinates

The relationship between the Cartesian (x, y, z) and spherical (ρ, φ, θ) coordinates are:

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases} ; \qquad \begin{cases} r = \rho \sin\varphi \\ z = \rho \cos\varphi \\ \theta = \theta \end{cases} .$$

4.2 Vectors in space

A vector in space $\vec{v} \in \mathbb{R}^3$ is determined by its:

- **length**, $\|\vec{v}\|$ or $|\vec{v}|$ (magnitude, absolute value) which is a nonnegative number;
- **direction** - a straight line which represents all the straight lines parallel to the given one;
- **sense** in which the given straight line is directed.

The vectors are added by either the triangle law or the parallelogram law.

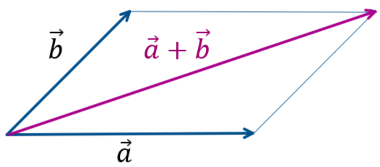


Figure 4.4: The parallelogram law

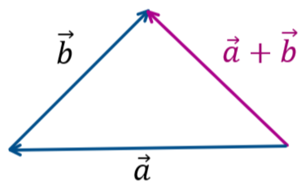


Figure 4.5: The triangle rule

The set of all vectors in space is denoted by $\mathcal{V}_3$.

In talking about vectors, numbers are often called scalars.

Consider now the axes Ox, Oy, Oz, mutually perpendicular, forming a right-handed rectangular Cartesian co-ordinate frame. Let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the unit vectors for this system. In vector spaces notations, $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$ and $\vec{k} = (0,0,1)$ and they represent the canonical basis of $\mathbb{R}^3$.

Every vector $\vec{v}$ can be written, uniquely, in the form

$$\vec{v} = a\vec{i} + b\vec{j} + c\vec{k} = (a,b,c),$$

where a, b, c are scalars (the components of $\vec{v}$).

Any ordered pair of points $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_A)$ of the space define one and only one vector

$$\overrightarrow{AB} = (x_B - x_A)\overrightarrow{i} + (y_B - y_A)\overrightarrow{j} + (z_B - z_A)\overrightarrow{k}.$$

Let the point $M(x_M, y_M, z_M) \in \mathbb{R}^3$. Then the vector

$$\overrightarrow{OM} = x_M\overrightarrow{i} + y_M\overrightarrow{j} + z_M\overrightarrow{k}$$

is called the *position vector* of the point M.

For $\overrightarrow{v_1} = a_1\overrightarrow{i} + b_1\overrightarrow{j} + c_1\overrightarrow{k}$ and $\overrightarrow{v_2} = a_2\overrightarrow{i} + b_2\overrightarrow{j} + c_2\overrightarrow{k}$ we have:

- $\|\overrightarrow{v_1}\| = \sqrt{a_1^2 + b_1^2 + c_1^2}$ - **magnitude, length, absolute value**;
- $\overrightarrow{v_1} + \overrightarrow{v_2} = (a_1 + a_2)\overrightarrow{i} + (b_1 + b_2)\overrightarrow{j} + (c_1 + c_2)\overrightarrow{k}$ - addition of two vectors;
- $\alpha\overrightarrow{v_1} = \alpha a_1\overrightarrow{i} + \alpha b_1\overrightarrow{j} + \alpha c_1\overrightarrow{k}$, $\alpha \in \mathbb{R}$ - multiplication of a vector by a scalar;
- $\overrightarrow{v_1} \parallel \overrightarrow{v_2} \Leftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Leftrightarrow \exists \alpha \in \mathbb{R}^*$ such that $\overrightarrow{v_1} = \alpha\overrightarrow{v_2}$.

The set $\mathcal{V}_3$ is a vector space over the field of the real numbers, where the internal operation is the addition of the vectors, and the external operation is the multiplication by scalars, both defined above.

Scalar product

One associates with any two vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ a number called their **scalar product** (inner product) and denoted by $\overrightarrow{v_1} \cdot \overrightarrow{v_2}$.

$$\overrightarrow{v_1} \cdot \overrightarrow{v_2} \overset{def}{=} \|\overrightarrow{v_1}\| \cdot \|\overrightarrow{v_2}\| \cdot \cos\alpha,$$

where $\alpha \in [0, \pi]$ is the angle between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

Properties. For all $\overrightarrow{v_1}, \overrightarrow{v_2} \in \mathcal{V}_3$, $\alpha \in \mathbb{R}$ we have:

1. $\overrightarrow{v_1} \cdot \overrightarrow{v_2} = \overrightarrow{v_2} \cdot \overrightarrow{v_1}$ - *comutativity*;
2. $\overrightarrow{v_1} \cdot (\overrightarrow{v_2} + \overrightarrow{v_3}) = \overrightarrow{v_1} \cdot \overrightarrow{v_2} + \overrightarrow{v_1} \cdot \overrightarrow{v_3}$ - *distributivity over the addition of the vectors*;
3. $(\alpha\overrightarrow{v_1}) \cdot \overrightarrow{v_2} = \alpha(\overrightarrow{v_1} \cdot \overrightarrow{v_2})$;
4. $\overrightarrow{v_1} \cdot \overrightarrow{v_1} \geqslant 0$, $\overrightarrow{v_1} \cdot \overrightarrow{v_1} = 0 \Longleftrightarrow \overrightarrow{v_1} = \overrightarrow{0}$.

By the definition of the scalar product we have:

- $\cos\alpha = \dfrac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{\|\overrightarrow{v_1}\| \cdot \|\overrightarrow{v_2}\|}$.
- $\overrightarrow{v_1} \perp \overrightarrow{v_2} \Leftrightarrow \overrightarrow{v_1} \cdot \overrightarrow{v_2} = 0$.
- $\overrightarrow{v} \cdot \overrightarrow{v} = \|\overrightarrow{v}\|^2$.
- $\mathrm{pr}_{\overrightarrow{v_1}}\overrightarrow{v_2} = |\cos\alpha| \cdot \|v_2\|$, where $\alpha = \sphericalangle(\overrightarrow{v_1}, \overrightarrow{v_2})$.

  The absolute value for cos is required when the angle between the vectors is grater than $\dfrac{\pi}{2}$.

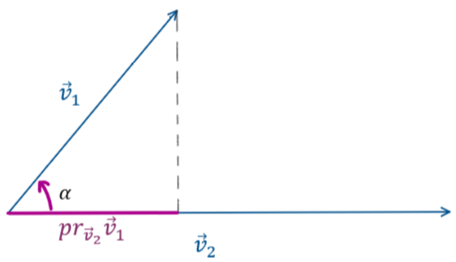


Figure 4.6: The projection when the angle $\alpha \in (0, \frac{\pi}{2})$

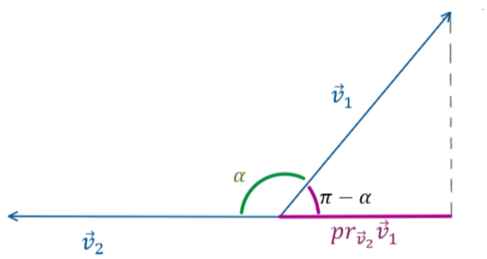


Figure 4.7: The projection when the angle $\alpha \in (\frac{\pi}{2}, \pi)$

If $\overrightarrow{v_1} = a_1\overrightarrow{i} + b_1\overrightarrow{j} + c_1\overrightarrow{k}$ and $v_2 = a_2\overrightarrow{i} + b_2\overrightarrow{j} + c_2\overrightarrow{k}$, we have the following formula for computing the scalar product of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$:

$$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = a_1a_2 + b_1b_2 + c_1c_2.$$

Vector Product

The **vector product** of the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ is the vector $\overrightarrow{v_1} \times \overrightarrow{v_2}$, characterized by:

- **the length** $|\overrightarrow{v_1} \times \overrightarrow{v_2}| = |\overrightarrow{v_1}| \cdot |\overrightarrow{v_2}| \cdot \sin \alpha$;
- **the direction** $\overrightarrow{v_1} \times \overrightarrow{v_2}$ is perpendicular to both $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$;
- **the sense** such that the triad of vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_1} \times \overrightarrow{v_2}\}$ is oriented like the triad $\{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\}$.

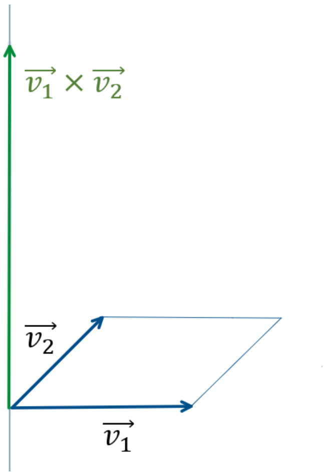


Figure 4.8: The vector product

Properties. For all $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3} \in \mathcal{V}_3$ and $\alpha \in \mathbb{R}$ we have:

1. $\overrightarrow{v_1} \times \overrightarrow{v_2} = -\overrightarrow{v_2} \times \overrightarrow{v_1}$;
2. $(\alpha\overrightarrow{v_1}) \times \overrightarrow{v_2} = \alpha(\overrightarrow{v_1} \times \overrightarrow{v_2})$;
3. $\overrightarrow{v_1} \times (\overrightarrow{v_2} + \overrightarrow{v_3}) = \overrightarrow{v_1} \times \overrightarrow{v_2} + \overrightarrow{v_1} \times \overrightarrow{v_3}$;
4. $\overrightarrow{v_1} \parallel \overrightarrow{v_2} \Leftrightarrow \overrightarrow{v_1} \times \overrightarrow{v_2} = \overrightarrow{0}$;

5. The magnitude of the vector product equals the numerical value of the area of the parallelogram constructed on $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, $\mathcal{A}_{\overrightarrow{v_1}\overrightarrow{v_2}} = |\overrightarrow{v_1} \times \overrightarrow{v_2}|$.

By the definition of the vector product we have

$\overrightarrow{v_1} \times \overrightarrow{v_2}$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overrightarrow{k}$
$\overrightarrow{i}$	$\overrightarrow{0}$	$\overrightarrow{k}$	$-\overrightarrow{j}$
$\overrightarrow{j}$	$-\overrightarrow{k}$	$\overrightarrow{0}$	$\overrightarrow{i}$
$\overrightarrow{k}$	$\overrightarrow{j}$	$-\overrightarrow{i}$	$\overrightarrow{0}$

If $\overrightarrow{v_1} = a_1\overrightarrow{i} + b_1\overrightarrow{j} + c_1\overrightarrow{k}$ and $\overrightarrow{v_2} = a_2\overrightarrow{i} + b_2\overrightarrow{j} + c_2\overrightarrow{k}$, we have the following formula for computing the vector product of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$

$$\overrightarrow{v_1} \times \overrightarrow{v_2} = (b_1c_2 - c_1b_2)\overrightarrow{i} + (c_1a_2 - a_1c_2)\overrightarrow{j} + (a_1b_2 - a_2b_1)\overrightarrow{k}$$

or, using a symbolic determinant:

$$\overrightarrow{v_1} \times \overrightarrow{v_2} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

Triple scalar product

The triple scalar product of the vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$ is defined by

$$(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = \overrightarrow{v_1} \cdot (\overrightarrow{v_2} \times \overrightarrow{v_3}).$$

Properties. For all $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3} \in \mathcal{V}_3$ and $\alpha \in \mathbb{R}$ we have:

1. $(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = (\overrightarrow{v_3}, \overrightarrow{v_1}, \overrightarrow{v_2}) = (\overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_1})$;

2. $(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = -(\overrightarrow{v_2}, \overrightarrow{v_1}, \overrightarrow{v_3})$;

3. $(\alpha\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = \alpha(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3})$.

Geometry applications.

1. The absolute value of the triple scalar product equals the numerical value of the volume of the parallelepiped constructed on $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$.

2. The volume of the tetrahedron constructed on the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ equals $\frac{1}{6}|(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3})|$.

3. $(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = 0 \Leftrightarrow \overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ are parallel to the same plane (the vectors are coplanar).

For $\overrightarrow{v_1} = a_1\overrightarrow{i} + b_1\overrightarrow{j} + c_1\overrightarrow{k}$, $\overrightarrow{v_2} = a_2\overrightarrow{i} + b_2\overrightarrow{j} + c_2\overrightarrow{k}$ and $\overrightarrow{v_3} = a_3\overrightarrow{i} + b_3\overrightarrow{j} + c_3\overrightarrow{k}$ the formula for computing the triple scalar product of $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$ is:

$$(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Triple vector product

The triple vector product of the vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$ is the vector $\overrightarrow{v_1} \times (\overrightarrow{v_2} \times \overrightarrow{v_3})$.

It has no important geometrical meaning, but is expressed by the Gibbs formula which is of use for applications:

$$\overrightarrow{v_1} \times (\overrightarrow{v_2} \times \overrightarrow{v_3}) = (\overrightarrow{v_1} \cdot \overrightarrow{v_3})\overrightarrow{v_2} - (\overrightarrow{v_1} \cdot \overrightarrow{v_2})\overrightarrow{v_3} = \begin{vmatrix} \overrightarrow{v_2} & \overrightarrow{v_3} \\ \overrightarrow{v_1} \cdot \overrightarrow{v_2} & \overrightarrow{v_1} \cdot \overrightarrow{v_3} \end{vmatrix}.$$

4.3 Solved problems

Problem 4.1. Consider the points $A(2,0)$, $B(-3,0)$, $C(-1,1)$, $D(0,-4)$, $E(3,-3\sqrt{3})$ in $\mathbb{R}^2$. Convert rectangular coordinates to polar coordinates.

Solution:

- $A(2,0) \implies x = 2,\ y = 0 \implies \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases} \implies \begin{cases} r = \sqrt{2^2+0^2} \\ \tan\theta = \dfrac{0}{2} \end{cases} \implies$
$\begin{cases} r = 2 \\ \theta = 0 \end{cases} \implies$ the polar coordinates of A are $(2,0)$.

- $B(-3,0) \implies x = -3,\ y = 0 \implies \begin{cases} r = \sqrt{(-3)^2+0^2} \\ \tan\theta = \dfrac{0}{-3} \end{cases} \iff \begin{cases} r = 3 \\ \theta = \pi \end{cases} \implies$
the polar coordinates of B are $(3,\pi)$.

- $C(-1,1) \implies x = -1,\ y = 1 \implies \begin{cases} r = \sqrt{(-1)^2+1^2} \\ \tan\theta = \dfrac{1}{-1}, \quad \theta \in \left(\dfrac{\pi}{2},\pi\right) \end{cases} \implies$
$\begin{cases} r = \sqrt{2} \\ \theta = \pi - \arctan(1) = \dfrac{3\pi}{4} \end{cases} \implies$ the polar coordinates of C are $\left(\sqrt{2}, \dfrac{3\pi}{4}\right)$.

- $D(0,-4) \implies x = 0,\ y = -4 \implies \begin{cases} r = \sqrt{0^2+(-4)^2} \\ \theta = \dfrac{3\pi}{2} \end{cases} \implies$ the polar coordinates of D are $\left(4, \dfrac{3\pi}{2}\right)$.

- $E(3,-3\sqrt{3}) \implies x = -2,\ y = 2\sqrt{3} \implies \begin{cases} r = \sqrt{(3)^2+(-3\sqrt{3})^2} \\ \tan\theta = \dfrac{-3\sqrt{3}}{3}, \quad \theta \in \left(\dfrac{3\pi}{2}, 2\pi\right) \end{cases}$
$\implies \begin{cases} r = 6 \\ \theta = 2\pi - \arctan(\sqrt{3}) = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3} \end{cases}$
$\implies$ the polar coordinates of E are $\left(6, \dfrac{5\pi}{3}\right)$.

Problem 4.2. Let $A\left(3,\dfrac{3\pi}{4}\right)$, $B\left(5,\dfrac{4\pi}{3}\right)\in\mathbb{R}^2$. Convert the polar coordinates of the given points to cartesian coordinates.

Solution:

- $A\left(3,\dfrac{3\pi}{4}\right)\Longrightarrow r=3,\ \theta=\dfrac{3\pi}{4}\Longrightarrow\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}\Longrightarrow\begin{cases}x=3\cos\dfrac{3\pi}{4}\\ y=3\sin\dfrac{3\pi}{4}\end{cases}\Longrightarrow$ $\begin{cases}x=3\left(-\dfrac{\sqrt{2}}{2}\right)\\ y=3\dfrac{\sqrt{2}}{2}\end{cases}$ $\Longrightarrow$ the cartesian coordinates of A are $\left(-\dfrac{3\sqrt{2}}{2},\dfrac{3\sqrt{2}}{2}\right)$.

- $B\left(5,\dfrac{4\pi}{3}\right)\Longrightarrow r=5,\ \theta=\dfrac{4\pi}{3}\Longrightarrow\begin{cases}x=5\cos\dfrac{4\pi}{3}\\ y=5\sin\dfrac{4\pi}{3}\end{cases}\Longrightarrow\begin{cases}x=5\left(-\dfrac{\sqrt{3}}{2}\right)\\ y=5\left(-\dfrac{1}{2}\right)\end{cases}$ $\Longrightarrow$ the cartesian coordinates of B are $\left(-\dfrac{5\sqrt{3}}{2},-\dfrac{5}{2}\right)$.

Problem 4.3. Let $A\left(4,\dfrac{2\pi}{3},\dfrac{7\pi}{4}\right)\in\mathbb{R}^3$. Convert the spherical coordinates of A to cylindrical and cartesian coordinates.

Solution:

$$A\left(4,\frac{2\pi}{3},\frac{7\pi}{4}\right)\Longrightarrow\rho=4,\ \varphi=\frac{2\pi}{3},\theta=\frac{7\pi}{4}\Longrightarrow\begin{cases}x=\rho\sin\varphi\cos\theta\\ y=\rho\sin\varphi\sin\theta\\ z=\rho\cos\varphi\end{cases}\Longrightarrow$$

$\begin{cases}x=4\sin\dfrac{2\pi}{3}\cos\dfrac{7\pi}{4}\\ y=4\sin\dfrac{2\pi}{3}\sin\dfrac{7\pi}{4}\\ z=4\cos\dfrac{2\pi}{3}\end{cases}\Longrightarrow\begin{cases}x=4\cdot\dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2}\\ y=4\cdot\dfrac{\sqrt{3}}{2}\cdot\left(-\dfrac{\sqrt{2}}{2}\right)\\ z=4\cdot\left(-\dfrac{1}{2}\right)\end{cases}$ $\Longrightarrow$ the cartesian coordinates of A are $(\sqrt{6},-\sqrt{6},-2)$.

For the cylindrical coordinates we need to determine (r, θ, z). We know $\theta = \dfrac{7\pi}{6}$ and $z = -2$. We calculate $r = \sqrt{x^2 + y^2} = \sqrt{12} = 2\sqrt{3} \Longrightarrow$ the cylindrical coordinates of A are $\left(2\sqrt{3}, \dfrac{7\pi}{4}, -2\right)$.

Problem 4.4. Determine the cylindrical and spherical coordinates of $A \in \mathbb{R}^3$ $A(-\sqrt{2}, \sqrt{2}, 2\sqrt{3})$, given in the cartesian coordinates.

Solution:

$$A(-\sqrt{2}, \sqrt{2}, 2\sqrt{3}) \Longrightarrow \begin{cases} x = -\sqrt{2} \\ y = \sqrt{2} \\ z = 2\sqrt{3} \end{cases} \Longrightarrow \begin{cases} \rho = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2 + (2\sqrt{3})^2} \\ \cos\varphi = \dfrac{2\sqrt{3}}{\rho}, \ \varphi \in \left(0, \dfrac{\pi}{2}\right) \\ \tan\theta = \dfrac{\sqrt{2}}{-\sqrt{2}}, \ \theta \in \left(\dfrac{\pi}{2}, \pi\right) \end{cases}$$

$$\Longrightarrow \begin{cases} \rho = 4 \\ \cos\varphi = \dfrac{2\sqrt{3}}{4} \\ \theta = \pi - \arctan 1, \end{cases} \Longrightarrow \begin{cases} \rho = 4 \\ \varphi = \arccos\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{6} \\ \theta = \pi - \arctan 1 = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}, \end{cases} \Longrightarrow$$

the cartesian coordinates of A are $\left(4, \dfrac{\pi}{6}, \dfrac{3\pi}{4}\right)$.

For the cylindrical coordinates we need to determine (r, θ, z). We know $\theta = \dfrac{3\pi}{4}$, and $z = 2\sqrt{3}$. We calculate $r = \sqrt{x^2 + y^2} = \sqrt{4} = 2 \Longrightarrow$ the cylindrical coordinates of A are $\left(2, \dfrac{3\pi}{4}, 2\sqrt{3}\right)$.

Problem 4.5. Let $M\left(5, \dfrac{5\pi}{3}, -3\right) \in \mathbb{R}^3$ given in cylindrical coordinates. Determine the cartesian and spherical coordinates of M.

Solution:

$$M\left(5, \dfrac{5\pi}{3}, -4\right) \Longrightarrow \begin{cases} r = 5 \\ \theta = \dfrac{5\pi}{3} \\ z = -4 \end{cases} \Longrightarrow \begin{cases} x = 5\cos\dfrac{5\pi}{3} \\ y = 5\sin\dfrac{5\pi}{3} \\ z = -4 \end{cases} \Longrightarrow \begin{cases} x = 5 \cdot \dfrac{1}{2} \\ y = 5 \cdot \left(-\dfrac{\sqrt{3}}{2}\right) \\ z = -4 \end{cases}$$

$\Longrightarrow$ the cartesian coordinates of M are $\left(\frac{5}{2}, -\frac{5\sqrt{3}}{2}, -4\right)$.

For the spherical coordinates we need to determine (ρ, φ, θ). We know $\theta = \frac{5\pi}{3}$. We calculate $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{41}$ and $\cos\varphi = \frac{-4}{\rho} = -\frac{4}{\sqrt{41}}$, $\varphi \in \left(\frac{\pi}{2}, \pi\right) \Longrightarrow$ $\varphi = \pi - \arccos\frac{4}{\sqrt{29}}$.

The spherical coordinates of M are $\left(\sqrt{41}, \pi - \arccos\frac{4}{\sqrt{41}}, \frac{5\pi}{3}\right)$.

Problem 4.6. Decompose $\overrightarrow{v} = \overrightarrow{i} + 3\overrightarrow{j} - 5\overrightarrow{k}$ as a linear combination of the vectors $\overrightarrow{a} = \overrightarrow{i} + 2\overrightarrow{j}$, $\overrightarrow{b} = \overrightarrow{i} + 2\overrightarrow{k}$, $\overrightarrow{c} = 2\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$.

Solution:

The decomposition is $\overrightarrow{v} = \alpha\overrightarrow{a} + \beta\overrightarrow{b} + \gamma\overrightarrow{c} \Longleftrightarrow$
$\overrightarrow{i} + 3\overrightarrow{j} - 5\overrightarrow{k} = \alpha(\overrightarrow{i} + 2\overrightarrow{j}) + \beta(\overrightarrow{i} + 2\overrightarrow{k}) + \gamma(2\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}) \Longleftrightarrow$

$$\begin{cases} \alpha + \beta + 2\gamma = 1 \\ 2\alpha - \gamma = 3 \\ 2\beta + \gamma = -5 \end{cases}$$

The system has the solution $\alpha = 2$, $\beta = -3$, $\gamma = 1$ so, the decomposition is $\overrightarrow{v} = 2\overrightarrow{a} - 3\overrightarrow{b} + \overrightarrow{c}$.

Problem 4.7. Let $\overrightarrow{a} = 2\overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k}$, $\overrightarrow{b} = \overrightarrow{j} - 2\overrightarrow{k}$, $\overrightarrow{c} = \overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k}$. Calculate:

a) $\overrightarrow{a} + \overrightarrow{b}$.

b) $\|\overrightarrow{c}\|$.

c) $\overrightarrow{b} \cdot \overrightarrow{c}$.

d) $\overrightarrow{a} \times \overrightarrow{b}$.

e) The angle between $\overrightarrow{a}$ and $\overrightarrow{b}$.

f) $\overrightarrow{a} \cdot (\overrightarrow{b} + \overrightarrow{c})$.

g) $(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c}$.

h) The projection of $\overrightarrow{a}$ on $\overrightarrow{b}$, $\text{pr}_{\overrightarrow{b}} \overrightarrow{a}$.

Solution:

a) $\overrightarrow{a} + \overrightarrow{b} = (2\overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k}) + (\overrightarrow{j} - 2\overrightarrow{k}) = 2\overrightarrow{i} - 2\overrightarrow{j} - \overrightarrow{k}$.

b) $\|\overrightarrow{c}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$.

c) $\overrightarrow{b} \cdot \overrightarrow{c} = 0 \cdot 1 + 1 \cdot (-1) + (-2) \cdot 2 = -5$.

d) $\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & -3 & 1 \\ 0 & 1 & -2 \end{vmatrix} = 5\overrightarrow{i} + 4\overrightarrow{j} + 2\overrightarrow{k}$.

e) $\cos \sphericalangle(\overrightarrow{a}, \overrightarrow{b}) = \dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{\|\overrightarrow{a}\| \cdot \|\overrightarrow{b}\|} = \dfrac{2 \cdot 0 + (-3) \cdot 1 + 1 \cdot (-2)}{\sqrt{4+9+1}\sqrt{0+1+4}} = -\dfrac{5}{\sqrt{70}} < 0 \Longrightarrow$

$\varphi \in \left(\dfrac{\pi}{2}, \pi\right)$.

$\sphericalangle(\overrightarrow{a}, \overrightarrow{b}) = \pi - \arccos \dfrac{5}{\sqrt{70}}$.

f) $\overrightarrow{b} + \overrightarrow{c} = \overrightarrow{i}$.

$\overrightarrow{a} \cdot (\overrightarrow{b} + \overrightarrow{c}) = (2\overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k}) \cdot (\overrightarrow{i}) = 2$.

g) $(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c} = \overrightarrow{c} \cdot (\overrightarrow{a} \times \overrightarrow{b}) = (\overrightarrow{c}, \overrightarrow{a}, \overrightarrow{b}) = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -3 & 1 \\ 0 & 1 & -2 \end{vmatrix} = 5$.

h) $\varphi = \sphericalangle(\overrightarrow{a}, \overrightarrow{b}) \Longrightarrow \text{pr}_{\overrightarrow{b}} \overrightarrow{a} = |\cos \varphi| \|\overrightarrow{a}\|$.

$\cos \varphi = \dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{\|\overrightarrow{a}\| \cdot \|\overrightarrow{b}\|} = \dfrac{-5}{\sqrt{70}} \Longrightarrow \varphi \in \left(\dfrac{\pi}{2}, \pi\right)$.

$$\text{pr}_{\overrightarrow{b}}\overrightarrow{a} = \sqrt{14}\cdot\frac{5}{\sqrt{70}} = \sqrt{5}.$$

Problem 4.8. Consider the points $A(-2,1,0)$, $B(2,0,3)$, $C(-2,0,4)$, $D(-4,1,3)$. Determine:

a) Area of the triangle $\triangle ABC$.

b) The distance between the point D and the line BC.

c) $\sphericalangle(\overrightarrow{AB},\overrightarrow{CD})$.

d) The volume of the tetrahedron $ABCD$.

e) The height of the tetrahedron $ABCD$ having as basis the plane ABC.

Solution:

a) $\mathcal{A}_{\triangle ABC} = \dfrac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\|.$

$\overrightarrow{AB} = 4\overrightarrow{i} - \overrightarrow{j} + 3\overrightarrow{k};$

$\overrightarrow{AC} = -\overrightarrow{j} + 4\overrightarrow{k}.$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 4 & -1 & 3 \\ 0 & -1 & 4 \end{vmatrix} = -\overrightarrow{i} - 16\overrightarrow{j} - 4\overrightarrow{k}.$$

$$\mathcal{A}_{\triangle ABC} = \frac{1}{2}\sqrt{1+256+16} = \frac{1}{2}\sqrt{273}.$$

b) $d(A, BC) = h$, where h is the height on BC of the triangle .

$\mathcal{A}_{\triangle ABC} = \dfrac{h\cdot BC}{2}.$

$\overrightarrow{BC} = -4\overrightarrow{i} + \overrightarrow{k} \Longrightarrow \|\overrightarrow{BC}\| = \sqrt{17}.$

$$h = \frac{\mathcal{A}_{\triangle ABC}}{\|\overrightarrow{BC}\|} = \sqrt{\frac{273}{17}}.$$

c) $\cos \sphericalangle(\overrightarrow{AB}, \overrightarrow{CD}) = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\|\overrightarrow{AB}\| \cdot \|\overrightarrow{CD}\|}$.

$\overrightarrow{CD} = -2\vec{i} + \vec{j} - \vec{k}$;

$\cos \sphericalangle(\overrightarrow{AB}, \overrightarrow{CD}) = \dfrac{-12}{\sqrt{26}\sqrt{6}} = -\dfrac{6}{\sqrt{39}} < 0 \Longrightarrow \varphi \in \left(\frac{\pi}{2}, \pi\right) \Longrightarrow$

$= \pi - \arccos \dfrac{6}{\sqrt{39}}$.

d) $$\mathcal{V}_{ABCD} = \frac{1}{6}|(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})| = \frac{1}{6}|\begin{vmatrix} 4 & -1 & 3 \\ 0 & -1 & 4 \\ -2 & 0 & 3 \end{vmatrix}| = \frac{1}{6}|-10| = \frac{5}{3}.$$

e) $$\mathcal{V}_{ABCD} = \frac{\mathcal{A}_{\triangle ABC} \cdot h}{3} \Longrightarrow h = \frac{3\mathcal{V}_{ABCD}}{\mathcal{A}_{ABC}} = \frac{3 \cdot \frac{5}{3}}{\frac{\sqrt{273}}{2}} = \frac{10}{\sqrt{273}}.$$

Problem 4.9. Consider the vectors $\vec{u} = \vec{i} - \lambda\vec{j} + 3\vec{k}$, $\vec{v} = \lambda\vec{i} - \vec{j} + \vec{k}$ and $\vec{w} = 3\vec{i} + \vec{j} - \vec{k}$. Determine $\lambda \in \mathbb{R}$ such that the vectors $\vec{u}, \vec{v}, \vec{w}$ are coplanar.

Solution:

$$\vec{u}, \vec{v}, \vec{w} \text{ are coplanar} \Longleftrightarrow (\vec{u}, \vec{v}, \vec{w}) = 0 \Longleftrightarrow \begin{vmatrix} 1 & -\lambda & 3 \\ \lambda & -1 & 1 \\ 3 & 1 & -1 \end{vmatrix} = 0$$

$\Longleftrightarrow 9 - \lambda^2 = 0 \Longleftrightarrow \lambda \in \{-3, 3\}$.

Problem 4.10. Determine $\lambda \in \mathbb{R}$ such that the vectors $\vec{a} = \vec{i} + 2\lambda\vec{j} - (\lambda - 1)\vec{k}$ and $\vec{b} = (3 - \lambda)\vec{i} + \vec{j} + 2\vec{k}$ are perpendicular.

Solution:

$\vec{a} \perp \vec{b} \Longleftrightarrow \vec{a} \cdot \vec{b} = 0 \Longleftrightarrow 1(3 - \lambda) + 2\lambda \cdot 1 - (\lambda - 1)2 = 0 \Longleftrightarrow \lambda = 5$.

Problem 4.11. Determine the angle between $\vec{u}$ and $\vec{v}$ if $\|\vec{u}\| = 2$, $\|\vec{v}\| = 4$ and $(2\vec{u} + \vec{v}) \perp (3\vec{u} - 2\vec{v})$.

Solution: Let $\alpha = \sphericalangle(\overrightarrow{u}, \overrightarrow{v})$.

$(2\overrightarrow{u} + \overrightarrow{v}) \perp (3\overrightarrow{u} - 2\overrightarrow{v}) \Longleftrightarrow (2\overrightarrow{u} + \overrightarrow{v}) \cdot (3\overrightarrow{u} - 2\overrightarrow{v}) = 0 \Longleftrightarrow$

$6\overrightarrow{u} \cdot \overrightarrow{u} - 4\overrightarrow{u} \cdot \overrightarrow{v} + 3\overrightarrow{v} \cdot \overrightarrow{u} - 2\overrightarrow{v} \cdot \overrightarrow{v} = 0.$

Applying the definition and the properties of the scalar product we have:

$6\|u\|^2 - \|\overrightarrow{u}\| \cdot \|\overrightarrow{v}\| \cdot \cos\alpha - 2\|\overrightarrow{v}\|^2 = 0 \Longleftrightarrow$

$6 \cdot 4 - 2 \cdot 4 \cdot \cos\alpha - 2 \cdot 16 = 0 \Longleftrightarrow$

$24 - 8\cos\alpha - 32 = 0 \Longleftrightarrow \cos\alpha = -1 \Longleftrightarrow \alpha = \pi.$

Problem 4.12. Consider $\overrightarrow{a} = 5\overrightarrow{p} - 3\overrightarrow{q}$ and $\overrightarrow{b} = \overrightarrow{p} + 2\overrightarrow{q}$, such that $\|\overrightarrow{p}\| = 3$, $\|\overrightarrow{q}\| = 2$ and the angle between $\overrightarrow{p}$ and $\overrightarrow{q}$ equals $\frac{\pi}{3}$.

a) Determine the length of the two diagonals of the parallelogram with the edges $\overrightarrow{a}$ and $\overrightarrow{b}$.

b) Find the angle between the diagonals of the parallelogram with $\overrightarrow{a}$ and $\overrightarrow{b}$ as the edges.

c) Calculate the area of the parallelogram constructed on $\overrightarrow{a}$ and $\overrightarrow{b}$.

Solution:

a) We know from the parallelogram and triangle rule that the two diagonals of a parallelogram is the sum and the difference of the two vectors on which the parallelogram is constructed.

Let $\overrightarrow{d_1} = \overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{d_2} = \overrightarrow{a} - \overrightarrow{b}$

$\overrightarrow{d_1} = 5\overrightarrow{p} - 3\overrightarrow{q} + \overrightarrow{p} + 2\overrightarrow{q} = 6\overrightarrow{p} - \overrightarrow{q}.$

$\overrightarrow{d_2} = 5\overrightarrow{p} - 3\overrightarrow{q} - (\overrightarrow{p} + 2\overrightarrow{q}) = 4\overrightarrow{p} - 5\overrightarrow{q}.$

In what follows we will use the formula $\overrightarrow{v} \cdot \overrightarrow{v} = \|\overrightarrow{v}\|^2$ to determine the length of the diagonals.

We evaluate:

$$\begin{aligned}
\|\overrightarrow{d_1}\|^2 &= \|6\overrightarrow{p} - \overrightarrow{q}\|^2 = (6\overrightarrow{p} - \overrightarrow{q}) \cdot (6\overrightarrow{p} - \overrightarrow{q}) \\
&= 36\overrightarrow{p} \cdot \overrightarrow{p} - 6\overrightarrow{p} \cdot \overrightarrow{q} - 6\overrightarrow{q} \cdot \overrightarrow{p} + \overrightarrow{q} \cdot \overrightarrow{q} \\
&= 36\|\overrightarrow{p}\|^2 - 12\|\overrightarrow{p}\| \cdot \|\overrightarrow{q}\| \cdot \cos(\widehat{\overrightarrow{p}, \overrightarrow{q}}) + \|\overrightarrow{q}\|^2 \\
&= 36 \cdot 9 - 12 \cdot 3 \cdot 2 \cdot \cos\frac{\pi}{3} + 4 \\
&= 292
\end{aligned}$$

So, $\|\overrightarrow{d_1}\| = 2\sqrt{73}$.

$$\begin{aligned}
\|\overrightarrow{d_2}\|^2 &= \|4\overrightarrow{p} - 5\overrightarrow{q}\|^2 = (4\overrightarrow{p} - 5\overrightarrow{q}) \cdot (4\overrightarrow{p} - 5\overrightarrow{q}) \\
&= 16\overrightarrow{p} \cdot \overrightarrow{p} - 20\overrightarrow{p} \cdot \overrightarrow{q} - 20\overrightarrow{q} \cdot \overrightarrow{p} + 25\overrightarrow{q} \cdot \overrightarrow{q} \\
&= 16\|\overrightarrow{p}\|^2 - 40\|\overrightarrow{p}\| \cdot \|\overrightarrow{q}\| \cdot \cos(\widehat{\overrightarrow{p}, \overrightarrow{q}}) + 25\|\overrightarrow{q}\|^2 \\
&= 16 \cdot 9 - 40 \cdot 3 \cdot 2 \cdot \cos\frac{\pi}{3} + 25 \cdot 4 \\
&= 124
\end{aligned}$$

So, $\|\overrightarrow{d_1}\| = 2\sqrt{31}$.

b) Let $\sphericalangle(\overrightarrow{d_1}, \overrightarrow{d_2}) = \varphi \Longrightarrow \cos\varphi = \dfrac{\overrightarrow{d_1} \cdot \overrightarrow{d_2}}{\|\overrightarrow{d_1}\| \cdot \|\overrightarrow{d_2}\|}$.

$$\begin{aligned}
\overrightarrow{d_2} \cdot \overrightarrow{d_2} &= (6\overrightarrow{p} - \overrightarrow{q}) \cdot (4\overrightarrow{p} - 5\overrightarrow{q}) \\
&= 24\|\overrightarrow{p}\|^2 - 34\|\overrightarrow{p}\| \cdot \|\overrightarrow{q}\| \cdot \cos(\widehat{\overrightarrow{p}, \overrightarrow{q}}) + 5\|\overrightarrow{q}\|^2 \\
&= 24 \cdot 9 - 34 \cdot 3 \cdot 2 \cdot \frac{1}{2} + 20 \\
&= 134
\end{aligned}$$

$$\cos\varphi = \frac{134}{2\sqrt{73} \cdot 2\sqrt{31}} \Longrightarrow \varphi = \arccos\frac{67}{2\sqrt{73 \cdot 31}}.$$

c) $\mathcal{A}_{\overrightarrow{a}, \overrightarrow{b}} = \dfrac{\|\overrightarrow{d_1}\| \cdot \|\overrightarrow{d_2}\| \cdot \sin \sphericalangle(\overrightarrow{d_1}, \overrightarrow{d_2})}{2}$.

We determine $\sin\varphi = \sqrt{1-\cos^2\varphi} = \sqrt{1 - \dfrac{67^2}{4 \cdot 31 \cdot 73}} = \dfrac{39\sqrt{3}}{2\sqrt{31 \cdot 73}}$.

Then,

$$\mathcal{A}_{\overrightarrow{a}, \overrightarrow{b}} = \frac{2\sqrt{73} \cdot 2\sqrt{31} \cdot \frac{39\sqrt{3}}{2\sqrt{31 \cdot 73}}}{2} = 39\sqrt{3}.$$

Problem 4.13. Prove that $\overrightarrow{a} = 12\overrightarrow{i} + 3\overrightarrow{j} - 4\overrightarrow{k}$ and $\overrightarrow{b} = 3\overrightarrow{i} + 4\overrightarrow{j} + 12\overrightarrow{k}$ are two of the edges of a cube. Determine $\overrightarrow{c}$ such that $\overrightarrow{c}$ is the third edge of this cube.

Solution:

$\overrightarrow{a}$ and $\overrightarrow{b}$ are two of the edges of a cube iff:

1. $\|\overrightarrow{a}\| = \|\overrightarrow{b}\|$
2. $\overrightarrow{a} \perp \overrightarrow{b}$

$\|\overrightarrow{a}\| = \sqrt{12^2 + 3^2 + (-4)^2} = 13$, $\|\overrightarrow{b}\| = \sqrt{3^2 + 4^2 + 12^2} = 13$ so the first condition is fulfilled.

$\overrightarrow{a} \perp \overrightarrow{b} \Longleftrightarrow \overrightarrow{a} \cdot \overrightarrow{b} = 0 \Longleftrightarrow 12 \cdot 3 + 3 \cdot 4 + (-4) \cdot 12 = 0 \Longleftrightarrow 36 + 12 - 48 = 0$ which is true, so $\overrightarrow{a} \perp \overrightarrow{b}$.

The third edge of the cube $\overrightarrow{c}$ must fulfill

1. $\|\overrightarrow{c}\| = \|\overrightarrow{a}\| = \|\overrightarrow{b}\| = 13$,
2. $\begin{cases} \overrightarrow{c} \perp \overrightarrow{a} \\ \overrightarrow{c} \perp \overrightarrow{b} \end{cases} \Longrightarrow \overrightarrow{c} \parallel \overrightarrow{a} \times \overrightarrow{b}$.

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 12 & 3 & -4 \\ 3 & 4 & 12 \end{vmatrix} = 52\overrightarrow{i} - 156\overrightarrow{j} + 39\overrightarrow{k}.$$

Because $\|\overrightarrow{a} \times \overrightarrow{b}\| = \mathcal{A}_{\overrightarrow{a},\overrightarrow{b}} = 13^2$, and $\|\overrightarrow{c}\| = 13$, then,
$\overrightarrow{c} = \pm\frac{1}{13}(52\overrightarrow{i} - 156\overrightarrow{j} + 39\overrightarrow{k}) = \pm(4\overrightarrow{i} - 12\overrightarrow{j} + 3\overrightarrow{k})$.

4.4 Problems

Problem 4.14. Consider the points $A(-3,0)$, $B(0,-4)$, $C(4,-4)$, $D(-7,0)$, $E(0,6)$ $F(\sqrt{3},-3)$ in $\mathbb{R}^2$. Convert rectangular coordinates to polar coordinates.

Problem 4.15. Let $A\left(4,\frac{5\pi}{4}\right)$, $B\left(2,\frac{7\pi}{6}\right)$ and $C\left(9,\frac{\pi}{3}\right)$ points in $\mathbb{R}^2$. Convert the polar coordinates of the given points to cartesian coordinates.

Problem 4.16. The point P is on a sphere of radius 4, O the centre of the sphere. The angle between OP and Oz axis is $30°$, the angle between OP' ($P' = \text{pr}_{xOy}P$) and Ox is $60°$. Determine the cylindrical and cartesian coordinates of P.

Problem 4.17. Consider the points $A\left(8,\frac{\pi}{6},\frac{\pi}{3}\right)$, $B\left(12,\frac{4\pi}{3},\frac{5\pi}{6}\right)$, $C\left(4,\frac{3\pi}{4},\frac{2\pi}{3}\right)$ in $\mathbb{R}^3$. Convert the spherical coordinates of the given points to cylindrical and cartesian coordinates.

Problem 4.18. Determine the cylindrical coordinates of $A(2,2\sqrt{3},5)$, $B(4,-4,6)$, $C(-3,\sqrt{3},-4)$, $D(-3\sqrt{3},-9,0)$, $E(0,0,4)$ given in the cartesian coordinates.

Problem 4.19. Convert the cartesian coordinates of $A(2\sqrt{3},6,4)$, $B(0,-6\sqrt{3},6)$ and $C(-16,0,0)$ to spherical coordinates.

Problem 4.20. Consider the points $A\left(3,\frac{\pi}{3},-3\right)$, $B\left(5,\frac{3\pi}{4},-6\right)$, $C\left(4,\frac{7\pi}{4},2\right)$, $D\left(5,\frac{\pi}{3},2\right)$ in $\mathbb{R}^3$, given in cylindrical coordinates. Determine the cartesian and spherical coordinates of these points.

Problem 4.21. Decompose $\overrightarrow{v} = 2\overrightarrow{i} - 3\overrightarrow{j} - 5\overrightarrow{k}$ as a sum of the vectors $\overrightarrow{v_1} = 3\overrightarrow{i} + \overrightarrow{k}$, $\overrightarrow{v_2} = -\overrightarrow{j} - \overrightarrow{k}$ and $\overrightarrow{v_3} = -\overrightarrow{i} + 3\overrightarrow{k}$

Problem 4.22. Decompose $\overrightarrow{v} = \overrightarrow{i} + \overrightarrow{j} + 2\overrightarrow{k}$ as a linear combination of the vectors $\overrightarrow{a} = 2\overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{b} = -\overrightarrow{i} + 3\overrightarrow{k}$ and $\overrightarrow{c} = -\overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$.

Problem 4.23. Let $\overrightarrow{a} = 2\overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{b} = -\overrightarrow{i} + 3\overrightarrow{k}$, $\overrightarrow{c} = -\overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$. Prove that $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ is a basis for $\mathcal{V}_3$. Determine the coordinates of $\overrightarrow{d} = 2\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$ using the basis $\{\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\}$.

Problem 4.24. Consider three points $A(1, -1, 2)$, $B(2, 1, 0)$ and $C(3, 2, -6)$ in space. Find:

a) the length of the vector $\overrightarrow{AB}$.

b) the scalar product $\overrightarrow{AB} \cdot \overrightarrow{BC}$

c) the area of ΔABC.

Problem 4.25. Consider the vectors $\overrightarrow{u} = 2\overrightarrow{i} + 3\overrightarrow{j} + 2\overrightarrow{k}$, $\overrightarrow{v} = -\overrightarrow{i} + 4\overrightarrow{j} + \overrightarrow{k}$, $\overrightarrow{w} = -2\overrightarrow{j} + 2\overrightarrow{k}$. Calculate:

a) $\overrightarrow{v} + \overrightarrow{w}$.

b) $2\overrightarrow{u} - 3\overrightarrow{w}$.

c) $\overrightarrow{u} \cdot \overrightarrow{v}$.

d) $\|\overrightarrow{v}\|$.

e) $\overrightarrow{u} \times \overrightarrow{w}$.

f) the area of the parallelogram construct on the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$.

g) the hight of the parallelogram with the edges $\overrightarrow{u}$ and $\overrightarrow{v}$, considering $\overrightarrow{v}$ as the basis.

h) the volume of the tetrahedron with the edges $\overrightarrow{u}, \overrightarrow{v}$ and $\overrightarrow{w}$.

Problem 4.26. Consider the points $A(0,1,-1)$, $B(1,1,3)$, $C(2,1,-2)$, $D(3,-1,2)$. Calculate:

a) The volume of the tetrahedron $ABCD$.

b) The distance between the point D and the plane ABC.

c) $\sphericalangle(\overrightarrow{AB}, \overrightarrow{CD})$.

Problem 4.27. Determine $\lambda \in \mathbb{R}$ such that $\overrightarrow{a} = 2\overrightarrow{i} + 3\overrightarrow{j} + (2\lambda - 3)\overrightarrow{k}$ and $\overrightarrow{b} = (2 + 3\lambda)\overrightarrow{i} + \lambda\overrightarrow{j} - 2\overrightarrow{k}$ are perpendicular.

Problem 4.28. Find the angle between the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ if $\|\overrightarrow{a}\| = 4$, $\|\overrightarrow{b}\| = 2$ and $(3\overrightarrow{a} + 5\overrightarrow{b}) \perp (\overrightarrow{a} - 2\overrightarrow{b})$.

Problem 4.29. Prove that $\overrightarrow{a} = 6\overrightarrow{i} + 2\overrightarrow{j} - 3\overrightarrow{k}$ and $\overrightarrow{b} = -3\overrightarrow{i} + 6\overrightarrow{j} - 2\overrightarrow{k}$ are two of the edges of a cube. Determine $\overrightarrow{c}$ such that $\overrightarrow{c}$ is the third edge of this cube.

Problem 4.30. Consider the vectors $\overrightarrow{a} = (2,3,1)$, $\overrightarrow{b} = (1,1,-2)$, $\overrightarrow{c} = (-2,1,2)$. Calculate:

a) $\overrightarrow{a} \cdot (\overrightarrow{b} + \overrightarrow{c})$;

b) $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c})$;

c) $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})$;

d) $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$.

Problem 4.31. Consider four points A, B, C, D in space. Prove that:

a) $\overrightarrow{DA} \cdot \overrightarrow{BC} + \overrightarrow{DB} \cdot \overrightarrow{CA} + \overrightarrow{DC} \cdot \overrightarrow{AB} = 0$.

b) If $DA \perp BC$ and $DB \perp CA$ then $DC \perp AB$.

Problem 4.32. Let G be the weight center of the triangle ABC. Prove that:

a) $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \overrightarrow{0}$.

b) If M is an arbitrary point then $3\overrightarrow{MG} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}$.

Problem 4.33. Prove the Lagrange's identity

$$\|\overrightarrow{a} \times \overrightarrow{b}\|^2 + (\overrightarrow{a} \cdot \overrightarrow{b})^2 = \|\overrightarrow{a}\|^2 \|\overrightarrow{b}\|^2,$$

for any vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

Problem 4.34. Determine the vector $\overrightarrow{w}$ such that $\|\overrightarrow{w}\| = 3$, $\overrightarrow{w}$ is perpendicular on the axis Oz and makes a $45°$ angle with the positive direction of Ox.

Problem 4.35. Find the angle between:

a) the vector $\overrightarrow{v} = -\frac{\sqrt{3}}{2}\overrightarrow{i} + \frac{1}{2}\overrightarrow{j}$ and Ox axis.

b) $\overrightarrow{AB}$ and $\overrightarrow{AC}$ where $A(3, 1, -2)$, $B(2, 1, -1)$ and $C(3, 0, -1)$.

Problem 4.36. Let $\overrightarrow{v} = 3\overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k}$ and $\overrightarrow{u} = \overrightarrow{j} - 2\overrightarrow{k}$. Determine the height of the parallelogram with the edges $\overrightarrow{v}$ and $\overrightarrow{u}$, considering $\overrightarrow{v}$ as the basis.

Problem 4.37. Let $\overrightarrow{a} = 3\overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k}$, $\overrightarrow{b} = \overrightarrow{j} - 2\overrightarrow{k}$ and $\overrightarrow{c} = \overrightarrow{j} + 4\overrightarrow{k}$. Determine the height of the parallelepiped with the edges $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$, considering the parallelogram with the edges $\overrightarrow{a}$ and $\overrightarrow{b}$ as the basis.

Problem 4.38. If $\overrightarrow{a} = 3\overrightarrow{i} - \overrightarrow{j} + \alpha\overrightarrow{k}$, $\overrightarrow{b} = \overrightarrow{j} + 2\overrightarrow{k}$ and $\overrightarrow{c} = 3\overrightarrow{i} - \overrightarrow{k}$, determine $\alpha \in \mathbb{R}$ such that the vector $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$ is parallel to the plane yOz.

Problem 4.39. If $\overrightarrow{a} = \overrightarrow{i} + \overrightarrow{j} + 2\overrightarrow{k}$, $\overrightarrow{b} = 2\overrightarrow{i} - \overrightarrow{j} + \lambda\overrightarrow{k}$ and $\overrightarrow{c} = \overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}$, determine $\lambda \in \mathbb{R}$ such that the vector $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$ is parallel to the plane xOy.

Problem 4.40. Consider the vectors $\overrightarrow{a} = \overrightarrow{i} - m\overrightarrow{j} + 3\overrightarrow{k}$, $\overrightarrow{b} = m\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$ and $\overrightarrow{c} = 3\overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$. Determine $m \in \mathbb{R}$ such that the vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are coplanar. For $m = 1$, calculate the volume of the parallelepiped with the edges $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$.

Problem 4.41. Consider the triangle ABC, $AA_1 \perp BC$, $A_1 \in BC$, $BB_1 \perp AC$, $B_1 \in AC$, $AA_1 \cap BB_1 = \{H\}$. Prove that $CH \perp AB$.

Problem 4.42. Consider the vectors $\overrightarrow{v} = 3\overrightarrow{a} + 2\overrightarrow{b}$ and $\overrightarrow{w} = 2\overrightarrow{a} - \overrightarrow{b}$ such that $\|\overrightarrow{a}\| = 2$, $\|\overrightarrow{b}\| = 3$ and the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ equals $\frac{\pi}{3}$.

a) Determine the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$.

b) Find the projection of $\overrightarrow{w}$ on $\overrightarrow{v}$.

c) Calculate the area of the parallelogram with the edges $\overrightarrow{v}$ and $\overrightarrow{w}$.

Problem 4.43. Consider $\overrightarrow{a} = \overrightarrow{m} + 2\overrightarrow{n}$ and $\overrightarrow{b} = \overrightarrow{m} - 3\overrightarrow{n}$ such that $\|\overrightarrow{m}\| = 5$, $\|\overrightarrow{n}\| = 3$ and the angle between $\overrightarrow{n}$ and $\overrightarrow{m}$ equals $\frac{\pi}{2}$.

a) Determine the length of the two diagonals of the parallelogram with the edges $\overrightarrow{a}$ and $\overrightarrow{b}$.

b) Find the angle between the diagonals of the parallelogram with $\overrightarrow{a}$ and $\overrightarrow{b}$ as the edges.

c) Calculate the area of the parallelogram with $\overrightarrow{a}$ and $\overrightarrow{b}$ as the edges.

Problem 4.44. Prove the identity of Jacobi

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) + \overrightarrow{c} \times (\overrightarrow{a} \times \overrightarrow{b}) + \overrightarrow{b} \times (\overrightarrow{c} \times \overrightarrow{a}) = \overrightarrow{0},$$

for any vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$.

5

Straight lines and Planes in space

5.1 Planes in space

We can determine the equation of a plane in several situations.

Plane determined by a point and a normal vector

Let $M_0(x_0, y_0, z_0)$ be a point in space and let $\overrightarrow{n} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k} \neq \overrightarrow{0}$ a vector. Let (P) be the plane passing through M_0 and is perpendicular to $\overrightarrow{n}$.

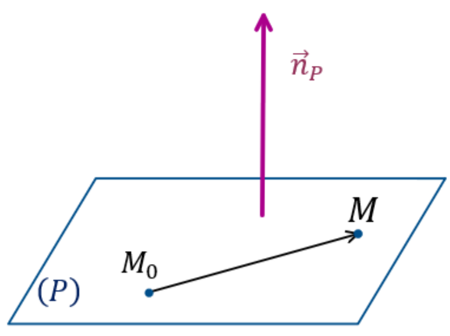


Figure 5.1: Plane determined by a point and a normal vector

The point $M(x, y, z)$ will lie in the plane (P) if and only if the vector $\overrightarrow{n}$ is perpendicular to $\overrightarrow{M_0M}$.

$\overrightarrow{n} \perp \overrightarrow{M_0M} \Longleftrightarrow \overrightarrow{n} \cdot \overrightarrow{M_0M} = 0 \Longleftrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$

So, the equation of the plane passing through the point $M_0(x_0, y_0, z_0)$ and having as normal vector $\overrightarrow{n} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k} \neq \overrightarrow{0}$ is

$$(P) : a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

If we denote by $d = -ax_0 - by_0 - cz_0$ we obtain the general equation of a plane in space

$$(P) : ax + by + cz + d = 0.$$

The vector

$$\overrightarrow{n_P} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$$

is called **normal** to the plane and is a vector having the direction perpendicular to the plane (P).

The point $A(x_A, y_A, z_A)$ is on the plane (P) if $ax_A + by_A + cz_A + d = 0$.

Remark 5.1. *In particular, the equations of the planes xOy, xOz, yOz are:*

- $xOy : z = 0$;
- $xOz : y = 0$;
- $yOz : x = 0$.

Plane determined by three non-collinear points

Let $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$, $C(x_C, y_C, z_C)$ be three non-collinear points in space. Let (P) be the plane determined by these three points. We also consider $M(x, y, z)$ an arbitrary point of (P).

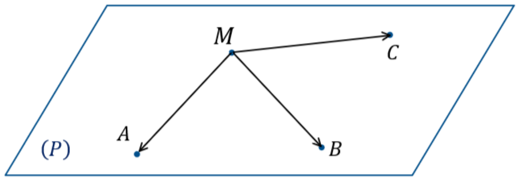


Figure 5.2: Plane determined by three non-collinear points

Hence, the points A, B, C and M are coplanar which implies that the triple scalar product of the vectors $\overrightarrow{MA}$, $\overrightarrow{MB}$ and $\overrightarrow{MC}$ is zero. Using the analytical expression of the triple scalar product one has:

$$\begin{vmatrix} x_A - x & y_A - y & z_A - z \\ x_B - x & y_B - y & z_B - z \\ x_C - x & y_C - y & z_C - z \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z & 1 \\ x_A - x & y_A - y & z_A - z & 0 \\ x_B - x & y_B - y & z_B - z & 0 \\ x_C - x & y_C - y & z_C - z & 0 \end{vmatrix} = 0.$$

By adding the first row to the second, to the third and to the fourth, we will obtain the equation of the plane determined by the non-collinear points $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$, $C(x_C, y_C, z_C)$:

$$(P) : \begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0.$$

Plane determined by a point and two non-collinear vectors

Let (P) be the plane that passes through the point $M_0(x_0, y_0, z_0)$ and is parallel to two non-collinear vectors $\overrightarrow{v_1} = a_1 \overrightarrow{i} + b_1 \overrightarrow{j} + c_1 \overrightarrow{k}$ and $\overrightarrow{v_2} = a_2 \overrightarrow{i} + b_2 \overrightarrow{j} + c_2 \overrightarrow{k}$.

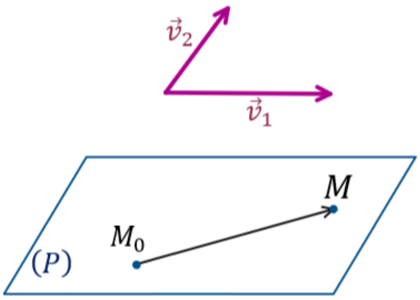


Figure 5.3: Plane determined by the point and two non-collinear vectors

An arbitrary point $M(x, y, z)$ lies on the plane (P) if and only if the vectors $\overrightarrow{M_0M}$, $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are coplanar, that is $(\overrightarrow{M_0M}, \overrightarrow{v_1}, \overrightarrow{v_2}) = 0$ which lead us to

$$(P) : \begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

5.2 Straight lines in space

Let d be the line passing through the point $A(x_A, y_A, z_A)$ and is parallel to the vector $\overrightarrow{v_d} \neq \overrightarrow{0}$, $\overrightarrow{v_d} = l\,\overrightarrow{i} + m\,\overrightarrow{j} + n\,\overrightarrow{k}$. Then a point $M(x, y, z)$ is on the line d if the vector $\overrightarrow{MA}$ is parallel to $\overrightarrow{v_d}$.

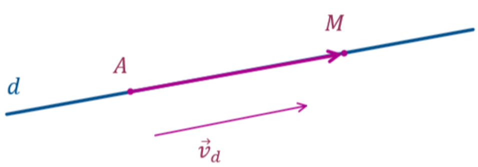


Figure 5.4: Line determined by a point and a director vector

So, $A \in d \iff \exists t \in \mathbb{R}$ such that $\overrightarrow{AM} = t\overrightarrow{v_d} \iff$

$$(x - x_A)\overrightarrow{i} + (y - y_A)\overrightarrow{j} + (z - z_A)\overrightarrow{k} = t(l\,\overrightarrow{i} + m\,\overrightarrow{j} + n\,\overrightarrow{k}) \iff$$

$$d: \begin{cases} x = lt + x_A \\ y = mt + y_A \\ z = nt + z_A \end{cases}, t \in \mathbb{R}$$

which are the **parametric equations** of the line d.

The vector $\overrightarrow{v_d}$ is also called the **director vector** of the line d while the components of $\overrightarrow{v_d}$ i.e. (l, m, n) are called the **director ratios** of this direction. Any other numbers proportional to (l, m, n) are also direction ratios for the same direction.

When $t \in \mathbb{R}$ we have the line d, when $t \in [a, b]$ then we will obtain a line segment form the point where $t = a$ to the point where $t = b$.

When we eliminate t form the parametric equations, we obtain the **cartesian equations** of the line d:

$$d: \frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n}.$$

Remark 5.2. *If one of the component of the director vector is zero, then, in the cartesian equations of the line the corresponding numerator is also zero.*

Example 5.3. *The equations of the Ox axis are:*

$$Ox: \frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

or, equivalent

$$Ox: \begin{cases} x = t \\ y = 0 \\ z = 0. \end{cases}, t \in \mathbb{R} \iff Ox: \begin{cases} y = 0 \\ z = 0. \end{cases}$$

Similarly

$$Oy : \frac{x}{0} = \frac{y}{1} = \frac{z}{0} \iff Oy : \begin{cases} x = 0 \\ y = t \\ z = 0. \end{cases}, t \in \mathbb{R} \iff Oy : \begin{cases} x = 0 \\ z = 0. \end{cases}$$

$$Oz : \frac{x}{0} = \frac{y}{0} = \frac{z}{1} \iff Oz : \begin{cases} x = 0 \\ y = 0 \\ z = t. \end{cases}, t \in \mathbb{R} \iff Oz : \begin{cases} x = 0 \\ y = 0. \end{cases}$$

Equations of the line joining two points

The line joining the points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ is $d = AB$.

Figure 5.5: Line joining two points

If we write the equations of the line passing through A and having the director vector $\overrightarrow{AB} = (x_B - x_A)\overrightarrow{i} + (y_B - y_A)\overrightarrow{j} + (z_B - z_A)\overrightarrow{k}$, we get

$$d : \frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}.$$

Line determined by the intersection of two planes

The equations of the line d determined by the intersection of two planes (P_1) and (P_2) are:

$$d : \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}.$$

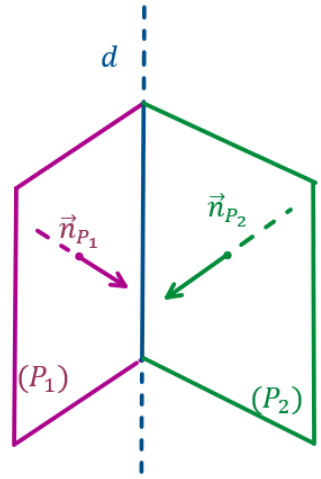


Figure 5.6: Line determined by the intersection of two planes

The normal vectors to (P_1) and (P_2) are $\overrightarrow{n_1} = a_1\overrightarrow{i} + b_1\overrightarrow{j} + c_1\overrightarrow{k}$ and $\overrightarrow{n_2} = a_2\overrightarrow{i} + b_2\overrightarrow{j} + c_2\overrightarrow{k}$, respectively. They are both perpendicular to d, so d is parallel to $\overrightarrow{n} = \overrightarrow{n_1} \times \overrightarrow{n_2}$.

Therefore, $\overrightarrow{v_d} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$.

The direction-ratios of $\overrightarrow{v_d}$ are $\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right)$.

5.3 Relative positions

Relative positions between two planes

Let $(P_1): a_1x + b_1y + c_1z + d_1 = 0$ and $(P_2): a_2x + b_2y + c_2z + d_2 = 0$ be two planes in space.

- $(P_1) \parallel (P_2) \Longleftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \neq \dfrac{d_1}{d_2} \Longleftrightarrow \overrightarrow{n_{P_1}} \parallel \overrightarrow{n_{P_2}}$.

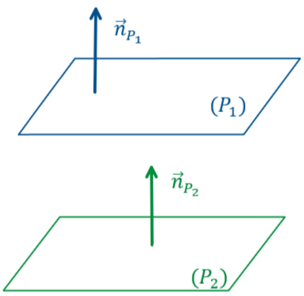


Figure 5.7: Two parallel planes

- $(P_1) \perp (P_2) \Longleftrightarrow \overrightarrow{n_{P_1}} \perp \overrightarrow{n_{P_2}} \Longleftrightarrow \begin{cases} \overrightarrow{n_{P_1}} \parallel (P_2) \\ \overrightarrow{n_{P_2}} \parallel (P_1) \end{cases}$.

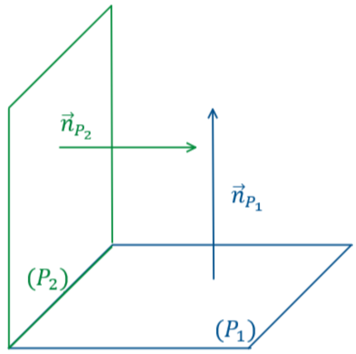


Figure 5.8: Two perpendicular planes

- $(P_1) = (P_2) \iff \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$.

Relative positions between two straight lines

Let

$$d_1 : \frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$

and

$$d_2 : \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

be two lines in space.

The director vector of d_1 is $\overrightarrow{v_{d_1}} = l_1 \overrightarrow{i} + m_1 \overrightarrow{j} + n_1 \overrightarrow{k}$ and the point $M_1(x_1, y_1, z_1) \in d_1$ while the director vector of d_2 is $\overrightarrow{v_{d_2}} = l_2 \overrightarrow{i} + m_2 \overrightarrow{j} + n_2 \overrightarrow{k}$ and the point $M_2(x_2, y_2, z_2) \in d_2$.

- the lines are **parallel**, $d_1 \parallel d_2 \iff \overrightarrow{v_{d_1}} \parallel \overrightarrow{v_{d_2}} \iff \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

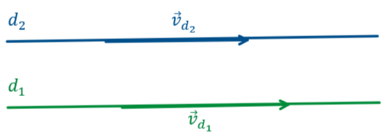


Figure 5.9: Two parallel lines

- the lines coincide, $d_1 = d_2 \iff \begin{cases} \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} \\ \frac{x_2 - x_1}{l_1} = \frac{y_2 - y_1}{m_1} = \frac{z_2 - z_1}{n_1} \end{cases}$.

- the lines are **perpendicular**, $d_1 \perp d_2 \iff \overrightarrow{v_{d_1}} \perp \overrightarrow{v_{d_2}} \iff \overrightarrow{v_{d_1}} \cdot \overrightarrow{v_{d_2}} = 0 \iff l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

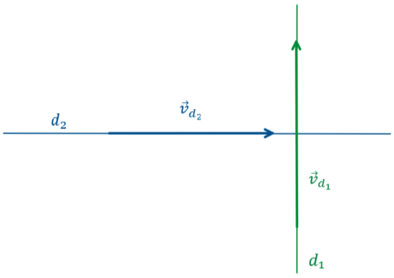


Figure 5.10: Two perpendicular lines

- the lines are **coplanar** if the vectors $\overrightarrow{v_{d_1}}$, $\overrightarrow{v_{d_2}}$ and $\overrightarrow{M_1M_2}$ are parallel to the same plane (are coplanar), which is equivalent to

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

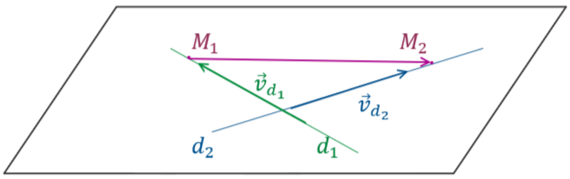


Figure 5.11: Two coplanar lines

- Two lines that are not coplanar are **skew** lines. Skew lines are lines that do not intersect and are not parallel. The lines d_1 and d_2 are skew if:

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \neq 0.$$

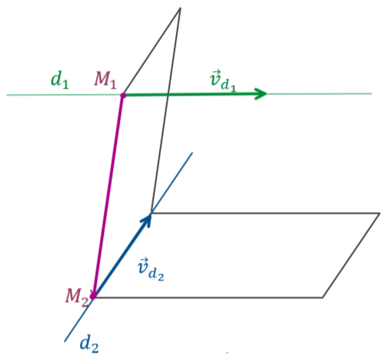


Figure 5.12: Two skew lines

Relative positions between straight lines and planes

Let $d : \dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$ be a line in space and $(P) : ax + by + cz + d = 0$ a plane in space.

We have:

- $d \parallel (P) \iff \overrightarrow{v_d} \perp \overrightarrow{n_P} \iff al + bm + cn = 0.$

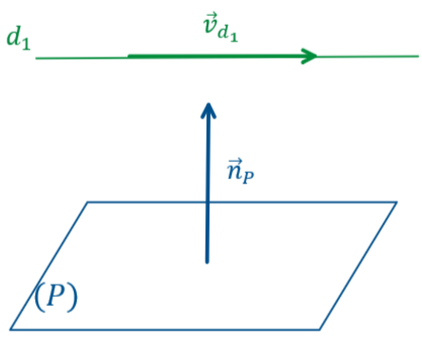


Figure 5.13: A line parallel to a plane

- $d \cap (P) = \{M\} \iff \overrightarrow{v_d} \not\perp \overrightarrow{n_P} \iff al + bm + cn \neq 0.$

- $d \subset (P) \iff \begin{cases} \overrightarrow{v_d} \perp \overrightarrow{n_P} \\ M_0(x_0, y_0, z_0) \in d \Longrightarrow M_0 \in (P) \end{cases}$

$$\iff \begin{cases} al + bm + cn = 0 \\ ax_0 + by_0 + cz_0 + d = 0. \end{cases}$$

5.4 Angles and distances

Distances

- The distance from the point $M(x_0, y_0, z_0)$ to the plane $(P) : ax+by+cz+d = 0$ is

$$\text{dist}(M, d) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

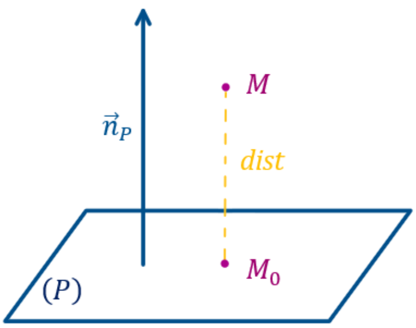


Figure 5.14: Distance from a point to a plane

- The distance between two parallel planes (P_1) and (P_2) is the distance from $M_2(x_2, y_2, z_2) \in (P_2)$ to the plane (P_1).

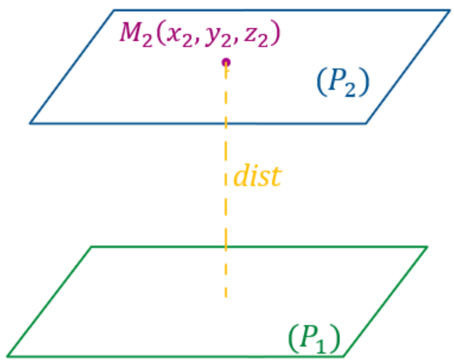


Figure 5.15: Distance between two parallel planes

- The distance from the point A to the line d is

$$\text{dist}(A, d) = \frac{\|\overrightarrow{v_d} \times \overrightarrow{M_0A}\|}{\|\overrightarrow{v_d}\|},$$

where $M_0 \in d$.

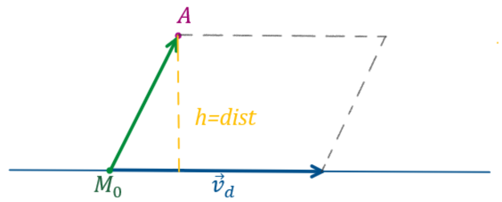


Figure 5.16: Distance from a point to a line

- The distance between two parallel lines d_1 and d_2 is

$$\text{dist}(d_1, d_2) = \text{dist}(M_2, d_1),$$

where $M_2 \in d_2$.

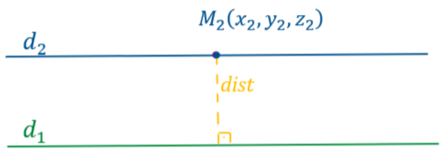


Figure 5.17: Distance between two parallel lines

- The distance between the skew lines d_1 and d_2 is

$$\text{dist}(d_1, d_2) = \frac{|(\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}, \overrightarrow{M_1M_2})|}{\|\overrightarrow{v_{d_1}} \times \overrightarrow{v_{d_2}}\|},$$

where $M_1 \in d_1$ and $M_2 \in d_2$.

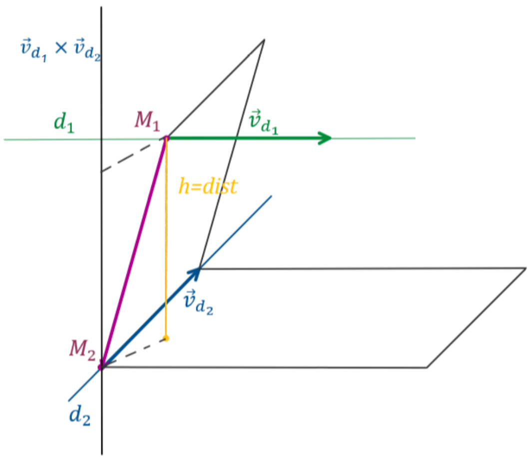


Figure 5.18: Distance between skew lines

- If $d \parallel (P)$, the distance between the line d and the plane (P), is

$$\text{dist}(d, (P)) = \text{dist}(A, (P)),$$

where $A \in d$.

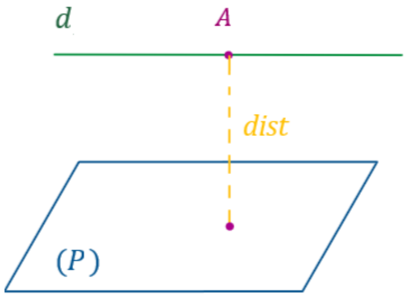


Figure 5.19: Distance between a line parallel to a plane

Angles

Let $d_1 : \dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $d_2 : \dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ be two lines in space and

$(P_1) : a_1x + b_1y + c_1z + d_1 = 0$ and $(P_2) : a_2x + b_2y + c_2z + d_2 = 0$ two planes in space.

- The angle between the planes (P_1) and (P_2)

  $\measuredangle((P_1), (P_2)) = \measuredangle(\overrightarrow{n_{P_1}}, \overrightarrow{n_{P_2}}) = \alpha,\ \alpha \in [0, \frac{\pi}{2}]$.

$$\cos\alpha = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

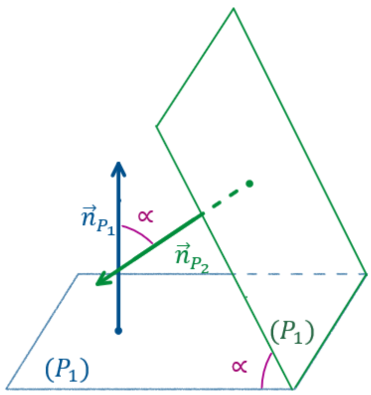


Figure 5.20: The angle between two planes

- The angle between the lines d_1 and d_2 is

$\sphericalangle(d_1, d_2) = \sphericalangle(\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}) = \varphi,\ \varphi \in [0, \frac{\pi}{2}]$.

$$\cos\varphi = \frac{|l_1 l_2 + m_1 m_2 + n_1 n_2|}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}.$$

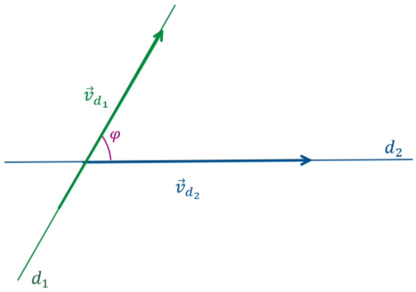


Figure 5.21: The angle between two lines

- The angle between the line d_1 and the plane (P_1) is

$\sphericalangle(d_1, (P_1)) = 90^\circ - \sphericalangle(\overrightarrow{v_{d_1}}, \overrightarrow{n_{P_1}}) = \theta,\ \theta \in [0, \frac{\pi}{2}]$.

$\cos(\sphericalangle(\overrightarrow{v_{d_1}}, \overrightarrow{n_{P_1}})) = \sin(90^\circ - \sphericalangle(\overrightarrow{v_{d_1}}, \overrightarrow{n_{P_1}})) = \sin\theta$,

$$\sin\theta = \frac{|l_1 a_1 + m_1 b_1 + n_1 c_1|}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{a_1^2 + b_1^2 + c_1^2}}.$$

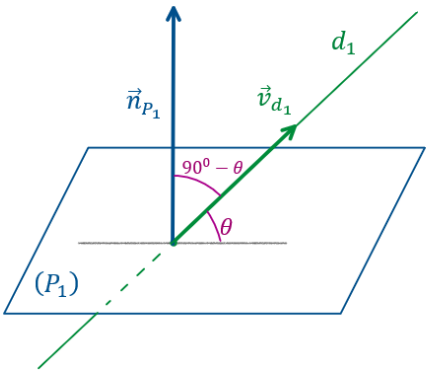


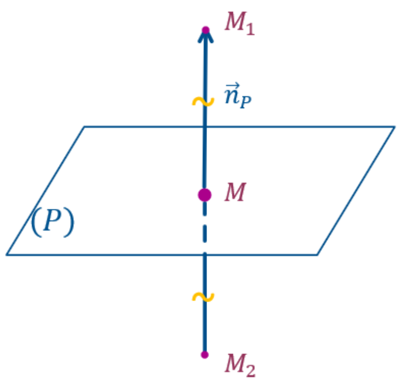
Figure 5.22: The angle between a plane and a line

5.5 Solved Problems

Problem 5.1. Write the equation of the plane (P) if $M_1(1,-2,-1)$ and $M_2(3,4,1)$ are symmetrical about the plane (P).

Solution:

The information in our problem can be represented as the following figure shows.



If M_1 and M_2 are symmetrical about the plane (P) then $M_1M_2 \perp (P) \Longrightarrow \overrightarrow{M_1M_2} \parallel \overrightarrow{n_P} \Longrightarrow \overrightarrow{n_P} = \frac{1}{2}(2\overrightarrow{i} + 6\overrightarrow{j} + 2\overrightarrow{k}) = \overrightarrow{i} + 3\overrightarrow{j} + \overrightarrow{k}$.

We can also deduce that the middle of the line segment $[M_1M_2]$ denoted by M is on the plane (P). The coordinates of M are $M(2,1,0)$.

We write the equation of the plane passing through $M(2,1,0)$ and having as normal vector $\overrightarrow{n}_P = \overrightarrow{i} + 3\overrightarrow{j} + \overrightarrow{k}$, therefore the equation of the plane (P) is

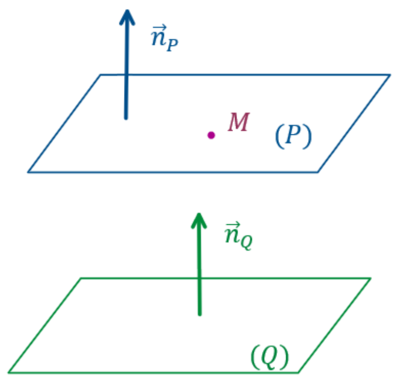
$(P): x - 2 + 3(y-1) + z = 0 \Longleftrightarrow$

$(P): x + 3y + z - 5 = 0.$

Problem 5.2. Determine the equation of the plane (P) which passes through $M(-4,-1,3)$ and is parallel to the plane $(Q): x + 2y - 3z - 7 = 0$.

Solution:

A sketch of the problem is represented below.



$(P) \parallel (Q) \Longrightarrow \overrightarrow{n_P} \parallel \overrightarrow{n_Q} \Longrightarrow \overrightarrow{n_P} = \alpha(\overrightarrow{i} + 2\overrightarrow{j} - 3\overrightarrow{k})$. For $\alpha = 1$, $\overrightarrow{n_P} = \overrightarrow{n_Q}$.

The equation of the plane passing through $M(-4, -1, 3)$ and having the normal $\overrightarrow{n_P}$ is

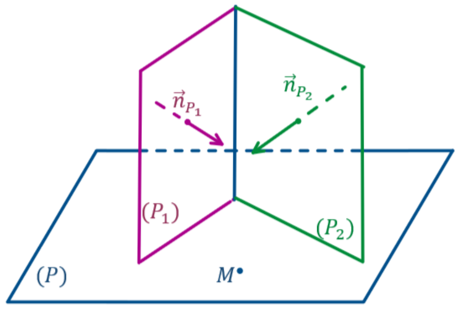
$(P) : (x + 4) + 2(y + 1) - 3(z - 3) = 0 \Longleftrightarrow$

$(P) : x + 2y - 3z + 15 = 0.$

Problem 5.3. Determine the equation of the plane (P) which passes through $M(2, 1, 0)$ and (P) is perpendicular on both $(P_1) : x - y + z - 7 = 0$ and $(P_2) : 2x + z - 3 = 0$.

Solution:

A sketch of the problem is represented below.



$(P) \perp (P_1) \Longrightarrow \overrightarrow{n_{P_1}} \parallel (P) \Longrightarrow \overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k} \parallel (P),$

$(P) \perp (P_2) \Longrightarrow \overrightarrow{n_{P_2}} \parallel (P) \Longrightarrow 2\overrightarrow{i} + \overrightarrow{k} \parallel (P).$

The equation of the plane passing through $M(2,1,0)$ and having two vectors parallel to the plane, $\overrightarrow{n_{P_1}}$ and $\overrightarrow{n}_{P_2}$, is

$$(P) : \begin{vmatrix} x-2 & y-1 & z \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow$$

$(P) : -x + y + 2z + 1 = 0.$

Problem 5.4. Determine the parametric equations of the line d which passes through $A(2,-3,1)$ and $B(4,1,1)$.

Solution:

We substitute in the equations of the line joining two points the coordinates of A and B, i.e.

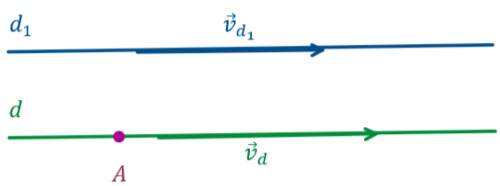
$$d : \frac{x-2}{4-2} = \frac{y-(-3)}{1-(-3)} = \frac{z-1}{1-1} = t \Longleftrightarrow$$

$$d : \begin{cases} x = 2t + 2 \\ y = 4t - 3 \\ z = 1 \end{cases}, t \in \mathbb{R}.$$

Problem 5.5. Write the equations of the line d which passes through $A(1,-2,5)$ and is parallel to the line $d_1 : \dfrac{x-2}{3} = \dfrac{y+2}{4} = \dfrac{z+2}{-5}$.

Solution:

A sketch of the problem is represented below.



$d \parallel d_1 \Longrightarrow \overrightarrow{v_d} \parallel \overrightarrow{v_{d_1}} \Longrightarrow \overrightarrow{v_d} \parallel 3\overrightarrow{i} + 4\overrightarrow{j} - 5\overrightarrow{k} \Longrightarrow \overrightarrow{v_d} = \alpha(3\overrightarrow{i} + 4\overrightarrow{j} - 5\overrightarrow{k})$. For $\alpha = 1$ the director vector of d is $\overrightarrow{v_d} = 3\overrightarrow{i} + 4\overrightarrow{j} - 5\overrightarrow{k}$.

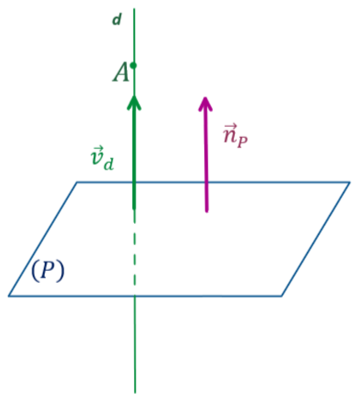
The equations of the line passing through the point $A(1, -2, 5)$ and having as director vector $\overrightarrow{v_d} = 3\overrightarrow{i} + 4\overrightarrow{j} - 5\overrightarrow{k}$ are

$$d : \frac{x-1}{3} = \frac{y+2}{4} = \frac{z-5}{-5}.$$

Problem 5.6. Write the equations of the line d which passes through $A(6, -2, -3)$ and is perpendicular on the plane $(P) : 2x - y + 7z - 9 = 0$.

Solution:

A sketch of the problem is represented below.



$d \perp (P) \Longrightarrow \overrightarrow{v_d} \parallel \overrightarrow{n_P} \Longrightarrow \overrightarrow{v_d} \parallel 2\overrightarrow{i} - \overrightarrow{j} + 7\overrightarrow{k} \Longrightarrow \overrightarrow{v_d} = \alpha(2\overrightarrow{i} - \overrightarrow{j} + 7\overrightarrow{k})$. For $\alpha = 1$ the director vector of d is $\overrightarrow{v_d} = 2\overrightarrow{i} - \overrightarrow{j} + 7\overrightarrow{k}$.

The equations of the line passing through the point $A(6, -2, -3)$ and having as director vector $\overrightarrow{v_d} = 2\overrightarrow{i} - \overrightarrow{j} + 7\overrightarrow{k}$ are

$$d : \frac{x-6}{2} = \frac{y+2}{-1} = \frac{z+3}{7}.$$

Problem 5.7. Determine the parametric and cartesian equations of the line at the intersection of the planes $(P_1) : 2x - 3y + z - 1 = 0$ and $(P_2) : -x + 3z + 5 = 0$.

Solution:

$$(P_1) \cap (P_2) = d : \begin{cases} 2x - 3y + z - 1 = 0 \\ -x + 3z + 5 = 0 \end{cases}.$$

$$\overrightarrow{v_d} \parallel \overrightarrow{n_{P_1}} \times \overrightarrow{n_{P_2}} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & -3 & 1 \\ -1 & 0 & 3 \end{vmatrix} = -9\overrightarrow{i} - 7\overrightarrow{j} - 3\overrightarrow{k}.$$

So, we can chose $\overrightarrow{v_d} = 9\overrightarrow{i} + 7\overrightarrow{j} + 3\overrightarrow{k}$.

The coordinates of a point $A(x_A, y_A, z_A)$ of the line d are a solution of the system $\begin{cases} 2x - 3y + z - 1 = 0 \\ -x + 3z + 5 = 0 \end{cases}$. Choosing $z = 0$ we obtain $x = 5$ and $y = 3$. So, a point from the line is $A(5, 3, 0)$.

The cartesian equations of the line passing through the point $A(5, 3, 0)$ and having as director vector $\overrightarrow{v_d} = 9\overrightarrow{i} + 7\overrightarrow{j} + 3\overrightarrow{k}$ are

$$d : \frac{x-5}{9} = \frac{y-3}{7} = \frac{z}{3}.$$

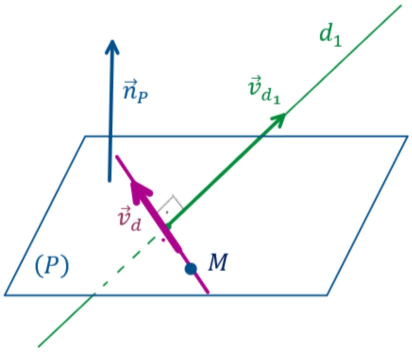
The parametric equations of the line d are:

$$d : \begin{cases} x = 9t + 5 \\ y = 7t + 3 \\ z = 3t \end{cases}, t \in \mathbb{R}.$$

Problem 5.8. Determine the equations of the line d which lies in the plane $(P) : x - y + 3z - 5 = 0$, d is perpendicular to the line $d_1 : \begin{cases} x = 3 + t \\ y = -t + 1 \\ z = 5 \end{cases}, t \in \mathbb{R}$, and passes through $M(1, -1, 1)$.

Solution:

A sketch of the problem is represented below.



The director vector of the line d_1 is $\overrightarrow{v_{d_1}} = \overrightarrow{i} - \overrightarrow{j}$.

$d \subset (P) \Longrightarrow \overrightarrow{v_d} \perp \overrightarrow{n_P} \Longrightarrow \overrightarrow{v_d} \perp \overrightarrow{i} - \overrightarrow{j} + 3\overrightarrow{k}$.

$d \perp d_1 \Longrightarrow \overrightarrow{v_d} \perp \overrightarrow{v_{d_1}} \Longrightarrow \overrightarrow{v_d} \perp \overrightarrow{i} - \overrightarrow{j}$.

$$\left.\begin{array}{l}\overrightarrow{v_d} \perp \overrightarrow{n_P} \\ \overrightarrow{v_d} \perp \overrightarrow{v_{d_1}}\end{array}\right\} \Longrightarrow \overrightarrow{v_d} \parallel \overrightarrow{n_P} \times \overrightarrow{v_{d_1}} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -1 & 3 \\ 1 & -1 & 0 \end{vmatrix} = 3\overrightarrow{i} + 3\overrightarrow{j}.$$

We can chose $\overrightarrow{v_d} = \frac{1}{3}(3\overrightarrow{i} + 3\overrightarrow{j}) = \overrightarrow{i} + \overrightarrow{j}$

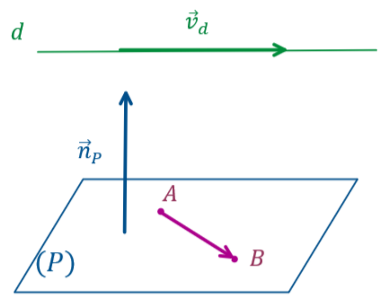
The cartesian equations of the line passing through the point $M(1, -1, 1)$ and having as director vector $\overrightarrow{v_d} = \overrightarrow{i} + \overrightarrow{j}$ are

$$d : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-1}{0} \iff d : \begin{cases} x - y - 2 = 0 \\ z = 1 \end{cases}.$$

Problem 5.9. Determine the equation of the plane (P) which passes through $A(2, -1, 3)$ and $B(-1, 1, 1)$, and is parallel to the line $d : \dfrac{x}{3} = \dfrac{y-1}{2} = z$.

Solution:

A sketch of the problem is represented below.



We present in what follows two methods of solving this problem.

The first method.

$(P) \parallel d \Longrightarrow \overrightarrow{n_P} \perp \overrightarrow{v_d} \Longrightarrow \overrightarrow{n_P} \perp 3\overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k}$.

$A, B \in (P) \Longrightarrow \overrightarrow{n_P} \perp \overrightarrow{AB} \Longrightarrow \overrightarrow{n_P} \perp -3\overrightarrow{i} + 2\overrightarrow{j} - 2\overrightarrow{j}$.

$$\left.\begin{array}{l} \overrightarrow{n_P} \perp \overrightarrow{v_d} \\ \overrightarrow{n_P} \perp \overrightarrow{AB} \end{array}\right\} \Longrightarrow \overrightarrow{n_P} \parallel \overrightarrow{v_d} \times \overrightarrow{AB} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 3 & 2 & 1 \\ -3 & 2 & -2 \end{vmatrix} = -6\overrightarrow{i} + 3\overrightarrow{j} + 12\overrightarrow{k}.$$

We can chose $\overrightarrow{n_P} = \frac{1}{3}(-6\overrightarrow{i} + 3\overrightarrow{j} + 12\overrightarrow{k}) = -2\overrightarrow{i} + \overrightarrow{j} + 4\overrightarrow{k}$.

The equation of the plane passing through $A(2, -1, 3)$ and having the normal $\overrightarrow{n_P}$ is

$(P) : -2(x - 2) + (y + 1) + 4(z - 3) = 0 \Longleftrightarrow$

$(P) : -2x + y + 4z - 7 = 0$.

The second method.

$(P) \parallel d \Longrightarrow (P) \parallel \overrightarrow{v_d} \Longrightarrow (P) \parallel 3\overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k}$.

$A, B \in (P) \Longrightarrow (P) \parallel \overrightarrow{AB} \Longrightarrow (P) \parallel -3\overrightarrow{i} + 2\overrightarrow{j} - 2\overrightarrow{j}$.

The equation of the plane passing through $M(2, -1, 3)$ and having two parallel vectors to the plane, $\overrightarrow{v_d}$ and $\overrightarrow{AB}$, is

$$(P) : \begin{vmatrix} x-2 & y+1 & z-3 \\ 3 & 2 & 1 \\ -3 & 2 & -2 \end{vmatrix} = 0 \Longleftrightarrow$$

$(P): -6(x-2)+3(y+1)+12(z-3)=0 \Longleftrightarrow$

$(P): -2x+y+4z-7=0.$

Problem 5.10. Determine the angle between the lines $d_1 : \dfrac{x-1}{3} = \dfrac{y-2}{2} = \dfrac{z+1}{-1}$ and $d_2 : \dfrac{x+2}{3} = \dfrac{y+1}{-1} = z-5$.

Solution:

Let $\sphericalangle(d_1, d_2) = \varphi$ be the angle between the lines d_1 and d_2.

$\overrightarrow{v_{d_1}} = 3\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}$,

$\overrightarrow{v_{d_2}} = 3\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$.

$\cos(\sphericalangle(\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}})) = \dfrac{\overrightarrow{v_{d_1}} \cdot \overrightarrow{v_{d_2}}}{\|\overrightarrow{v_{d_1}}\| \cdot \|\overrightarrow{v_{d_2}}\|} = \dfrac{3\cdot 3 + 2\cdot(-1) + 1\cdot(-1)}{\sqrt{9+4+1}\sqrt{9+1+1}} = \dfrac{6}{\sqrt{154}} \Longrightarrow$

$\varphi = \arccos \dfrac{6}{\sqrt{154}}.$

Problem 5.11. Let $(P): x - 3y + 2z - 5 = 0$ and $(Q): \alpha x - 3y + 2z - 5 = 0$ be two planes in space. Determine $\alpha \in \mathbb{R}$ such that $(P) \perp (Q)$.

Solution:

$(P) \perp (Q) \Longleftrightarrow \overrightarrow{n_P} \perp \overrightarrow{n_Q} \Longleftrightarrow \overrightarrow{n_P} \cdot \overrightarrow{n_Q} = 0.$

$\overrightarrow{n_P} = \overrightarrow{i} - 3\overrightarrow{j} + 2\overrightarrow{k}$,

$\overrightarrow{n_Q} = \alpha\overrightarrow{i} - 3\overrightarrow{j} + 2\overrightarrow{k}$.

$\overrightarrow{n_P} \cdot \overrightarrow{n_Q} = 0 \Longleftrightarrow \alpha + 9 + 4 = 0 \Longleftrightarrow \alpha = -13.$

Problem 5.12. Let $d_1 : \dfrac{x+2}{3} = \dfrac{y+3}{2} = \dfrac{z+1}{-2}$ and $d_2 : \dfrac{x-2}{a} = \dfrac{y-3}{-1} = \dfrac{z+1}{3}$ be two lines. Determine $a \in \mathbb{R}$ such that d_1 and d_2 are coplanar.

Solution:

d_1 and d_2 are coplanar $\Longleftrightarrow$ $\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}, \overrightarrow{AB}$ are coplanar $\Longleftrightarrow$ $(\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}, \overrightarrow{AB}) = 0$, where $A \in d_1$ and $B \in d_2$.

$\overrightarrow{v_{d_1}} = 3\overrightarrow{i} + 2\overrightarrow{j} - 2\overrightarrow{k}$,

$\overrightarrow{v_{d_2}} = a\overrightarrow{i} - \overrightarrow{j} + 3\overrightarrow{k}$.

$A(-2,-3,-1) \in d_1$ obtained for the ratios in the line equations equal to 0, and $B(2,3,-1) \in d_2$ obtained in the same way. The vector $\overrightarrow{AB}$ has the expresion $\overrightarrow{AB} = 4\overrightarrow{i} + 6\overrightarrow{j}$.

We determine $a \in \mathbb{R}$ such that $(\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}, \overrightarrow{AB}) = 0$.

$$\begin{vmatrix} 3 & 2 & -2 \\ a & -1 & 3 \\ 4 & 6 & 0 \end{vmatrix} = 0 \Longleftrightarrow -38 - 12a = 0 \Longleftrightarrow a = -\frac{19}{6}.$$

Problem 5.13. Let $d : \begin{cases} x = 2t - 1 \\ y = 2 - t \\ z = t - 2 \end{cases}, t \in \mathbb{R}$ be a line in space and $(P) : (\lambda - 1)x - y + 3z - 5 = 0$ a plane in space. Determine $\lambda \in \mathbb{R}$ such that $d \parallel (P)$. For λ determined, find the distance between d and (P).

Solution:

$d \parallel (P) \Longleftrightarrow \overrightarrow{v_d} \perp \overrightarrow{n_P} \Longleftrightarrow \overrightarrow{v_d} \cdot \overrightarrow{n_P} = 0$.

$\overrightarrow{v_d} = 2\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$,

$\overrightarrow{n_P} = (\lambda - 1)\overrightarrow{i} - \overrightarrow{j} + 3\overrightarrow{k}$.

$\overrightarrow{v_d} \cdot \overrightarrow{n_P} = 0 \Longleftrightarrow 2(\lambda - 1) + 1 + 3 = 0 \Longleftrightarrow \lambda = -1$.

The equation of the plane (P) for $\lambda = -1$ is $(P) : -2x - y + 3z - 5 = 0$ and $\overrightarrow{n_P} = -2\overrightarrow{i} - \overrightarrow{j} + 3\overrightarrow{k}$.

Since $d \parallel (P)$, $\text{dist}(d,(P)) = \text{dist}(A,P)$, where A is a point belonging to d.

We have the parametric equations of the line d, so choosing for example $t = 1$, we have $A(1,1,-1)$.

$\text{dist}(A,(P)) = \dfrac{|-2 \cdot 1 - 1 + 3 \cdot (-1) - 5|}{\sqrt{4+1+9}} = \dfrac{11}{\sqrt{14}} \Longrightarrow$

$\text{dist}(d,(P)) = \dfrac{11}{\sqrt{14}}$.

Problem 5.14. Let

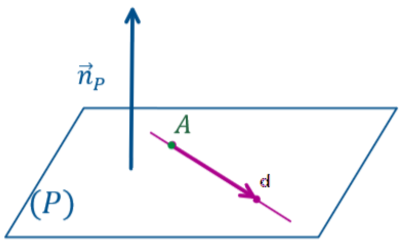
$$d : \frac{x-\beta}{1} = \frac{y-1}{4} = \frac{z+2}{\alpha-1}$$

be a line in space and

$$(P) : 2x - 3y + 5z + 13 = 0$$

a plane in space. Determine α, $\beta \in \mathbb{R}$ such that $d \subset (P)$.

Solution:



We can use one of the two conditions :

1. $d \subset (P) \iff \begin{cases} \overrightarrow{v_d} \perp \overrightarrow{n_P} \\ A \in d \Longrightarrow A \in (P) \end{cases}$ or
2. $d \subset (P) \iff A$ and $B \in d \Longrightarrow A$ and $B \in (P)$.

For this particular problem is easier to choose the first one.

$\overrightarrow{v_d} = \overrightarrow{i} + 4\overrightarrow{j} + (\alpha - 1)\overrightarrow{k}$,

$\overrightarrow{n_P} = 2\overrightarrow{i} - 3\overrightarrow{j} + 5\overrightarrow{k}$.

$\overrightarrow{v_d} \perp \overrightarrow{n_P} \iff \overrightarrow{v_d} \cdot \overrightarrow{n_P} = 0 \iff 2 - 12 + 5(\alpha - 1) = 0 \Longrightarrow \alpha = 3.$

$A(\beta, 1, -2) \in d \Longrightarrow A \in (P) \iff 2\beta - 3 - 10 + 13 = 0 \Longrightarrow \beta = 0.$

Problem 5.15. Let $d : \dfrac{x-2}{4} = \dfrac{y-1}{3} = \dfrac{z}{2}$ be a line in space and $M(7, 2, -3)$. Determine the distance from the point M to the line d.

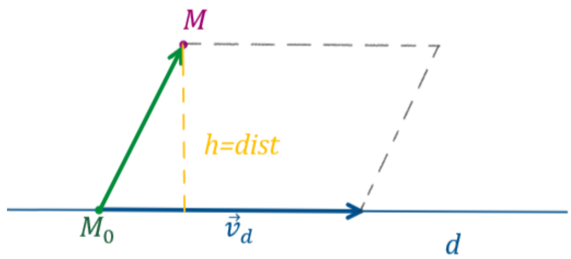
Solution:

$\text{dist}(M, d) = \dfrac{\|\overrightarrow{v_d} \times \overrightarrow{MM_0}\|}{\|\overrightarrow{v_d}\|}$, where $M_0 \in d$.

$\overrightarrow{v_d} = 4\overrightarrow{i} + 3\overrightarrow{j} + 2\overrightarrow{k}$,

$M_0 \in d, M_0(2,1,0) \Longrightarrow \overrightarrow{MM_0} = -5\overrightarrow{i} - \overrightarrow{j} + 3\overrightarrow{k}$.

$$\overrightarrow{v_d} \times \overrightarrow{MM_0} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 4 & 3 & 2 \\ -5 & -1 & 3 \end{vmatrix} = 11\overrightarrow{i} - 22\overrightarrow{j} + 11\overrightarrow{k} = 11(\overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}).$$



$$\text{dist}(M,d) = \frac{\|11(\overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k})\|}{\|4\overrightarrow{i} + 3\overrightarrow{j} + 2\overrightarrow{k}\|} = \frac{11\sqrt{6}}{\sqrt{29}}.$$

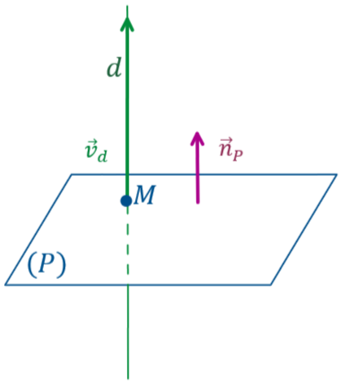
Problem 5.16. Consider $d : \begin{cases} x + 2y - z + 5 = 0 \\ 2x - 2z + 3 = 0 \end{cases}$ a line in space and $(P) : x + z - 3 = 0$ a plane in space. Determine the relative position of d and (P). If $d \parallel (P)$ find the distance between them, else determine the intersection of d and (P).

Solution:

$$d : \begin{cases} x + 2y - z + 5 = 0 \\ 2x - 2z + 3 = 0 \end{cases} \Longrightarrow \overrightarrow{v_d} \parallel \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2 & -1 \\ 2 & 0 & -2 \end{vmatrix} = -4\overrightarrow{i} - 4\overrightarrow{k}.$$

The normal to the plane (P) is the vector $\overrightarrow{n_P} = \overrightarrow{i} + \overrightarrow{k}$.

It is obviously that $\overrightarrow{v_d} \parallel \overrightarrow{n_P}$ since $\overrightarrow{v_d} = -4\overrightarrow{n_P}$, so the line d is perpendicular to the plane (P).



Let $\{M\} = d \cap (P)$. This implies that $\begin{cases} M(x_M, y_M, z_M) \in d \\ M(x_M, y_M, z_M) \in (P) \end{cases} \Longleftrightarrow$

$$\begin{cases} x_M + 2y_M - z_M + 5 = 0 \\ 2x_M - 2z_M + 3 = 0 \\ x_M + z_M - 3 = 0 \end{cases}.$$

The solutions of the linear system are the coordinates of the point M, i.e. $M(\frac{3}{4}, -\frac{7}{4}, \frac{9}{4})$.

Problem 5.17. Let $d_1 : \begin{cases} x = t + 1 \\ y = 2t - 2 \\ z = t - 3 \end{cases}, t \in \mathbb{R}$ and $d_2 : \dfrac{x+3}{2} = \dfrac{y}{3} = z$ be two lines in space.

a) Prove that the lines are skew lines.

b) Determine the distance between d_1 and d_2.

c) Determine the equations of the common perpendicular of d_1 and d_2.

Solution:

a) d_1 and d_2 are skew line if the lines are not coplanar $\Longleftrightarrow (\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}, \overrightarrow{M_1M_2}) \neq 0$ where $M_1 \in d_1$ and $M_2 \in d_2$.

$\overrightarrow{v_{d_1}} = \overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k}$.

$\overrightarrow{v_{d_2}} = 2\overrightarrow{i} + 3\overrightarrow{j} + \overrightarrow{k}$.

$M_1(1, -2, -3) \in d_1$ (we chose $t = 0$ in the parametric equations of d_1).

$M_2(-3, 0, 0) \in d_2$ (we determined x, y, z when each ratio in the cartesian equations of the line d_2 equals 0).

$\overrightarrow{M_1M_2} = -4\overrightarrow{i} + 2\overrightarrow{j} + 3\overrightarrow{k}$.

$$(\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}, \overrightarrow{M_1M_2}) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -4 & 2 & 3 \end{vmatrix} = 3 \neq 0 \Longrightarrow d_1 \text{ and } d_2 \text{ are skew lines.}$$

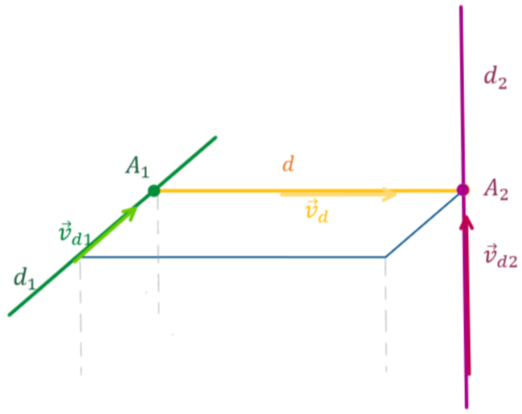
b) $$\text{dist}(d_1, d_2) = \frac{|(\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}, \overrightarrow{M_1M_2})|}{\|\overrightarrow{v_{d_1}} \times \overrightarrow{v_{d_2}}\|}$$

$$\overrightarrow{v_{d_1}} \times \overrightarrow{v_{d_2}} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix} = -\overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k} \Longrightarrow \|\overrightarrow{v_{d_1}} \times \overrightarrow{v_{d_2}}\| = \sqrt{3}.$$

$$\text{dist}(d_1, d_2) = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

c) Let d be the common perpendicular. Hence, $\begin{cases} d \perp d_1 \\ d \perp d_2 \end{cases} \Longrightarrow \begin{cases} \overrightarrow{v_d} \perp \overrightarrow{v_{d_1}} \\ \overrightarrow{v_d} \perp \overrightarrow{v_{d_2}} \end{cases} \Longrightarrow$

$\overrightarrow{v_d} \parallel \overrightarrow{v_{d_1}} \times \overrightarrow{v_{d_2}} = -\overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$.



$d = A_1A_2$ where $\{A_1\} = d \cap d_1$ and $\{A_2\} = d \cap d_2$.

Since we have the parametric equations of d_1 the coordinates of the point A_1 from d_1 are $A_1(t+1, 2t-2, t-3)$.

The parametric equations of d_2 are $d_2 : \begin{cases} x = 2m-3 \\ y = 3m \\ z = m \end{cases}$, $m \in \mathbb{R}$, so, the point $A_2 \in d_2$ has the coordinates $A_2(2m-3, 3m, m)$.

$\overrightarrow{A_1A_2} = (2m-t-4)\overrightarrow{i} + (3m-2t+2)\overrightarrow{j} + (m-t+3)\overrightarrow{k}$.

It's obvious that $\overrightarrow{v_d} \parallel \overrightarrow{A_1A_2} \Longleftrightarrow \dfrac{2m-t-4}{-1} = \dfrac{3m-2t+2}{1} = \dfrac{m-t+3}{-1} \Longleftrightarrow$

$\begin{cases} \dfrac{2m-t-4}{-1} = \dfrac{3m-2t+2}{1} \\ \dfrac{3m-2t+2}{1} = \dfrac{m-t+3}{-1} \end{cases} \Longleftrightarrow \begin{cases} 5m-3t = 2 \\ -4m+3t = 5 \end{cases} \Longleftrightarrow \begin{cases} m = 7 \\ t = 11 \end{cases} \Longrightarrow$

$A_1(12, 20, 8)$ and $A_2(11, 21, 7)$.

The equations of the common perpendicular are:

$$d = A_1A_2 : \frac{x-12}{-1} = \frac{y-20}{1} = \frac{z-8}{-1}.$$

Problem 5.18. Let $d_1 : \frac{x-1}{-5} = y - 2 = z$ and $d_2 : \frac{x-4}{-2} = \frac{y}{0} = z$ be two lines in space. Prove that d_1 and d_2 are skew lines and determine the common perpendicular of the two lines.

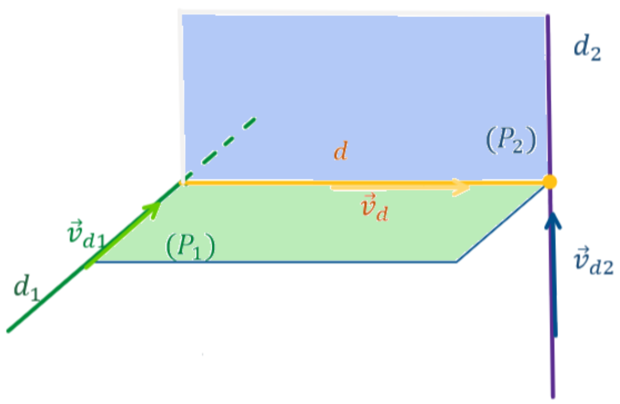
Solution:

Let $A(1, 2, 0) \in d_1$ and $B(4, 0, 0) \in d_2$, so $\overrightarrow{AB} = 3\vec{i} - 2\vec{j}$.

d_1 and d_2 are coplanar if $\overrightarrow{v_{d_1}}$, $\overrightarrow{v_{d_2}}$ şi $\overrightarrow{AB}$ are coplanar, but $(\overrightarrow{v_{d_1}}, \overrightarrow{v_{d_2}}, \overrightarrow{AB}) = \begin{vmatrix} -5 & 1 & 1 \\ -2 & 0 & 1 \\ 3 & -2 & 0 \end{vmatrix} = -3 \neq 0$, therefore d_1 and d_2 are skew lines.

Let d be the common perpendicular of d_1 and d_2.

$$\vec{v}_d = \overrightarrow{v_{d_1}} \times \overrightarrow{v_{d_2}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 1 & 1 \\ -2 & 0 & 1 \end{vmatrix} = \vec{i} + 3\vec{j} + 2\vec{k}.$$



Let (P_1) and (P_2) be the planes determined by d_1 and d, d_2 and d respectively. Therefore, d is at the intersection of (P_1) and (P_2).

$$(P_1):\begin{vmatrix} x-1 & y-2 & z \\ -5 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix}=0 \Longleftrightarrow (P_1): -x+11y-16z-21=0.$$

$$(P_2):\begin{vmatrix} x-4 & y & z \\ -2 & 0 & 1 \\ 1 & 3 & 2 \end{vmatrix}=0 \Longleftrightarrow (P_2): -3x+5y-6z+12=0.$$

The equations of d as intersection of (P_1) and (P_2) are

$$d:\begin{cases} -x+11y-16z-21=0 \\ -3x+5y-6z+12=0 \end{cases}.$$

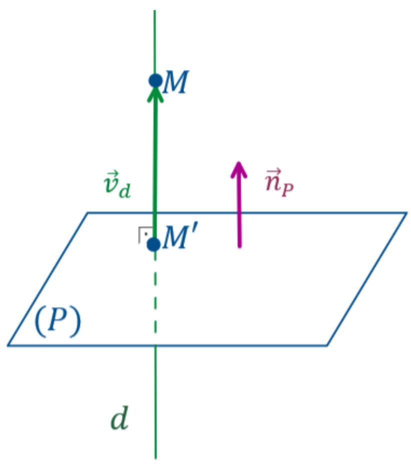
Solving the system we can write the parametric equations of the line d:

$$d:\begin{cases} x=\frac{t}{3}+\frac{53}{7} \\ y=t \\ z=\frac{2t}{3}-\frac{25}{14} \end{cases}, t\in\mathbb{R}.$$

Problem 5.19. Determine the projection of $M(1,-2,3)$ on the plane $(P): x-3y+z-5=0$.

Solution:

Let M' be the projection of M on (P), $M'=\text{pr}_{(P)}M$.



That implies that $MM' \perp (P) \Longrightarrow \overrightarrow{MM'} \parallel \vec{n}_P \Longrightarrow \overrightarrow{MM'} \parallel \vec{i} - 3\vec{j} + \vec{k}$.

We determine the equations of the line passing through M, perpendicular to the plane (P), which means that the line has the direction $\vec{i} - 3\vec{j} + \vec{k}$.

$d: \dfrac{x-1}{1} = \dfrac{y+2}{-3} = \dfrac{z-3}{1}$.

The point M' is at the intersection of d and (P).

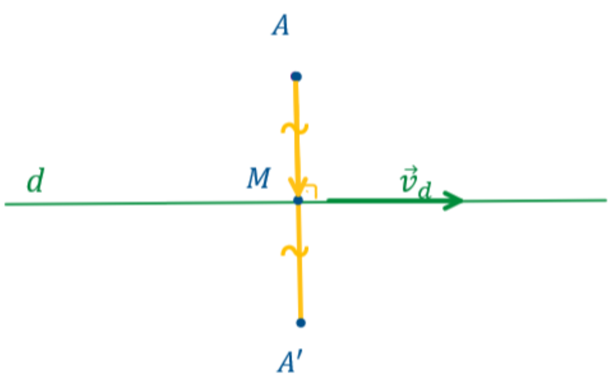
$$\begin{cases} M' \in d \Longleftrightarrow M'(t+1, -3t-2, t+3) \\ M' \in (P) \end{cases} \Longleftrightarrow \begin{cases} x_{M'} = t+1 \\ y_{M'} = -3t-2 \\ z_{M'} = t+3 \\ x_{M'} - 3y_{M'} + z_{M'} - 5 = 0 \end{cases}$$

$$\Longleftrightarrow t+1-3(-3t-2)+(t+3)-5=0 \Longleftrightarrow t = -\frac{5}{11} \Longrightarrow M'\left(\frac{6}{11}, -\frac{7}{11}, \frac{28}{11}\right).$$

Problem 5.20. Determine the symmetric of $A(-2,1,5)$ with respect to the line $d: \dfrac{x+1}{2} = \dfrac{y-2}{3} = \dfrac{z+1}{-2}$.

Solution:

If A' is the symmetric of A with respect to d then $AA' \perp d$ and M, the middle point of the line segment $[AA']$, is in the line d.



$M \in d \Longleftrightarrow M(2t-1, 3t+2, -2t-1)$ (from the parametric equations of the line d).

$\overrightarrow{AM} = (2t+1)\overrightarrow{i} + (3t+1)\overrightarrow{j} + (-2t-6)\overrightarrow{k}$.

$AM \perp d \iff \overrightarrow{AM} \cdot \overrightarrow{v_d} = 0 \iff 2(2t+1) + 3(3t+1) - 2(-2t-6) = 0 \implies t = -1 \implies M(-3,-1,1)$ (M is the projection of A on d, $M = \mathrm{pr}_d M$).

Since M is the middle of the line segment $[AA']$ we have

$$\begin{cases} x_M = \dfrac{x_A + x_{A'}}{2} \\ y_M = \dfrac{y_A + y_{A'}}{2} \\ z_M = \dfrac{z_A + z_{A'}}{2} \end{cases} \iff \begin{cases} -3 = \dfrac{-2 + x_{A'}}{2} \\ -1 = \dfrac{1 + y_{A'}}{2} \\ 1 = \dfrac{5 + z_{A'}}{2} \end{cases} \implies A'(-4,-3,-3).$$

Problem 5.21. Consider the line $d : \dfrac{x-3}{2} = \dfrac{y+1}{-2} = z - 3$ and the plane $(P) : x - 3y + z + 9 = 0$.

a) Determine the angle between the line and the plane.

b) Determine the projection of the line d on the plane (P).

Solution:

a) The director vector of the line d is $\overrightarrow{v_d} = 2\overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}$.

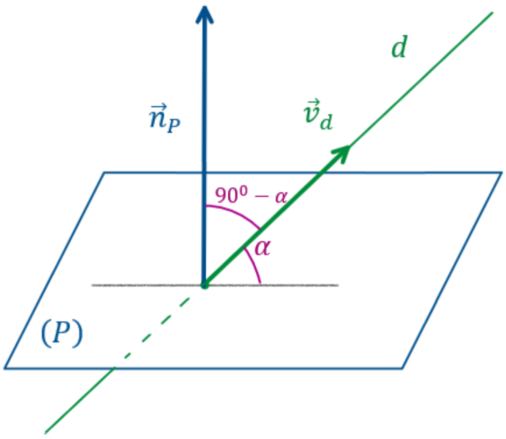
The normal to the plane is $\overrightarrow{n_P} = \overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k}$.

We study the relative position of d and (P).

(1) The line is perpendicular to the plane, $d \perp (P) \iff \overrightarrow{v_d} \parallel \overrightarrow{n_P}$.
$\dfrac{2}{1} \neq \dfrac{-2}{-3} \neq \dfrac{1}{1}$, so the line d is not perpendicular to the plane.

(2) The line is parallel to the plane, $d \parallel (P) \iff \overrightarrow{v_d} \perp \overrightarrow{n_P} \iff \overrightarrow{v_d} \cdot \overrightarrow{n_P} = 0$.
$\overrightarrow{v_d} \cdot \overrightarrow{n_P} = 2 \cdot 1 + (-2)(-3) + 1 \cdot 1 = 9 \neq 0$, so $d \nparallel (P)$.

(3) We have the third situation when the angle between d and (P), $\alpha \in (0, \frac{\pi}{2})$.

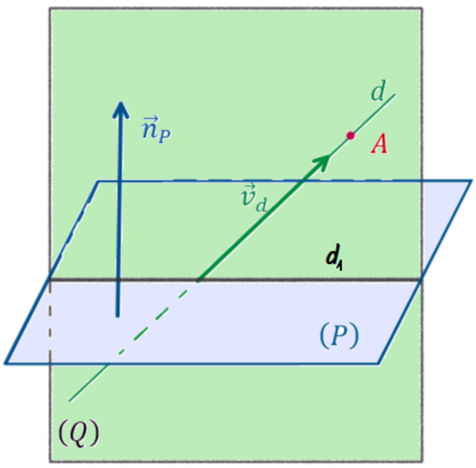


$$\alpha = 90^\circ - \measuredangle(\overrightarrow{v_d}, \overrightarrow{n_P}).$$

$$\cos(\measuredangle(\overrightarrow{v_d}, \overrightarrow{n_P})) = \frac{\overrightarrow{v_d} \cdot \overrightarrow{n_P}}{\|\overrightarrow{v_d}\| \cdot \|\overrightarrow{n_P}\|} = \frac{9}{3\sqrt{11}} = \frac{3}{\sqrt{11}}.$$

$$\cos(\measuredangle(\overrightarrow{v_d}, \overrightarrow{n_P})) = \sin(90^\circ - \measuredangle(\overrightarrow{v_d}, \overrightarrow{n_P})) = \sin\alpha = \frac{3}{\sqrt{11}} \Longrightarrow \alpha = \arcsin\frac{3}{\sqrt{11}}.$$

(b) The projection of a line d on a plane (P) is another line d_1 which lies on the plane (P) such that the plane (Q) determined by the lines d and d_1 is perpendicular to (P).



The equation of the plane determined by d and d_1, denoted by (Q), is a plane passing through any point of d, for example $A(3,-1,3) \in d$, and has two parallel directions:

- the normal to the plane (P): $(P) \perp (Q) \Longrightarrow \overrightarrow{n}_P = \overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k} \parallel (Q)$;
- the director vector of the line d: $d \subset (Q) \Longrightarrow \overrightarrow{v}_d = 2\overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k} \parallel (Q)$.

$$(Q): \begin{vmatrix} x-3 & y+1 & z-3 \\ 2 & -2 & 1 \\ 1 & -3 & 1 \end{vmatrix} = 0 \Longleftrightarrow (Q): x - y - 4z + 8 = 0.$$

$$d_1 = \mathrm{pr}_{(P)}d = (P) \cap (Q): \begin{cases} x - 3y + z + 9 = 0 \\ x - y - 4z + 8 = 0 \end{cases}.$$

Solving the system we can write the parametric equations of the line

$$d_1: \begin{cases} x = \dfrac{13}{2}t - \dfrac{15}{2} \\ y = \dfrac{5}{2}t + \dfrac{1}{2} \\ z = t \end{cases}, t \in \mathbb{R}.$$

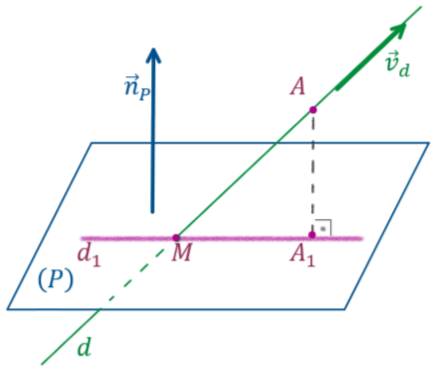
We present in what follows another method for solving this problem.

We know that the projections of all the points of d on (P) form the projection of d on (P), which in our case form a line, $d_1 = \mathrm{pr}_{(P)}d$. We only need two points of d_1 and we can write the equations of the line joining the two points.

Is obvious that the intersection of the plane and the line $\{M\} = d \cap (P)$ belongs to the line d_1.

The parametric equations of the line d are: $\begin{cases} x = 2t + 3 \\ y = -2t - 1 \\ z = t + 3 \end{cases}, t \in \mathbb{R}$

$\Longrightarrow M(2t+3, -2t-1, t+3)$.



$$\{M\} = d \cap (P) \Longleftrightarrow \begin{cases} x_M = 2t+3 \\ y_M = -2t-1 \\ z_M = t+3 \\ x_M - 3y_M + z_M + 9 = 0 \end{cases} \Longleftrightarrow 2t+3-3(-2t-1)+t+3+9 = 0$$

$\Longrightarrow t = -2 \Longrightarrow M(-1,3,1) \in d_1$.

For the second point we chose a point from the line d and then we will determine then its projection on (P).

For $t = 0$ in the parametric equations of d we obtain the point $A(3,-1,3) \in d$.

Let's find the coordinates of the projection of A on (P), $A_1 = \text{pr}_{(P)}A$.

$AA_1 \perp (P) \Longrightarrow \overrightarrow{AA_1} \parallel \overrightarrow{n}_P = \overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k} \Longrightarrow$

$$AA_1 : \frac{x-3}{1} = \frac{y+1}{-3} = \frac{z-3}{1} \Longleftrightarrow AA_1 : \begin{cases} x = t+3 \\ y = -3t-1 \\ z = t+3 \end{cases}, t \in \mathbb{R}.$$

$$\{A_1\} = AA_1 \cap (P) \Longleftrightarrow \begin{cases} x_{A'} = t+3 \\ y_{A_1} = -3t-1 \\ z_{A_1} = t+3 \\ x_{A_1} - 3y_{A_1} + z_{A_1} + 9 = 0 \end{cases} \Longleftrightarrow$$

$t+3-3(-3t-1)+t+3+9=0 \Longrightarrow t = -\dfrac{18}{11} \Longrightarrow A_1\left(\dfrac{15}{11}, \dfrac{43}{11}, \dfrac{15}{11}\right)$.

$d_1 = MA_1 : \dfrac{x+1}{\frac{15}{11}+1} = \dfrac{y-3}{\frac{43}{11}-3} = \dfrac{z-1}{\frac{15}{11}-1} \Longleftrightarrow$

$d_1 : \dfrac{x+1}{13} = \dfrac{y-3}{5} = \dfrac{z-1}{2}$.

Remark: We can notice that the parametric equations of

$d_1 : \begin{cases} x = 13p-1 \\ y = 5p+3 \\ z = 2p+1 \end{cases}$ $, p \in \mathbb{R}$ does not look like the ones obtained using the first method, i.e.

$$d_1 : \begin{cases} x = \dfrac{13}{2}t - \dfrac{15}{2} \\ y = \dfrac{5}{2}t + \dfrac{1}{2} \\ z = t \end{cases} , t \in \mathbb{R}.$$

But $z = t = 2p+1$, so we can express x and y using the parameter p in the latter form, i.e.

$x = \dfrac{13t-15}{2} = \dfrac{13(2p+1)}{2} - \dfrac{15}{2} = 13p-1$

$y = \dfrac{5t+1}{2} = \dfrac{5(2p+1)}{2} + \dfrac{1}{2} = 5p+3.$

We can observe that the parametric equations of a line are not unique.

5.6 Problems

Problem 5.22. Write the equation of the plane (P) such that:

a) $M(1,-2,-1) \in (P)$ and $Oz \perp (P)$.

b) $M(0,1,-3) \in (P)$, $Oz \parallel (P)$ and $Ox \parallel (P)$.

c) $M(2,-2,1) \in (P)$, $(P) \parallel (Q) : 2x - y + 6z - 5 = 0$.

Problem 5.23. Write the equation of the plane (P) if $A(2,-2,3)$ and $A'(4,2,-1)$ are symmetrical about the plane (P).

Problem 5.24. Write the equation of the plane which passes through the point $M(3,2,-1)$ and the axis Oy lies in (P).

Problem 5.25. Determine the equation of the plane (P) which contains the points $A(1,0,1)$ and $B(2,-1,1)$ and is parallel to Ox.

Problem 5.26. Write the equation of the plane which contains $A(1,2,-1)$ and is perpendicular to the line AB, $B(2,3,5)$. Calculate the distance between the point B and the plane (P).

Problem 5.27. Determine the equation of the plane (P) which passes through the point $A(1,0,1)$ and is perpendicular to the planes $(P_1) : 3x + y - 1 = 0$ and $(P_2) : x + y - z - 1 = 0$.

Problem 5.28. Determine the equation of the plane (P) which contains the points $A(-1,-2,0)$ and $B(1,1,2)$ and is perpendicular to the plane $(P) : x+2y+2z-4 = 0$.

Problem 5.29. Calculate the distance between the point $M(1,1,2)$ and the plane $(P) : x + 2y - 3z - 4 = 0$.

Problem 5.30. Calculate the angle between the planes $(P) : x + 3y + 2z + 4 = 0$ and $(Q) : 3x + 2y - z + 1 = 0$.

Problem 5.31. Write the equation of the plane which passes through the point $A(1, 2, 0)$ and is parallel to the plane $(P) : x - 2y + 3z - 5 = 0$.

Problem 5.32. Write the equation of the plane which contains the points $A(3, 0, 2)$ and $B(1, -1, 2)$ and is parallel to Ox.

Problem 5.33. Write the equation of the plane which contains the point $A(1, -1, 1)$ and is perpendicular to the planes $(P_1) : x-y+2z-3 = 0$ and $(P_2) : -x+2y-z = 0$. Find the angle formed by the planes (P_1) and (P_2).

Problem 5.34. Write the equation of the plane which is perpendicular to the plane $(P_1) : x + 2y + 2z - 4 = 0$ and contains the points $M_1(-1, -2, 0)$ and $M_2(1, 1, 2)$.

Problem 5.35. Consider the line $d : \frac{x-3}{4} = \frac{y+1}{-3} = z + 2$ in space.

(a) Write the direction vector of d.

(b) Write the parametric equations of d.

(c) Verify if the points $A(-1, 3, -4)$ and $B(-1, 2, -3)$ are points of d.

Problem 5.36. Consider $A(1, 2, 3), B(1, 1, 0)$ and $C(-1, 2, 1)$ three points in space. Write the canonical and the parametric equations of lines d_i if:

(a) $d_1 = AB$.

(b) d_2 passes through C and is parallel to the line d_1.

(c) $d_3 \perp d_1$, $d_3 \perp BC$ and passes through A.

(d) $d_4 : \begin{cases} x + 2y - 3z + 1 = 0 \\ -x + y + 2z - 5 = 0 \end{cases}$

Problem 5.37. Write the equations of the line which passes through the point $M(2,-5,3)$ and:

a) is parallel to Oz;

b) is parallel to the line $d : \dfrac{x-1}{4} = \dfrac{y-2}{-3} = \dfrac{z+3}{2}$.

c) is perpendicular to the plane $(P) : 3x - 7y + z - 23 = 0$.

Problem 5.38. Determine the equations of the line which is parallel to

$$d : \begin{cases} x = 3t - 4 \\ y = 5 - t \\ z = 4 \end{cases}, t \in \mathbb{R}$$

and passes through $A(1,6,-3)$.

Problem 5.39. Write the equations of the line AB if $A(-2,5,1)$ and $B(-2,2,5)$.

Problem 5.40. Write the equations of the line which lies in the plane $(P) : x + y - z - 2 = 0$, is perpendicular to the line $d_1 : \dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z-2}{-1}$ and passes through the point $M(1,0,-1)$.

Problem 5.41. Consider the lines d_1 and d_2 such that $d_1 \parallel \overrightarrow{v} = -\overrightarrow{i} + \overrightarrow{j}$ and $d_2 \parallel \overrightarrow{u} = \overrightarrow{i} + \overrightarrow{k}$.

(a) Calculate $m(\widehat{d_1, d_2})$.

(b) Write the equations of the line d_3 perpendicular to d_1 and perpendicular to d_2 and passes through $M(3,2,1)$.

Problem 5.42. Write the equations of the line which passes through $A(2,1,1)$ and is parallel to the line BC, $B(5,2,-1)$, $C(0,1,-2)$.

Problem 5.43. Write the equations of the line which passes through $A(-1,0,1)$ and is parallel to the line $d: \begin{cases} 2x-3y-z+4=0 \\ x-y+z-5=0 \end{cases}$.

Problem 5.44. Write the equations of the line which passes through $M(1,2,-2)$ and is perpendicular to the plane $(P): x-3y+z-5=0$.

Problem 5.45. Determine the line which lies in the plane $x-2y+z-5=0$, contains $A(1,-1,2)$ and is perpendicular to the line $d: \begin{cases} x=4+t \\ y=5+t \\ z=t-2 \end{cases}, t \in \mathbb{R}$.

Problem 5.46. Write the equation of the plane (P) which contains the line

$$d: \begin{cases} x=2-3t \\ y=4+t \\ z=1-2t \end{cases}, t \in \mathbb{R}$$

and passes through $A(-1,0,-1)$.

Problem 5.47. Determine the symmetric point of $M(-1,2,2)$ with respect to the plane $(P): x-y+2z+2=0$.

Problem 5.48. Determine the equation of the plane (P) which contains $M(1,-1,2)$ and is perpendicular to the line $d: \begin{cases} x+y-2z=0 \\ x-z+3=0 \end{cases}$.

Problem 5.49. Determine the equation of the plane (P) determined by the lines $d_1: \begin{cases} x-y+z-2=0 \\ 2x+y-z-1=0 \end{cases}$ and $d_2: \begin{cases} x+2y-2z-1=0 \\ x+y+z=0 \end{cases}$.

Problem 5.50. Find the projection of $M(1,2,-3)$ on the plane $(P): x-3y+z-2=0$ and calculate the distance from M to (P).

Problem 5.51. Find the projection of the line $d: \dfrac{x-3}{2} = \dfrac{y+1}{-2} = \dfrac{z-1}{3}$ on the plane $(P): x-y-z+1=0$.

Problem 5.52. Find the equation of the plane (P) which contains the line

$$d: \begin{cases} x-y+2z-6=0 \\ 2x+3y-z+3=0 \end{cases}$$

and is perpendicular to the plane $(Q): x+y-z+5=0$.

Problem 5.53. Determine the symmetric point of $A(2,4,-3)$ with respect to the line $d: x=2y=z$.

Problem 5.54. Find the distance between the point $M(1,2,-3)$ and the line

$$d: \frac{x-1}{2} = \frac{y+3}{-2} = z.$$

Problem 5.55. Determine the projection of $M(2,-1,2)$ on the line

$$d: \begin{cases} x = t+2 \\ y = 2t-1 \\ z = 3t+1 \end{cases}, t \in \mathbb{R}.$$

Problem 5.56. Write the equation of the line which passes through $M(2,3,1)$ and is parallel to the line $d: \dfrac{x+1}{2} = \dfrac{y}{-1} = \dfrac{z-2}{3}$.

Problem 5.57. Write the equation of the plane which passes through the point $M(1,-1,1)$ and is perpendicular to the line

(a) $d: \dfrac{x-3}{2} = y-2 = -z-2;$

(b) $d : \begin{cases} y - z + 4 = 0 \\ 2x - y = 0 \end{cases}$.

Problem 5.58. Find the distance between two lines d_1 and d_2 and the equations of the common perpendicular if it exists, for:

a) $d_1 : \dfrac{x-2}{3} = y + 1 = z$ and $d_2 : \begin{cases} 2x + y = 3 \\ z = 0 \end{cases}$;

b) $d_1 : \dfrac{x-1}{2} = \dfrac{y+1}{-2} = \dfrac{z}{3}$ and $d_2 : \begin{cases} x = 2t - 3 \\ y = 4 - 2t \\ z = 3t - 4 \end{cases}$;

c) $d_1 : \dfrac{x-2}{2} = \dfrac{y-1}{2} = z - 3$ and $d_2 : \dfrac{x+1}{-4} = \dfrac{y+1}{2} = \dfrac{z+3}{-1}$.

Problem 5.59. Consider the lines

$$d_1 : \frac{x-1}{2} = \frac{y+1}{0} = \frac{z}{-3}$$

and

$$d_2 : \frac{x+2}{3} = \frac{y-1}{\lambda} = \frac{z+1}{-1}.$$

Determine $\lambda \in \mathbb{R}$ such that the lines are coplanar and find the intersection point of d_1 and d_2.

Problem 5.60. Conisder the plane $(P) : 2x - 3y + 5z + 13 = 0$ and the line $d : \dfrac{x}{1} = \dfrac{y-1}{4} = \dfrac{z+2}{\alpha - 1}$. Determine $\alpha \in \mathbb{R}$ such that $d \subset (P)$.

Problem 5.61. Consider the plane $(P) : 2x + 3y - 2z + 1 = 0$ and the line $d : \dfrac{x-1}{3} = \dfrac{y+2}{\alpha + 2} = \dfrac{z+1}{1 - \alpha}$. Determine $\alpha \in \mathbb{R}$ such that $d \parallel (P)$.

Problem 5.62. Let $d_1 : \dfrac{x-1}{-5} = y - 2 = z$ and $d_2 : \dfrac{x-4}{-2} = \dfrac{y}{0} = \dfrac{z}{1}$ be two lines in space. Prove that d_1 and d_2 are skew and then determine the equations of their common perpendicular. Calculate the distance between d_1 and d_2 and the angle between the two lines.

Problem 5.63. Determine the relative position of

$$d_1 : \frac{x-1}{2} = \frac{y+1}{3} = z$$

and

$$d_2 : \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}$$

and then compute the distance between them.

Problem 5.64. Let $(P) : 2x + 2y - 3z - 1 = 0$ be a plane in space and

$$d : \frac{x+1}{2} = \frac{y-3}{2} = \frac{z-1}{1+\alpha}$$

be a line in space. Determine $\alpha \in \mathbb{R}$ such that:

(a) $(P) \parallel d$;

(b) $(P) \perp d$.

Problem 5.65. Let $A(2,4,1)$, $B(3,7,5)$ and $C(4,10,9)$ be three points in space. Prove that the points are collinear.

Problem 5.66. Determine if the points are coplanar:

(a) $A(-2,1,2)$, $B(-3,5,7)$, $C(-4,3,12)$ şi $D(1,1,1)$.

(b) $A(2,-1,0)$, $B(-1,-1,-5)$, $C(2,-2,-3)$ şi $D(-4,2,-1)$.

Problem 5.67. Determine the distance between $M(3,1,-1)$ and the line

$$d: \frac{x-1}{4} = \frac{y}{-5} = \frac{z+2}{3}.$$

Problem 5.68. Compute the distance between $M(-2,3,2)$ and the line

$$d: \frac{x-3}{1} = \frac{y+1}{2} = \frac{z-2}{-1}.$$

Problem 5.69. Compute the distance between the point $M(1,1,2)$ and the plane $(P): x+2y-3z-4=0$.

Problem 5.70. Determine the point M, $M \in d$, $d: \dfrac{x}{2} = \dfrac{y-1}{5} = z+1$ such that the distance between M and $(P_1): 4x+2y-z+6=0$ equals to the distance between M and $(P_2): 2x-4y+z-5=0$.

Problem 5.71. Determine the equations of the line d' which is the symmetric of d with respect to A if:

(a) $d: \begin{cases} x = t+2 \\ y = 2t-1 \\ z = 3t+1 \end{cases}, t \in \ ; A(-1,-1,2).$

(b) $d: \dfrac{x-1}{3} = \dfrac{y}{-2} = z+1 \ ; A(3,-2,4).$

Problem 5.72. Determine the equations of the line d which is perpendicular to the line $d_1: \dfrac{x+1}{2} = \dfrac{y}{-1} = \dfrac{z-2}{3}$, passes through $M(2,3,1)$ and the lines d and d_1 are coplanar.

Problem 5.73. Determine the equations of the planes (P_1) and (P_2) if both contain the line $d: \dfrac{x+1}{-2} = \dfrac{y+2}{3} = z+2$, $(P_1) \perp (P_2)$ and $A(2,-1,3) \in (P_2)$.

Problem 5.74. Determine the relative position of the plane (P) and the line d and then find the angle and the distance between the line and the plane if:

(a) $(P): x - y + z + 5 = 0;\;\; d: x + 1 = \dfrac{y-1}{-1} = \dfrac{z+3}{-2}.$

(b) $(P): 2x - 2y + 3z - 4 = 0;\;\; d: \begin{cases} x = 2t - 4 \\ y = 5t + 4 \\ z = 2t + 5 \end{cases},\; t \in \mathbb{R}.$

(c) $(P): x + 5y - z - 4 = 0;\;\; d: \begin{cases} x = 4 \\ y = t + 4 \\ z = 5 - 2t \end{cases},\; t \in \mathbb{R}.$

6

Conic sections, Cylinders and Quadric Surfaces

6.1 Conic sections

A conic section is a *curve* obtained as the intersection of a plane with a cone. A **cone** has two identically shaped parts called nappes. A right circular cone can be generated by revolving a line passing through the origin around the y-axis. The three types of conic section are the *hyperbola*, the *parabola*, the *ellipse* and the *circle* which is a special case of the ellipse.

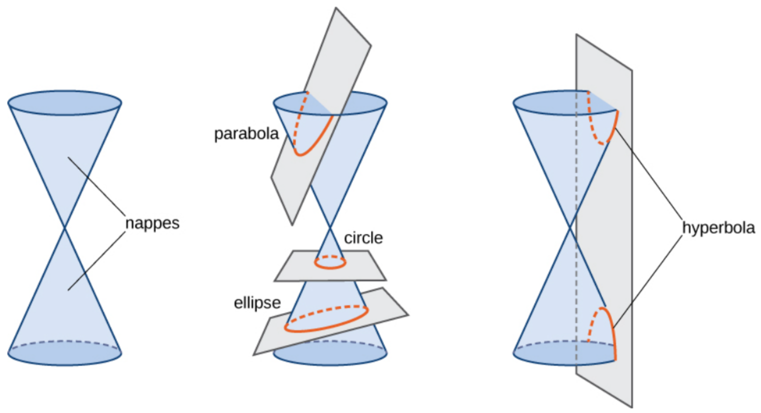


Figure 6.1: Conic sections

The general equation of a conic is

$$Ax^2 + Bxy + Cy^2 + Bx + Ex + F = 0.$$

Circle

Definition 6.1. *The **circle** is the set of the points that are a fixed distance from a center point.*

The distance is called **radius** and is denoted by **r**.

The general equation of a circle is

$$(\mathcal{C}) : (x - h)^2 + (y - k)^2 = r^2,$$

where $M(h, k)$ is the **center** of the circle and r is the **radius**. The circle is denoted by $\mathcal{C}((h, k), r)$.

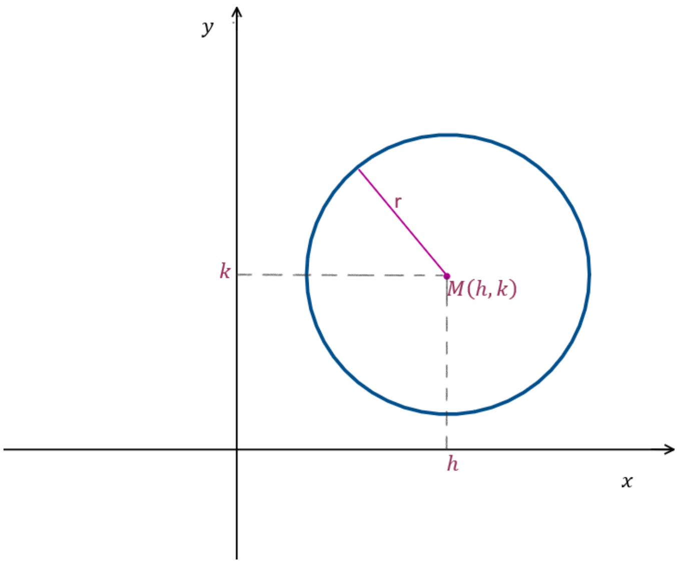


Figure 6.2: The circle

The parametric equations of the circle $\mathcal{C}((h,k),r)$ are:

$$(\mathcal{C}) : \begin{cases} x = h + r\cos\theta \\ y = k + r\sin\theta \end{cases}, \quad r \geqslant 0, \;\; \theta \in [0, 2\pi].$$

The equation of the circle with the center at the origin is $(\mathcal{C}) : x^2 + y^2 = r^2$.

The tangent line at a point to the circle $\mathcal{C}((h,k),r)$.

1. The tangent line at a point $M_0(x_0, y_0) \in (\mathcal{C})$ to the circle $(\mathcal{C})$ is

$$tg : (x-h)(x_0-h) + (y-k)(y_0-k) = r^2.$$

2. The tangent lines to the circle $(\mathcal{C})$ having the slope m are

$$tg : y - k = m(x-h) \pm r\sqrt{1+m^2}.$$

3. The tangent lines to the circle $(\mathcal{C})$ from an exterior point $M_0(x_0, y_0)$ is

$$tg : y - y_0 = m(x - x_0),$$

where $m \in \{m_1, m_2\}$, m_1 and m_2 are the solutions of the equation

$$y_0 - k = m(x_0 - h) \pm r\sqrt{1 + m^2}.$$

Ellipse

Definition 6.2. *The* ***ellipse*** *is a closed curve, the locus of a point such that the sum of the distance from that point to two other fixed points F_1, F_2 called foci of the ellipse, is constant.*

The general equation of an ellipse is:

$$(E) : \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

The parametric equations of the ellipse (E) are:

$$(E) : \begin{cases} x = h + a\cos\theta \\ y = k + b\sin\theta \end{cases}, \quad a, b \geqslant 0, \ \theta \in [0, 2\pi].$$

If $a > b$, then:

- The major axis is the longest width across the ellipse is on Ox (the ellipse is horizontally oriented), the length of the major axis is $2a$.
- The minor axis is the shortest width across the ellipse is on Oy, the length of the minor axis is $2b$.
- The center is the intersection of the two axes $M(h, k)$.

- The foci F_1, F_2 are on the major axis $F_1(h-c,k)$, $F_2(h+c,k)$, where $c = \sqrt{a^2-b^2}$.
- The vertices are the end points of the major axis $V_1(h-a,k)$, $V_2(h+a,k)$.
- The eccentricity is $\epsilon = \dfrac{c}{a} < 1$. The eccentricity of an ellipse refers to how flat or round the shape of the ellipse is. The more flattened the ellipse is, the greater the value of its eccentricity. The more circular, the smaller the value or closer to zero is the eccentricity.

If $a > b$ and $h = k = 0$ we have the ellipse horizontally oriented having the center at the origin $O(0,0)$:

$$(E) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

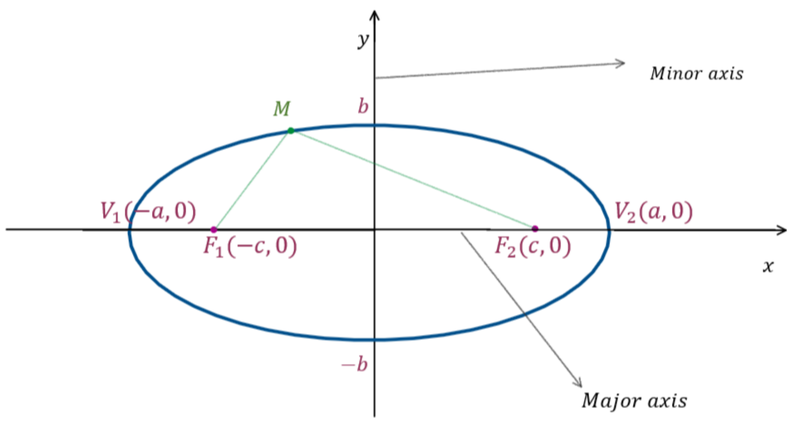


Figure 6.3: The ellipse horizontally oriented

If $b > a$ and $h = k = 0$ we have the ellipse vertically oriented having the center at the origin $O(0,0)$.

- The major axis is on Oy, the length of the major axis is $2b$.
- The minor axis is on Ox, the length of the minor axis is $2a$.
- The foci F_1, F_2 are on the major axis $F_1(0, -c)$, $F_2(0, c)$, where $c = \sqrt{b^2 - a^2}$.
- The vertices are $V_1(0, -b)$ and $V_2(0, b)$.

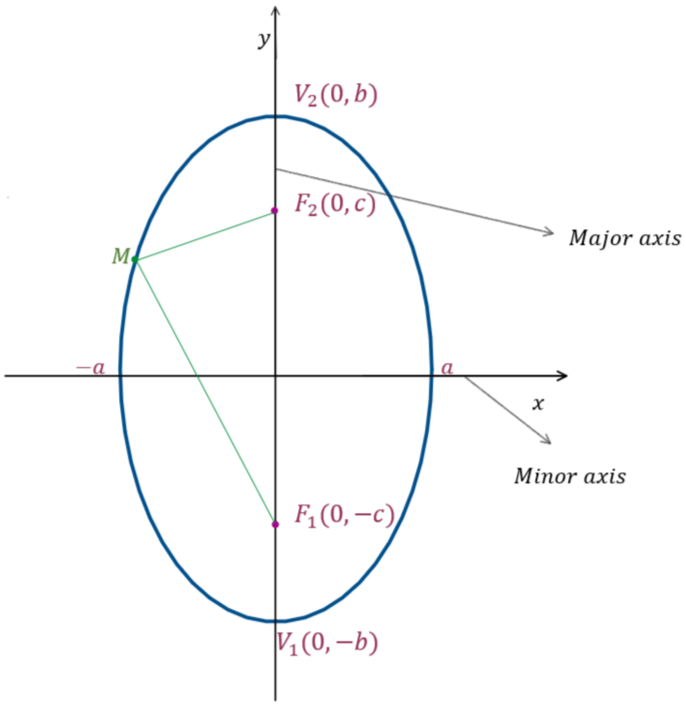


Figure 6.4: The ellipse vertically oriented

The tangent line at a point to the ellipse (E).

1. The tangent line at a point $M_0(x_0, y_0) \in (E)$ to the ellipse $(E) : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$tg : \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

2. The tangent lines to the ellipse (E) having the slope m are

$$y = mx \pm \sqrt{a^2m^2 + b^2}.$$

Hyperbola

Definition 6.3. *The **hyperbola** is the set of all points such that the difference between the distance to two focal points called foci F_1, F_2, is constant.*

If we choose the foci on Ox axis, $F_1(-c, 0)$, $F_2(c, 0)$, the equation of the hyperbola having $O(0, 0)$ as its center is:

$$(H) : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

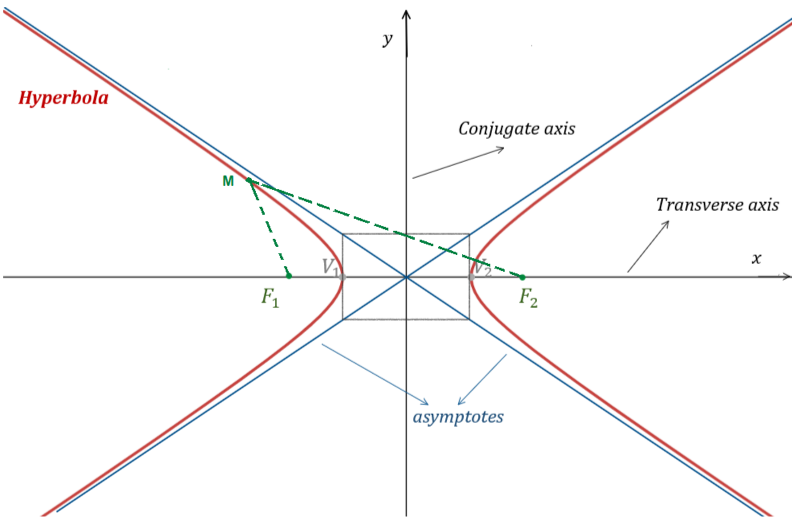


Figure 6.5: The hyperbola opens left and right

- The transverse axis is $F_1F_2 = Ox$.
- The conjugate axis is perpendicular on F_1F_2 and passes through the center, in this case is Oy axis.

- The rectangle having the edges $2a$ and $2b$ symmetric with respect to the hyperbola axes is called the fundamental rectangle of the hyperbola.
- The asymptotes of the hyperbola $y = \pm\dfrac{b}{a}x$ are the diagonals of the of the fundamental rectangle.
- The foci are $F_1(-c, 0)$, $F_2(c, 0)$, where $c = \sqrt{a^2 + b^2}$.
- The vertices, $V_1(-a, 0)$ and $V_2(a, 0)$, are intercepts of the transversal axis with the hyperbola.
- The eccentricity is $\epsilon = \dfrac{c}{a} > 1$.

The parametric equations of the hyperbola (H) are:

$$(H) : \begin{cases} x = \pm a \cosh t \\ y = b \sinh t \end{cases} , \quad t \in \mathbb{R},$$

where $\cosh t = \dfrac{e^t + e^{-t}}{2}$, $\sinh = \dfrac{e^t - e^{-t}}{2}$.
The tangent line at a point $M_0(x_0, y_0) \in (H)$ to the hyperbola (H) is

$$tg : \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1.$$

If we choose the foci on Oy then

$$(H) : \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

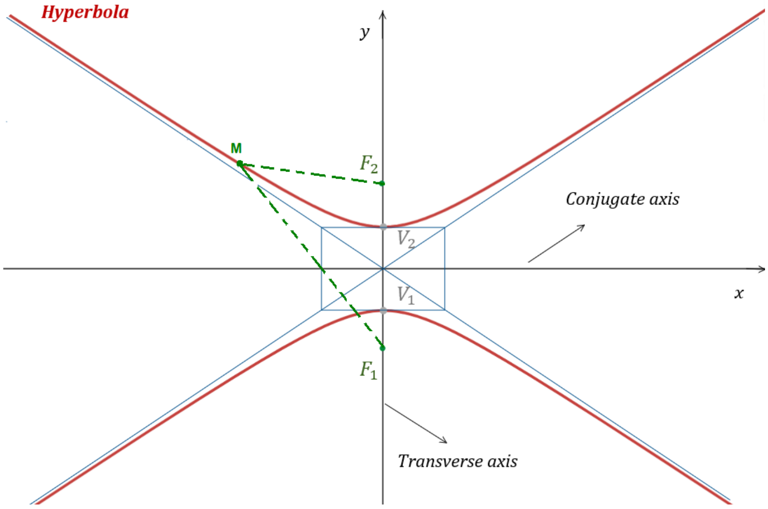


Figure 6.6: The hyperbola opens up and down

- The transverse axis is on Oy.
- The conjugate axis is on Ox.
- The branches open up and down.
- The foci are at $F_1(0, -c)$, $F_2(0, c)$, where $c = \sqrt{a^2 + b^2}$.
- The vertices are at $V_1(0, -b)$, $V_2(0, b)$.
- The asymptotes are of equations $y = \pm \frac{a}{b} x$.

The general equation of an hyperbola with the branches open left and right is:

$$(H) : \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

For this hyperbola we have:

- The center is at $C(h, k)$.

- The vertices are at $V(h \pm a, k)$.
- The foci are at $F_{1,2}(h \pm c, k)$, where $c = \sqrt{a^2 + b^2}$.
- The asymptotes are of equations $y - k = \pm\frac{b}{a}(x - h)$.
- The transverse axis is F_1F_2, a parallel line to Ox passing through C.
- The conjugate axis is a parallel line passing through C and perpendicular to the transverse axis.

The general equation of an hyperbola with the branches open up and down is:

$$(H): \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

For this hyperbola we have:

- The center is at $C(h, k)$.
- The vertices are at $V(h, k \pm a)$.
- The foci are at $F_{1,2}(h, k \pm c)$, where $c = \sqrt{a^2 + b^2}$.
- The asymptotes are of equations $y - k = \pm\frac{a}{b}(x - h)$.
- The transverse axis is F_1F_2, a parallel line to Oy.
- The conjugate axis is a parallel line passing through C and perpendicular to the transverse axis.

Rectangular hyperbola

A **rectangular hyperbola** is a hyperbola for which the asymptotes are perpendicular, also called an *equilateral hyperbola* or *right hyperbola*.

An equilateral hyperbola can be obtained when $a = b$, in this case the general equation is $x^2 - y^2 = a^2$ or $y^2 - x^2 = a^2$.

If the asymptotes are Ox and Oy axis, then the equation of the rectangular hyperbola is $xy = c^2$.

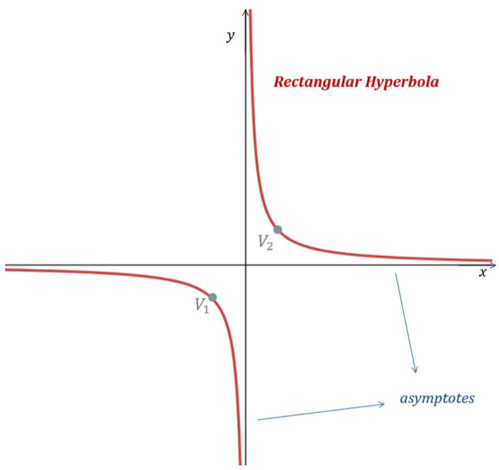


Figure 6.7: Rectangular hyperbola $xy = c^2$

- The rectangular hyperbola is the same shape as the standard hyperbola, but rotated by 45°.
- The asymptotes are Ox and Oy axes.
- The vertices of the rectangular hyperbola are $V_1(-c, -c)$, $V_2(c, c)$.
- The eccentricity is $\epsilon = \sqrt{2}$.
- The parametric equations of the rectangular hyperbola are

$$(H) : \begin{cases} x = ct \\ y = \dfrac{c}{t} \end{cases} , \quad t \in \mathbb{R}^*.$$

Parabola

Definition 6.4. *The **parabola** is the set of all points whose distance from a fixed point called the* focus *is equal the distance from a fixed line called the* directrix*.*

The point halfway between the focus and the directrix is called the *vertex* of the parabola.

There are four types of parabolas on the coordinate planes. They can open down, up, to the left and to the right.

1. $(P) : (x-h)^2 = 4p(y-k)$.

   The parabola opens up if $p > 0$ and opens down if $p < 0$.

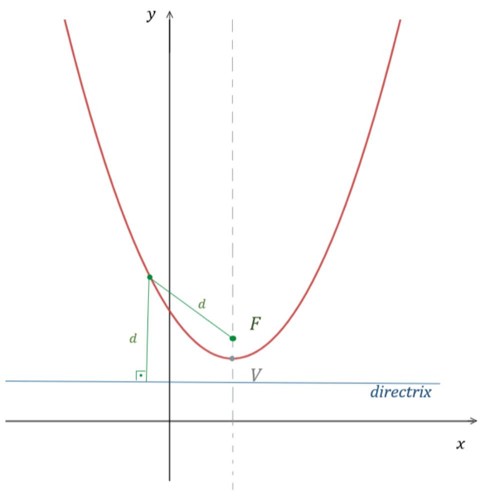


Figure 6.8: The parabola opens up

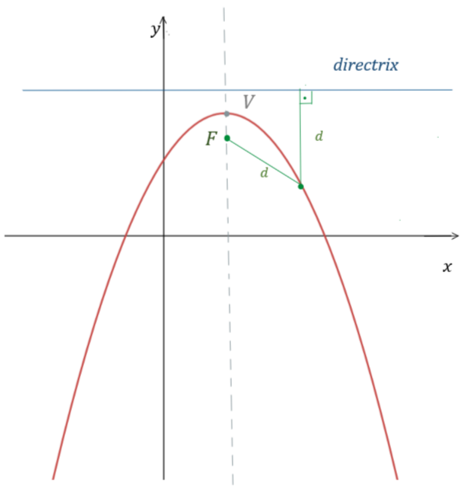


Figure 6.9: The parabola opens down

- The vertex is $V(h, k)$.
- The directrix is $y = k - p$.
- The focal point is $F(h, k + p)$.

- The parametric equations are: $\begin{cases} x = 2pt + h \\ y = pt^2 + k \end{cases}, \quad t \in \mathbb{R}.$

2. $(P) : (y - k)^2 = 4p(x - h)$.

   The parabola opens right if $p > 0$ and opens left if $p < 0$.

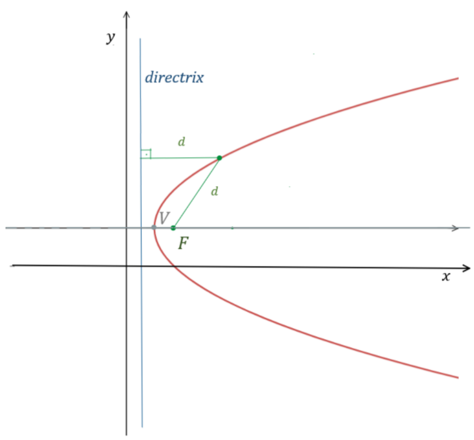


Figure 6.10: The parabola opens right

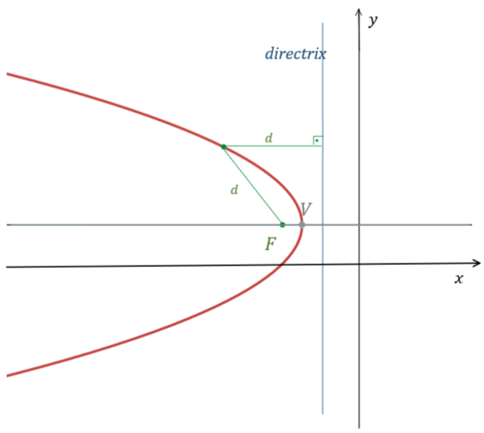


Figure 6.11: The parabola opens left

- The vertex is $V(h, k)$.
- The directrix is $x = h - p$.
- The focal point is $F(h + p, k)$.
- The parametric equations are $\begin{cases} x = pt^2 + h \\ y = 2pt + k \end{cases}, \quad t \in \mathbb{R}.$

6.2 Cylinders

Definition 6.5. *A* ***cylinder*** *is a surface that consists of all lines called* rulings, *that are parallel to a given line and pass through a given plane curve.*

Remark

- In $\mathbb{R}^3$ space, the equation of the cylinder has only two different variables. These equations give a trace of the curve on the coordinates plane denoted by the given variables.
- The curve is directed along the axis of the missing variable.
- The curve does not change along the direction axis.

Example 6.6. *Graph the surface of equation $z = x^2 + 1$.*

- *We don't have y in the equation, so we will look at the trace on the xOz plane.*
- *In xOz plane $z = x^2 + 1$ is a parabola opens up on Oz axis.*
- *We will obtain the full surface by assembling together the infinitely many parabolas traced in each plane $y = k$.*

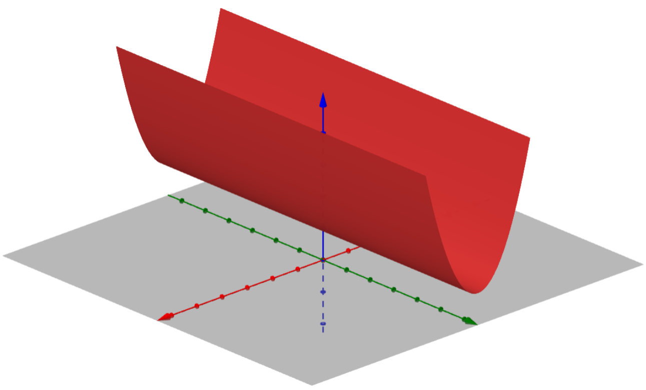


Figure 6.12: Cylinder $z = x^2 + 1$

Example 6.7. *Graph the surface of equation* $x^2 + y^2 = 4$.

- *We don't have* z *in the equation, so we will look at the trace on the* xOy *plane.*
- *In* xOy *plane* $x^2 + y^2 = 4$ *is a circle with the center at the origin and* $r = 2$.
- *We will obtain the full surface by assembling together the infinitely many circles traced in each plane* $z = k$.

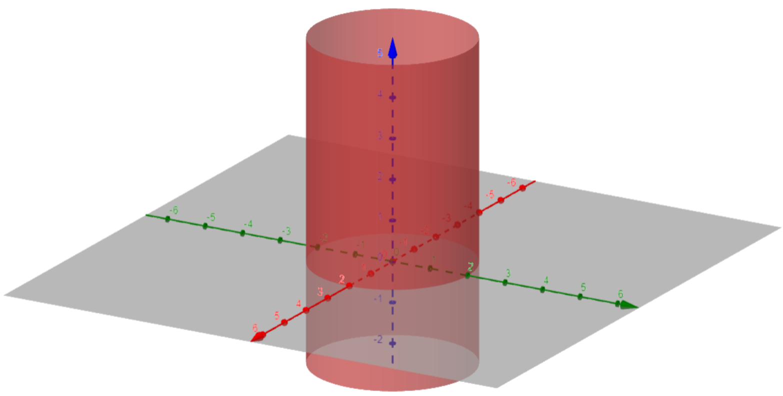


Figure 6.13: Cylinder $x^2 + y^2 = 4$

Example 6.8. *Graph the surface of equation* $yz = 6$.

- *We don't have* x *in the equation, so we will look at the trace on the* yOz *plane.*
- *In* yOz *plane* $yz = 6$ *is a rectangular hyperbola with the asymptotes* Oy *and* Oz.
- *We will obtain the full surface by assembling together the infinitely many hyperbolas traced in each plane* $x = k$.

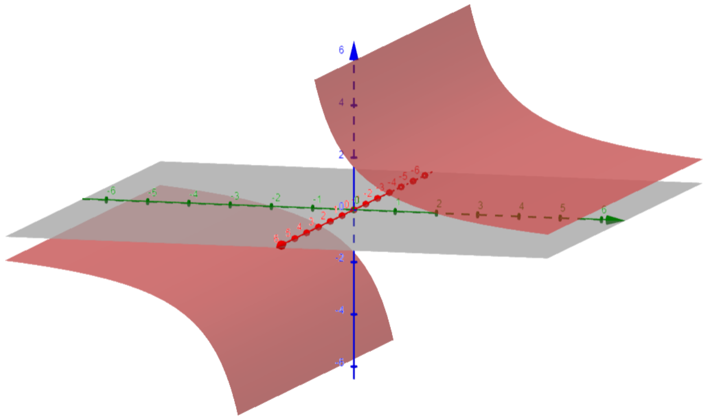


Figure 6.14: Cylinder $yz = 6$

6.3 Quadric Surfaces

A quadric surface is given by a degree two equation in the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

We will look at six basic surfaces that each of the following equations forms:

$$Ax^2 + By^2 + Cz^2 + J = 0$$

$$Ax^2 + By^2 + Iz = 0$$

By rotating and translating these we can obtain more general surfaces. The terms with xy, xz, zy are only when we dealing with a rotation of these quadric surfaces.

The six basic types of quadric surfaces are:

1. ellipsoid

2. hyperboloid of one sheet

3. hyperboloid of two sheets

4. elliptic cone

5. elliptic paraboloid

6. hyperbolic paraboloid

The degree and sign of the degree two terms as well as which terms are present help determine which of the six basic quadric surfaces is given.

To graph a quadric surface is it often helpful to graph the xy-trace, the xz-trace and the yz-trace. These are the intersection of the surface with the three planes of the coordinates system.

To determine the xy-trace we set $z = 0$.

To determine the xz-trace we set $y = 0$.

To determine the yz-trace we set $x = 0$.

A *trace* of a surface is the curve obtained by intersecting the surface with a plane parallel to the coordinate plane, i.e. $x =$ constant, $y =$ constant, $z =$ constant.

Ellipsoid

$$(E) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Characteristics:

- all three degree two terms are present;
- all three degree two terms are positive when equation equals 1;
- all three traces are ellipse;
- the intercepts are $x = \pm a$, $y = \pm b$, $z = \pm c$.

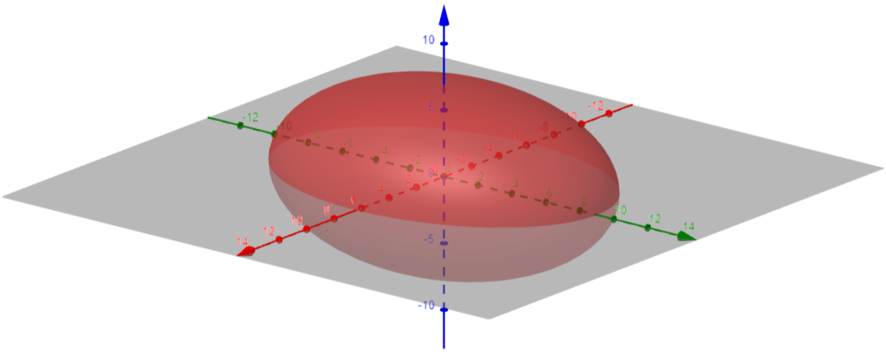


Figure 6.15: Ellipsoid

The tangent plane at $M_0(x_0, y_0, z_0) \in (E)$ to the ellipsoid is:

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} - 1 = 0.$$

Sphere

The sphere is a special case of an ellipsoid when $a = b = c$.

Definition 6.9. *The* ***sphere*** *is the locus of the points in space that are a fixed distance called the radius, r, from a point called the center.*

The equation of the sphere centered at $M_0(x_0, y_0, z_0)$ and having the radius r is:

$$(S) : (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Hyperboloid of one sheet

$$(H1S) : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Characteristics:

- all three degree two terms are present;

- two degree two terms are positive and one is negative when equation equals 1;
- one trace is an ellipse;
- two traces are hyperbolas;
- the axis of the hyperboloid of one sheet is parallel to the negative variable, i.e. is directed along the axis with "-".

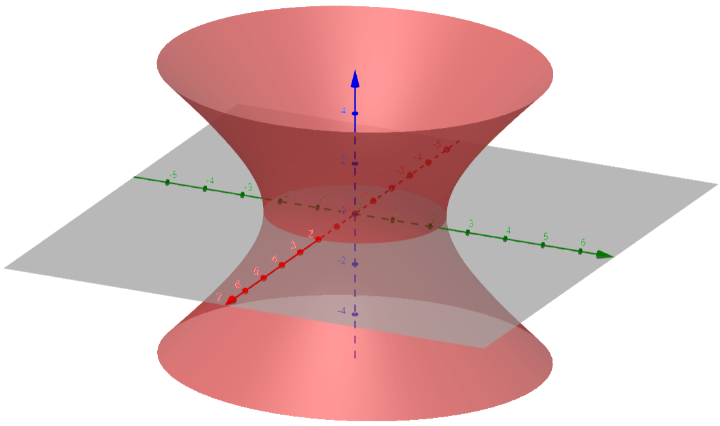


Figure 6.16: Hyperboloid of one sheet

The tangent plane at $M_0(x_0, y_0, z_0) \in (H1S)$ to the hyperboloid of one sheet is:

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} - 1 = 0.$$

Two sheets Hyperboloid

$$(H2S) : -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Characteristics:

- all three degree two terms are present;

- one degree two term is positive and two are negative when equation equals 1;
- one trace is an ellipse parallel to the xOy plane (to the plane determined by the negative variables);
- two traces are hyperbolas;
- the axis of the hyperboloid of two sheets is parallel to the positive variable axis.
- the intercepts are $z = \pm c$ (we set the negative variable 0).

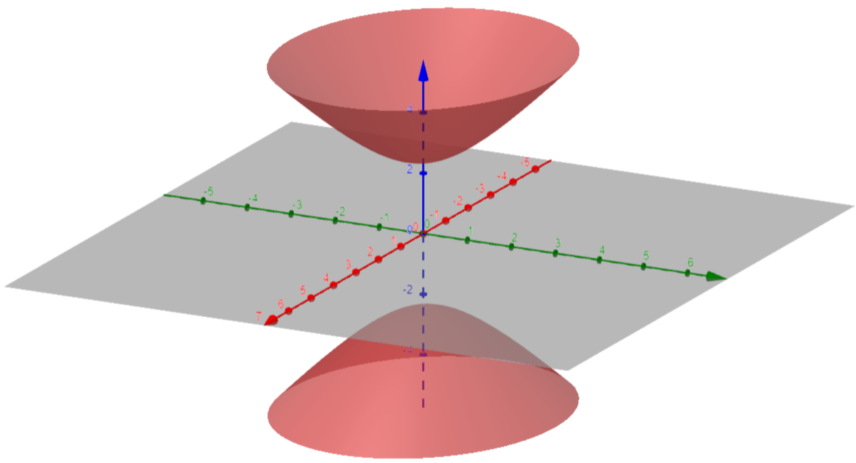


Figure 6.17: Hyperboloid of two sheets

The tangent plane at $M_0(x_0, y_0, z_0) \in (H2S)$ to the hyperboloid of two sheets is:

$$-\frac{xx_0}{a^2} - \frac{yy_0}{b^2} + \frac{zz_0}{c^2} - 1 = 0.$$

Elliptical cone

$$(EC): \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

If (x, y, z) is a solution of the general equation of the elliptical cone we have that $(\lambda x, \lambda y, \lambda z)$ is also a solution of the equation. That means that the surface is a union of lines through the origin.

Characteristics:

- all three degree two terms are present;
- two degree two terms are positive and one is negative when equation equals 0;
- one trace is a point or an ellipse parallel to the xOy plane (to the plane determined by the positive variables);
- two traces are two lines or two hyperbolas;
- the axis of the elliptical cone is parallel to the negative variable axis.

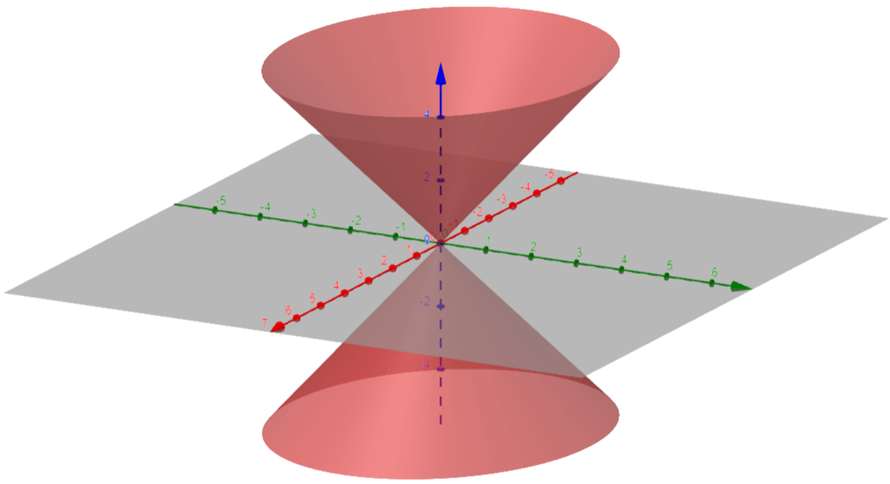


Figure 6.18: Elliptical cone

Elliptic paraboloid

$$(EP) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = cz.$$

Characteristics:

- two degree two terms are present and are positive;
- one degree one term is present;
- one trace is an ellipse;
- two traces are parabolas;
- the axis is parallel to degree one variable axis.
- direction:
  - if $c > 0$ the elliptic paraboloid opens towards positive direction of degree one variable axis.
  - if $c > 0$ the elliptic paraboloid opens towards negative direction of degree one variable axis.

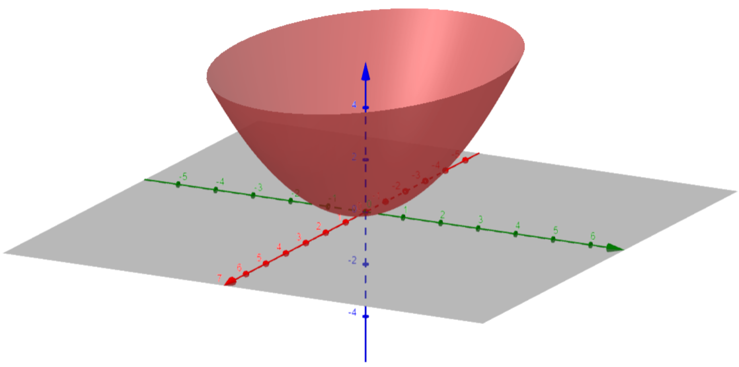


Figure 6.19: Elliptic paraboloid

The tangent plane at $M_0(x_0, y_0, z_0) \in (EP)$ to the elliptic paraboloid is:

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = \frac{c}{2}(z + z_0).$$

Hyperbolic paraboloid

$$(HP) : \frac{x^2}{a^2} - \frac{y^2}{b^2} = cz.$$

Characteristics:

- two degree two terms are present, one positive and one negative;
- one degree one term is present;
- one trace is a hyperbola;
- two traces are parabolas;
- the axis is parallel to degree one variable axis.

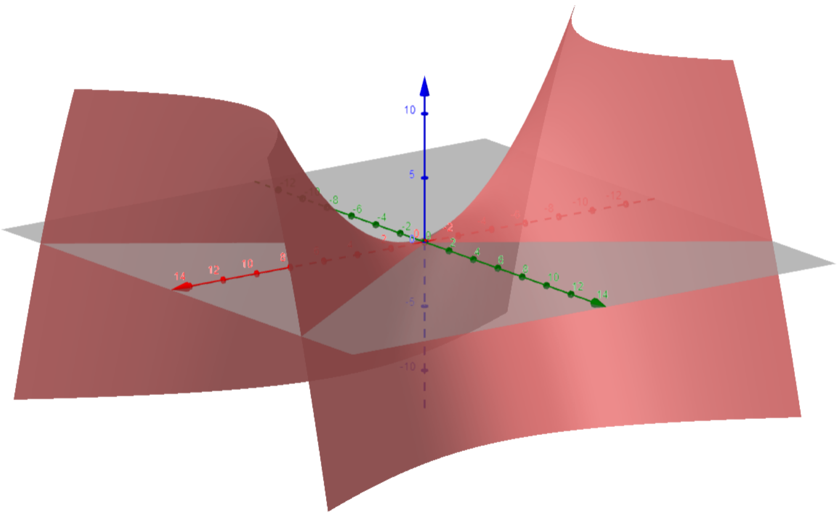


Figure 6.20: Hyperbolic paraboloid

The tangent plane at $M_0(x_0, y_0, z_0) \in (EP)$ to the hyperbolic paraboloid is:

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{c}{2}(z + z_0).$$

6.4 Solved problems

Problem 6.1. a) Put $2x^2 - 8x + 2y^2 + 4y - 8 = 0$ into a standard circle form.

b) Determine the radius and the center of the circle.

c) Write the parametric equations of this circle.

d) Draw the circle.

e) Find two points on the circle and plug them into the equation to make sure your drawing is correct.

f) Write the tangent line to the circle at one of the point previously determined.

Solution:

a) We divide by 2 the equation and we get:

$x^2 - 4x + y^2 + 2y - 4 = 0 \iff$

$x^2 - 4x + 4 + y^2 + 2y + 1 - 4 - 4 - 1 = 0 \iff$

$(x - 2)^2 + (y + 1)^2 = 9$

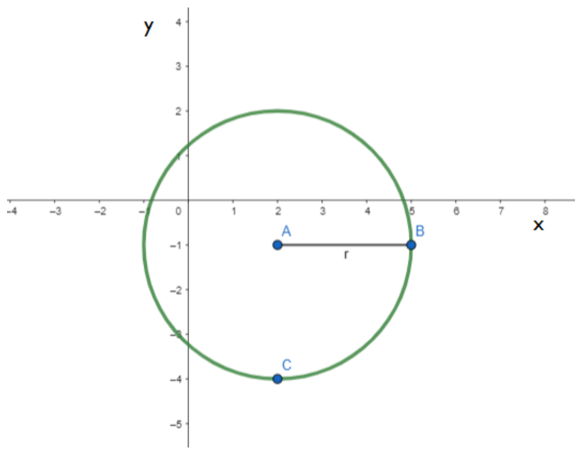
which is the standard circle form.

b) The circle $(\mathcal{C}) : (x - 2)^2 + (y + 1)^2 = 9$ has the radius $r = 3$ and the center at $A(2, -1)$.

c) The parametric equations of the circle are:

$$(\mathcal{C}) : \begin{cases} x = 2 + 3\cos t \\ y = -1 + 3\sin t \end{cases}, \quad t \in [0, 2\pi].$$

d) The graph of the circle is:



e) The points we chose are $B(5,-1)$ and $C(2,-4)$. $B \in (\mathcal{C})$ if $(5-2)^2+(-1+1)^2 = 9$ which is true and $C \in (\mathcal{C})$ if $(2-2)^2 + (-4+1)^2 = 9$ which is also true, so the chosen points are on the circle.

e) The tangent line at $B(5,-1)$ is

$tg : (x-2)(5-2) + (y+1)(-1+1) = 9 \Longleftrightarrow$

$tg : x = 5.$

Problem 6.2. For the ellipse

$$(E) : \frac{(x+1)^2}{9} + \frac{(y-2)^2}{25} = 1$$

find the major axis and its length, the minor axis and its length, the center, the vertices, the foci, write the parametric equations, find the intercepts of the ellipse and then graph it.

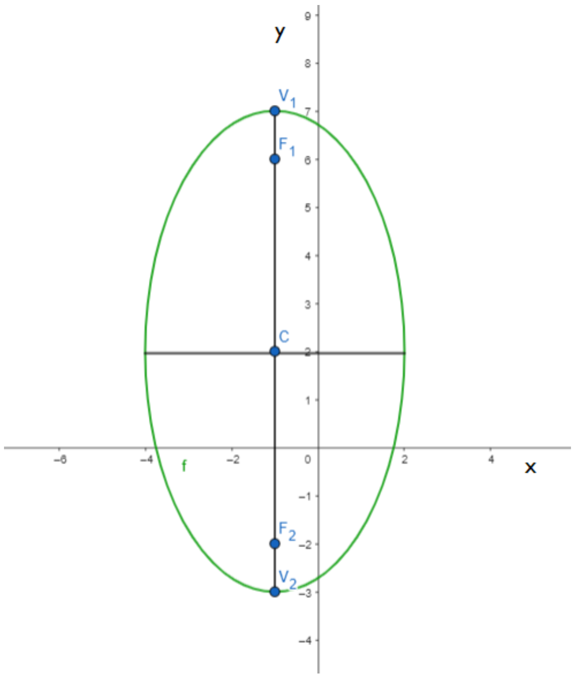
Solution:

We notice that $25 > 9$ so the major axis is along Oy, $a = 5$, $b = 3$. The length of the major axis is 10, the minor axis is Ox with the length 6.

The center is at $C(-1, 2)$.

The vertices are on a parallel line to Oy axis, $x = -1$, $V_1(-1, 7)$ and $V_2(-1, -3)$. (From the center of the ellipse $C(-1, 2)$ we go up and down $a = 5$ u.m.).

For the coordinates of the foci we need $c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = 4$. The foci are also on the major axis, $c = 4$ u.m. up and down from the center $F(-1, 2 \pm 4)$, $F_1(-1, 6)$ and $F_2(-1, -2)$.



The parametric equations of the ellipse are

$$(E) : \begin{cases} x = -1 + 3\cos t \\ y = 2 + 5\sin t \end{cases}, \quad t \in [0, 2\pi].$$

The intercepts of the ellipse are:

- $(E) \cap Ox: \; y = 0 \Longrightarrow \dfrac{(x+1)^2}{9} + \dfrac{4}{25} = 1 \Longleftrightarrow (x+1)^2 = 9 \cdot \dfrac{21}{25} \Longleftrightarrow$ $x = -1 \pm \dfrac{3\sqrt{21}}{5}$. We have two intercepts on Ox axis $A_1\left(-1 - \dfrac{3\sqrt{21}}{5}, 0\right)$ and $A_2\left(-1 + \dfrac{3\sqrt{21}}{5}, 0\right)$.
- $(E) \cap Oy: \; x = 0 \Longrightarrow \dfrac{1}{9} + \dfrac{(y-2)^2}{25} = 1 \Longleftrightarrow (y-2)^2 = \dfrac{200}{9} \Longleftrightarrow y = 2 \pm \dfrac{10\sqrt{2}}{3} \Longrightarrow$ we have two intercepts on Oy axis $B_1\left(0, 2 - \dfrac{10\sqrt{2}}{3}\right)$ and $B_2\left(0, 2 + \dfrac{10\sqrt{2}}{3}\right)$.

Problem 6.3. For the hyperbola

$$(H): x^2 - 4y^2 - 16 = 0$$

find the transverse and the conjugate axis, the center, the vertices, the foci, the asymptotes, the eccentricity and then graph it. Write the parametric equations of the hyperbola. Check if $M(2\sqrt{5}, -1)$ is on the hyperbola and write the equation of the tangent line at M.

Solution:

We notice that the equation is not in the standard form so we divide by 16, and we get $(H): \dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$.

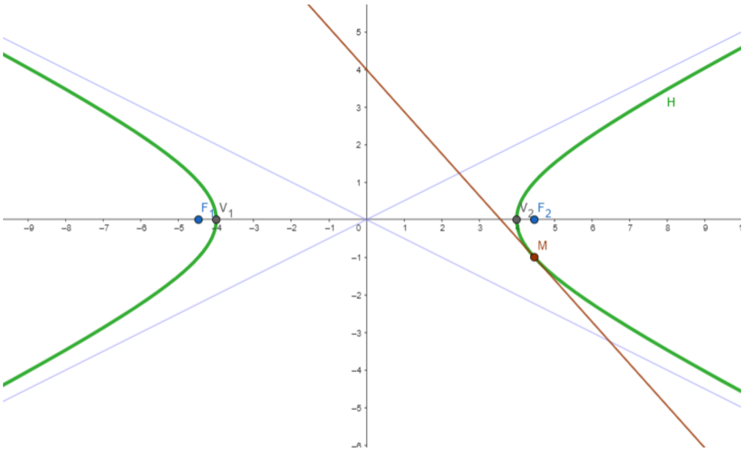
The center is at $O(0,0)$.

Because the x term is positive when the equation equals 1, the transverse axis is Ox, $a = 4$ and the conjugate axis is Oy, $b = 2$.

The vertex are on the transverse axis $V_1(-4,0)$ and $V_2(4,0)$.

For the coordinates of the foci, we determine $c = \sqrt{a^2+b^2} = \sqrt{16+4} = 2\sqrt{5}$. The foci are $F_1(-2\sqrt{5}, 0)$ and $F_1(2\sqrt{5}, 0)$.

The asymptotes are the lines $d_1: y = \frac{1}{2}x$ and $d_2: \; y = -\frac{1}{2}x$, while the eccentricity is $\epsilon = \dfrac{2\sqrt{5}}{4} = \dfrac{\sqrt{5}}{2}$.



The parametric equations of the hyperbola are

$$(E):\begin{cases} x = \pm 4\cosh t \\ y = 2\sinh t \end{cases}, \quad t \in \mathbb{R}.$$

$M(2\sqrt{5}, -1) \in (H)$ if $\dfrac{(2\sqrt{5})^2}{16} - \dfrac{(-1)^2}{4} = 1 \Longleftrightarrow \dfrac{20}{16} - \dfrac{1}{4} - 1 = 0$ which is true so, the tangent line to the hyperbola at M is

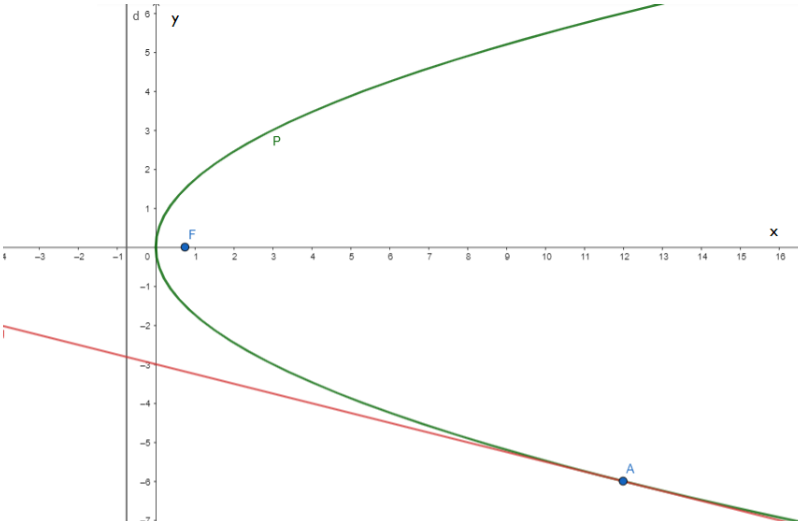
$tg : \dfrac{2\sqrt{5}x}{16} - \dfrac{-1 \cdot y}{4} = 1 \Longleftrightarrow$

$tg : \sqrt{5}x + 2y - 8 = 0.$

Problem 6.4. Let $(P) : y^2 = 3x$ be a parabola. Determine the vertex, the focal point, the directrix and then graph it. Write the parametric equations of the parabola. Determine the equation of the tangent line at $A(12, -6)$.

Solution:

The vertex of the parabola is at $O(0,0)$ and due to the fact that the second order term is at y, $p = \dfrac{3}{4}$, the focal point is on Ox axis at $F\left(\dfrac{3}{4}, 0\right)$. The directrix is a line parallel to Oy passing through $x = -p$ so the directrix is $x = -\dfrac{3}{4}$.



The parametric equations of the parabola are

$$(P) : \begin{cases} x = \dfrac{t^2}{3} \\ y = t \end{cases}, \quad t \in \mathbb{R}.$$

$A(12, -6) \in (P)$ since $(-6)^2 = 3 \cdot 12$. The tangent line to the parabola at A is
$tg : y \cdot (-6) = \dfrac{3}{2}(x + 12) \iff$
$tg : x + 4y + 12 = 0$.

Problem 6.5. Identify and then graph the conic having the equation:

a) $(C_1) : 3x^2 + y + 6x - 2 = 0$.

b) $(C_2) : x^2 + 4y^2 + 10x - 16y + 25 = 0$.

c) $(C_3) : x^2 + y^2 - 10x + 2y + 21 = 0$.

d) $(C_4) : 4y^2 - x^2 - 40y - 12x + 60 = 0$.

Solution:

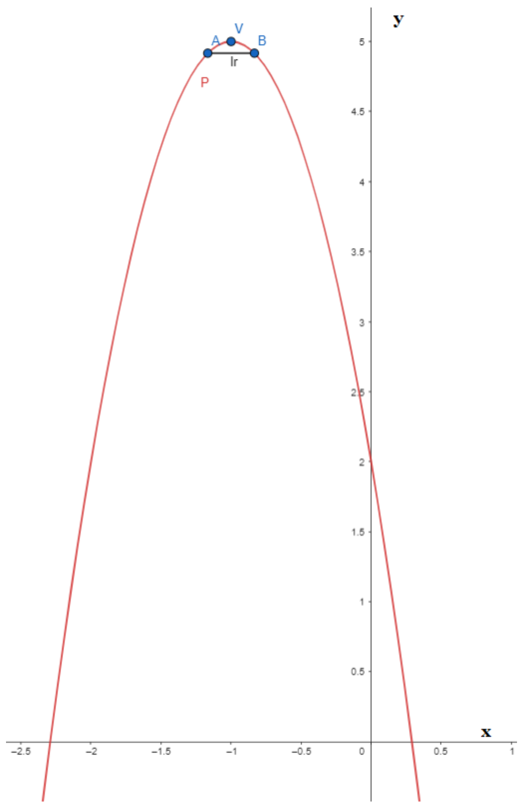
a) $(C_1): \ 3x^2 + y + 6x - 2 = 0 \Longleftrightarrow$

$(C_1): \ 3(x^2 + 2x + 1) = -y + 2 + 3 \Longleftrightarrow$

$(C_1): \ (x+1)^2 = -\dfrac{y-5}{3}$

which is the equation of a parabola along Oy axis, which opens down, having the vertex at $V(-1,5)$, and $p = -\dfrac{1}{12}$. The focal point is $F\left(-1, 5 - \frac{1}{12}\right)$.

When graphing the parabola, the number $4p$ which is called also *latus rectum* is very helpful since the points at the ends of the line segment parallel to the directrix having the length $|4p|$ and having the middle point the focal point, belong to the parabola. In our case the latus rectum is $1/3$ so the points $A\left(-1 - \frac{1}{6}, 5 - \frac{1}{12}\right)$ and $B\left(-1 + \frac{1}{6}, 5 - \frac{1}{12}\right)$ are on the parabola.



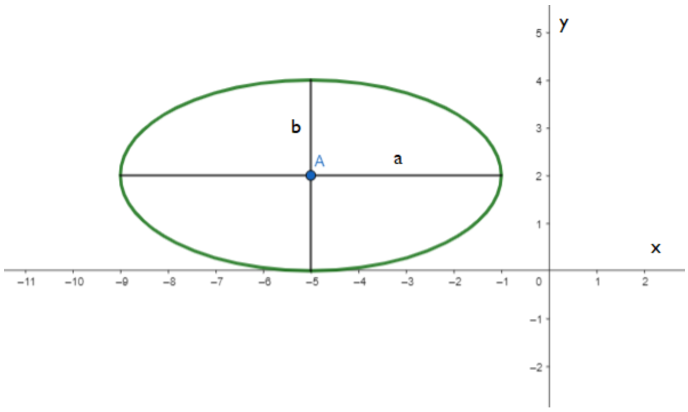
b) $(C_2): \ x^2 + 4y^2 + 10x - 16y + 25 = 0 \Longleftrightarrow$

$(C_2): \ x^2 + 10x + 25 + 4(y^2 - 4y + 4) - 16 = 0 \Longleftrightarrow$

$(C_2): \ (x+5)^2 + 4(y-2)^2 = 16 \Longleftrightarrow$

$(C_2): \ \dfrac{(x+5)^2}{16} + \dfrac{(y-2)^2}{4} = 1$

which is the equation of an ellipse centered at $A(-5, 2)$, the major axis on a line parallel to Ox, $a = 4$ and $b = 2$.

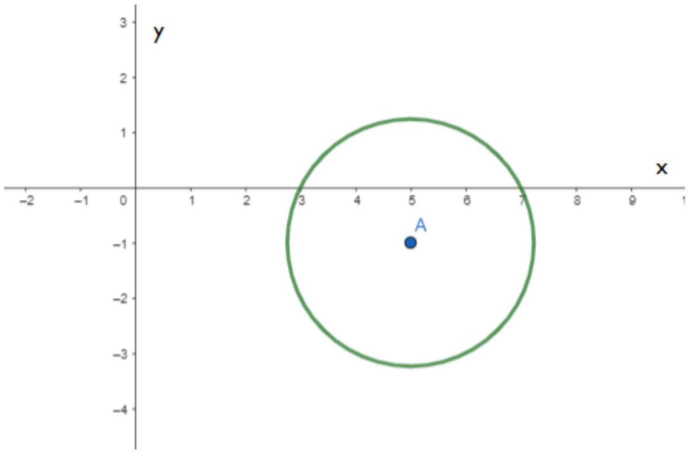


c) $(C_3): \ x^2 + y^2 - 10x + 2y + 21 = 0 \Longleftrightarrow$

$(C_3): \ x^2 - 10x + 25 + y^2 + 2y + 1 + 21 - 25 - 1 = 0 \Longleftrightarrow$

$(C_3): \ (x-5)^2 + (y+1)^2 = 5$

which is the equation of a circle centered at $A(5, -1)$ and having the radius $r = \sqrt{5}$.



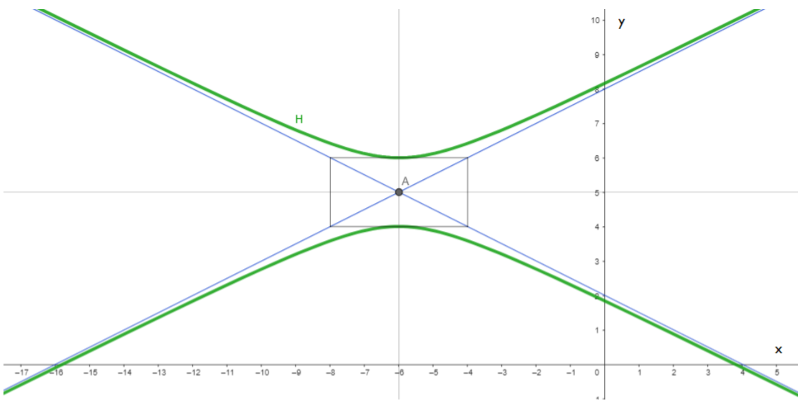
d) $(C_4): \ 4y^2 - x^2 - 40y - 12x + 60 = 0 \Longleftrightarrow$

$(C_4): \ 4(y^2 - 10y + 25) - (x^2 + 12x + 36) + 60 - 100 + 36 = 0 \Longleftrightarrow$

$(C_4): \ 4(y-5)^2 - (x+6)^2 = 4 \Longleftrightarrow$

$(C_4): \ (y-5)^2 - \dfrac{(x+6)^2}{4} = 1$

which is the equation of the hyperbola having the transverse axis a line passing through the center $A(-6,5)$ and parallel to Oy. The vertex are $V_1(-6,4)$ and $V_2(-6,6)$. $a = 1$ and $b = 2$, so the fundamental rectangle has the length of the edges 4 and 2 respectively and its diagonals are the asymptotes of the hyperbola.



Problem 6.6. Write the equation of the circle such that:

a) AB is the diameter of the circle, $A(-1,-2)$ and $B(5,-4)$.

b) The center of the circle is at $A(2,-3)$ and the line $d : 2x + 5y - 1 = 0$ is tangent to the circle.

Solution:

a) The center of the circle is the middle of the line segment $[AB]$ which is $M(2,-3)$ and the radius is $r = \dfrac{AB}{2} = \dfrac{\sqrt{6^2+2^2}}{2} = \sqrt{10}$.

The equation of the circle is:

$$(C) : (x-2)^2 + (y+3)^2 = 10.$$

b) The radius is perpendicular to the tangent, therefore $r = \text{dist}(A, d) = \dfrac{|4 + 5 \cdot (-3) - 1|}{\sqrt{4+25}} = \dfrac{12}{\sqrt{29}}$.

The equation of the circle is:

$$(C) : (x-2)^2 + (y+3)^2 = \frac{144}{29}.$$

Problem 6.7. Write the equation of the ellipse centered at $O(0,0)$ such that:

a) The distance between the foci is 6 and the eccentricity is $\varepsilon = \frac{3}{5}$.

b) The point $M(-2\sqrt{5}, 2)$ is on the ellipse, the major axis is on Ox and the length of the minor axis is 6.

Solution:

a) $2c = 6 \Longrightarrow c = 3$.

$\varepsilon = \frac{c}{a} = \frac{3}{5} \Longrightarrow a = \frac{3 \cdot 5}{3} = 5$.

$b = \sqrt{a^2 - c^2} = \sqrt{25 - 9} = 4$.

The equation of the ellipse is:

$$(E) : \frac{x^2}{25} + \frac{y^2}{16} = 1.$$

b) $2b = 6 \Longrightarrow b = 3 \Longrightarrow (E) : \frac{x^2}{a^2} + \frac{y^2}{9} = 1$.

$M(-2\sqrt{5}, 2) \in (E) \Longleftrightarrow \frac{20}{a^2} + \frac{4}{9} = 1 \Longleftrightarrow \frac{20}{a^2} = \frac{5}{9} \Longrightarrow a^2 = 36$.

The equation of the ellipse is:

$$(E) : \frac{x^2}{36} + \frac{y^2}{9} = 1.$$

Problem 6.8. Write the equation of the hyperbola centered at $O(0, 0)$ such that:

a) One vertex is at $V_1(0, -2)$ and the eccentricity is $\varepsilon = \frac{3}{2}$.

b) The point $M\left(\frac{9}{2}, -1\right)$ is on the hyperbola, the equations of the asymptotes are $y = \pm\frac{2}{3}x$, and the transverse axis is on Ox.

Solution:

a) If the vertex is $V_1(0, -2) \in Oy$ then the transverse axis is on Oy and $a = 2$.

$\varepsilon = \frac{c}{a} = \frac{3}{2} \Longrightarrow c = \frac{3 \cdot 2}{2} = 3$.

$b = \sqrt{c^2 - a^2} = \sqrt{9 - 4} = \sqrt{5}$.

The equation of the hyperbola is:

$$(E) : \frac{y^2}{4} - \frac{x^2}{5} = 1.$$

b) $y = \pm\frac{2}{3}x \Longrightarrow \frac{b}{a} = \frac{2}{3} \Longrightarrow a = 3k, b = 2k \Longrightarrow$

$(H) : \frac{x^2}{9k^2} - \frac{y^2}{4k^2} = 1.$

$M\left(\frac{9}{2}, -1\right) \in (H) \Longleftrightarrow \frac{81}{4 \cdot 9k^2} - \frac{1}{4k^2} = 1 \Longrightarrow k^2 = 2 \Longrightarrow a^2 = 18,\ b^2 = 8.$

The equation of the hyperbola is:

$$(H) : \frac{x^2}{18} - \frac{y^2}{8} = 1.$$

Problem 6.9. Write the equation of the parabola which is symmetrical about the Oy axis, the vertex is at the origin and passes through $A(9, 6)$.

Solution:

The general equation of the parabola is $(P) : x^2 = 4py$.

$A(9, 6) \in (P) \Longrightarrow 81 = 4 \cdot p \cdot 6 \Longrightarrow p = \frac{27}{8}.$

The equation of the parabola is:

$$(P) : x^2 = \frac{27}{2}y.$$

Problem 6.10. Write the equation of tangent to the parabola $y^2 = 2x$ perpendicular to the line $d : x - 2y + 4 = 0$.

Solution:

$tg \perp d \Longrightarrow m_{tg}m_d = -1 \Longrightarrow m_{tg} = \frac{-1}{\frac{1}{2}} = -2.$

The tangent line at the point $M(x_0, y_0)$ to the parabola is $tg : yy_0 = x + x_0 \Longrightarrow$ $m_{tg} = \frac{1}{y_0} \Longrightarrow y_0 = -\frac{1}{2}.$

$(x_0, y_0) \in (P) \Longrightarrow y_0^2 = 2x_0 \Longrightarrow x_0 = \frac{1}{8}.$

The equation of the tangent is

$tg : y(-\frac{1}{2}) = x + \frac{1}{8}$

$tg : 8x + 4y + 1 = 0.$

Problem 6.11. Determine the equations of the tangent planes to the sphere

$$(S): x^2 + y^2 + z^2 - 4x + 2y - 6z + 8 = 0$$

at the intersection points with the line

$$d: \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{2}.$$

Solution:

We write the equation of the sphere in the standard form.

$(S): x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 - 6z + 9 + 8 - 4 - 1 - 9 = 0 \Longleftrightarrow$

$(S): (x-2)^2 + (y+1)^2 + (z-3)^2 = 6.$

The tangent plane to the sphere at $M(x_0, y_0, z_0) \in (S)$ is

$(P_{tg}): (x-2)(x_0-2) + (y+1)(y_0+1) + (z-3)(z_0-3) = 6.$

Let $M = (S) \cap d$, $M \in d: \begin{cases} x = t+1 \\ y = -t \\ z = 2t+1. \end{cases}, t \in \mathbb{R} \Longrightarrow M(t+1, -t, 2t+1).$

$M \in (S) \Longrightarrow (t+1-2)^2 + (-t+1)^2 + (2t+1-3)^2 = 6 \Longleftrightarrow$

$6t^2 - 12t = 0 \Longrightarrow t(t-2) = 0 \Longrightarrow t_1 = 0,\ t_2 = 2.$

For $t = 0$ we have $M_1(1, 0, 1)$ and the equation of the tangent plane at M_1 is

$P_{tg}: (x-2)(1-2) + (y+1)(0+1) + (z-3)(1-3) = 6$

$P_{tg}: -x + y - 2z + 3 = 0.$

For $t = 2$ we have $M_2(3, -2, 5)$ and the equation of the tangent plane at M_2 is

$P_{tg}: (x-2)(3-2) + (y+1)(-2+1) + (z-3)(5-3) = 6$

$P_{tg}: x - y + 2z - 15 = 0.$

Problem 6.12. Sketch the appropriate traces then sketch and identify each of the following surfaces.

a) $(S_1) : 9x^2 - 4y^2 + 9z^2 = 0;$

b) $(S_2) : 9x^2 + 4y^2 + z^2 = 36;$

c) $(S_3) : 4x^2 + 4y^2 - z^2 + 4 = 0;$

d) $(S_4) : x^2 + 4y^2 = 4;$

e) $(S_5) : z = x^2 + 4y^2 - 4;$

f) $(S_6) : 9x^2 - 4y^2 + 9z^2 = 36;$

g) $(S_7) : x^2 - y^2 + 4z - 4 = 0.$

Solution:

a) $(S_1) :\ 9x^2 - 4y^2 + 9z^2 = 0$. If we divide the equation by 36 we get $(S_1) : \dfrac{x^2}{4} - \dfrac{y^2}{9} + \dfrac{z^2}{4} = 0$, which is an elliptical cone along Oy axis.

xy trace: $z = 0 \Longrightarrow \dfrac{x^2}{4} - \dfrac{y^2}{9} = 0 \Longleftrightarrow 9x^2 = 4y^2 \Longleftrightarrow y = \pm\dfrac{3}{2}x$ which are two lines.

xz trace: $y = 0 \Longrightarrow x = z = 0 \Longrightarrow O(0, 0, 0)$.

yz trace: $x = 0 \Longrightarrow -\dfrac{y^2}{9} + \dfrac{z^2}{4} = 0 \Longleftrightarrow 9z^2 = 4y^2 \Longleftrightarrow y = \pm\dfrac{3}{2}z$ which are two lines in yz plane.

If $x = \pm 2 \Longrightarrow -\dfrac{y^2}{9} + \dfrac{z^2}{4} = -1 \Longleftrightarrow \dfrac{y^2}{9} - \dfrac{z^2}{4} = 1$ which is an hyperbola having the transverse axis on Oy.

If $y = \pm 3 \Longrightarrow \dfrac{x^2}{4} + \dfrac{z^2}{4} = 1 \Longleftrightarrow x^2 + z^2 = 4$ which is a circle with the radius 2 in the planes $y = 3$ and $y = -3$.

If $z = \pm 2 \Longrightarrow -\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ which is a hyperbola having the transverse axis on Oy.

The traces are represented in the next figures.

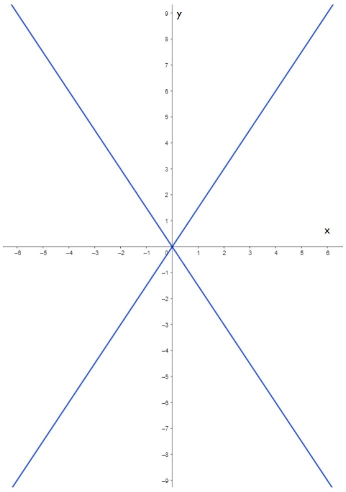


Figure 6.21: xy trace

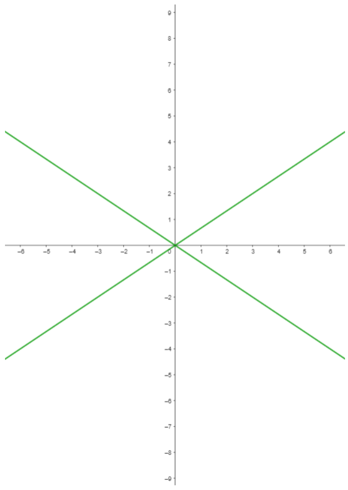


Figure 6.22: yz trace

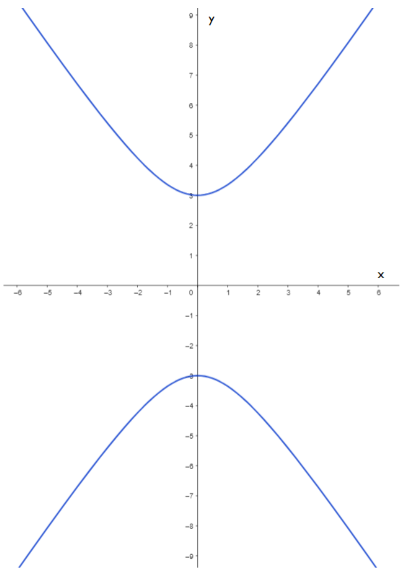


Figure 6.23: $z = \pm 2$ trace

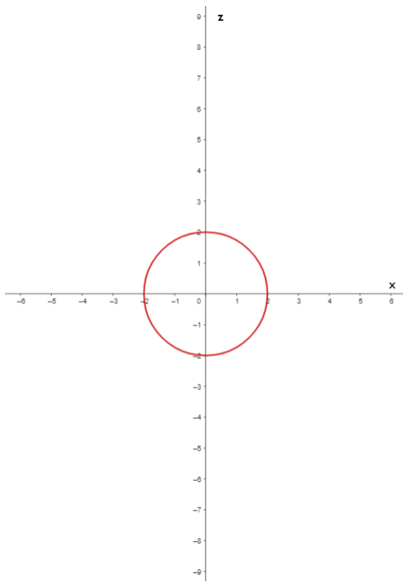


Figure 6.24: $y = \pm 3$ trace

The sketch of the elliptical cone is as the next figure shows.

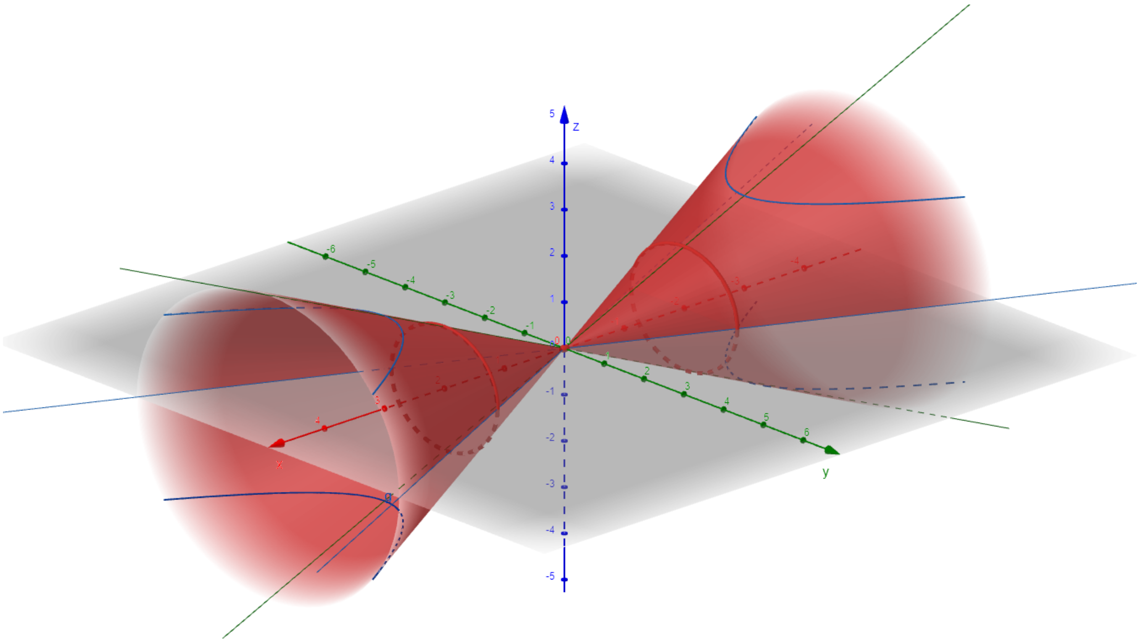


Figure 6.25: (S_1) - elliptical cone

b) $(S_2): 9x^2 + 4y^2 + z^2 = 36$, so we divide by 36 the equation and we get

$(S_2): \dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{36} = 1$, which is an ellipsoid.

xy trace: $z = 0 \Longrightarrow \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ an ellipse having the major axis on Oy.

xz trace: $y = 0 \Longrightarrow \dfrac{x^2}{4} + \dfrac{z^2}{36} = 1$ an ellipse having the major axis on Oz.

yz trace: $x = 0 \Longrightarrow \dfrac{y^2}{9} + \dfrac{z^2}{36} = 1$ an ellipse having the major axis on Oz.

The traces are represented in the next figures.

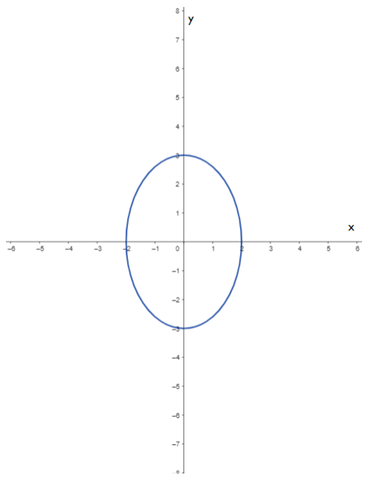


Figure 6.26: xy trace

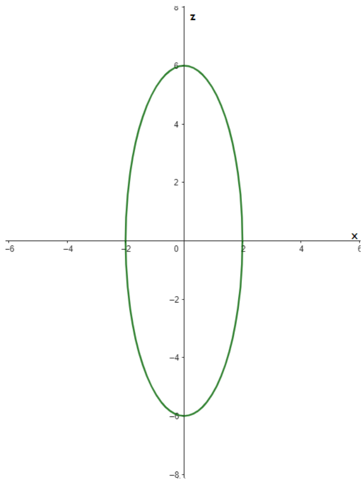


Figure 6.27: xz trace

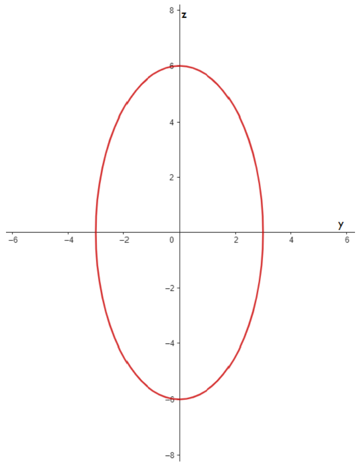


Figure 6.28: yz trace

The sketch of the ellipsoid is as the next figure shows.

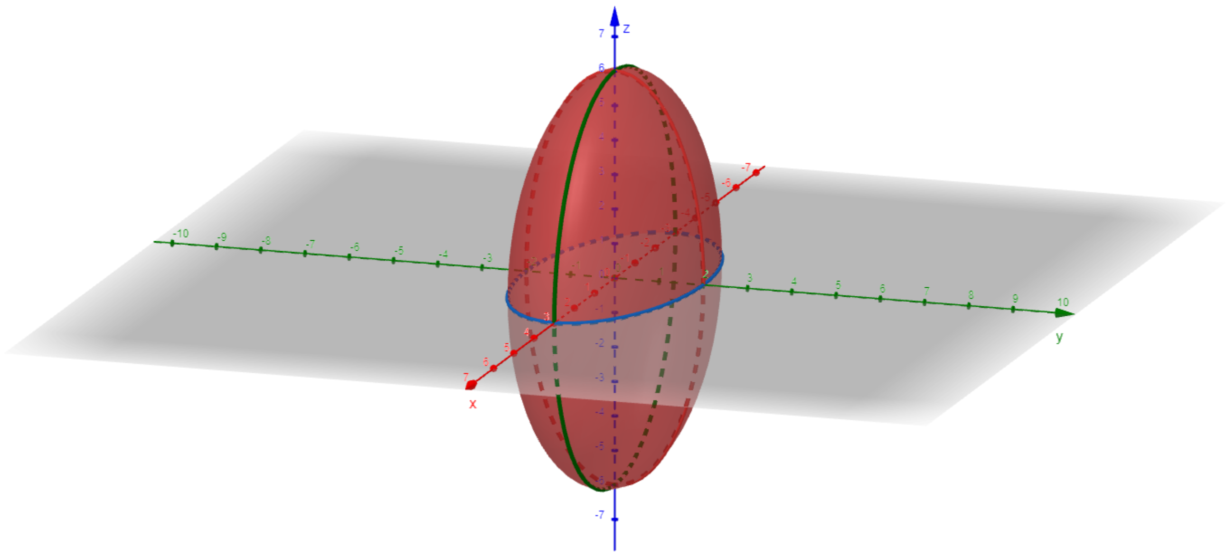


Figure 6.29: (S_2) - ellipsoid

c) $(S_3) : 4x^2 + 4y^2 - z^2 + 4 = 0$. If we divide by -4 the equation we get

$(S_3) : -x^2 - y^2 + \dfrac{z^2}{4} = 1$, which is a two sheets hyperboloid along Oz axis.

xy trace: $z = 0 \Longrightarrow -x^2 - y^2 = 1$ is $\varnothing$.

If $z = \pm 2 \Longrightarrow x^2 + y^2 = 0 \Longrightarrow x = y = 0 \Longrightarrow O(0, 0)$.

If $z = \pm 4 \Longrightarrow x^2 + y^2 = 3 \Longrightarrow$ a circle having the radius $\sqrt{3}$.

If $z = \pm 6 \Longrightarrow x^2 + y^2 = 8 \Longrightarrow$ a circle having the radius $\sqrt{8}$.

xz trace: $y = 0 \Longrightarrow -x^2 + \dfrac{z^2}{4} = 1$ an hyperbola having the transverse axis on Oz, and the vertex $V_{1,2}(0, 0, \pm 2)$.

yz trace: $x = 0 \Longrightarrow -y^2 + \dfrac{z^2}{4} = 1$ an hyperbola having the transverse axis on Oz, and the vertex $V_{1,2}(0, 0, \pm 2)$.

The traces are represented in the next figures.

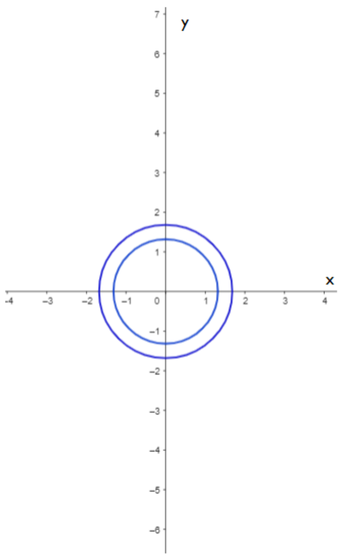


Figure 6.30: $z = \pm 4$ and $z = \pm 6$ traces

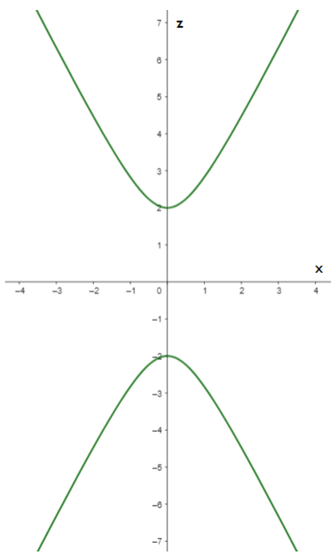


Figure 6.31: xz trace

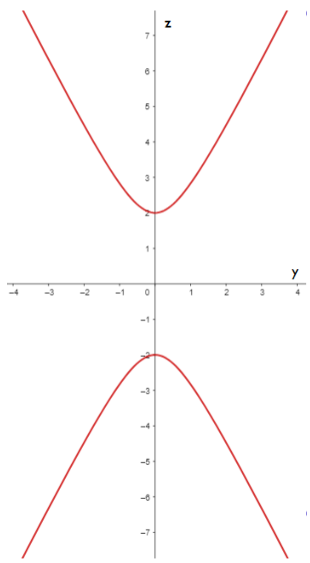


Figure 6.32: yz trace

The sketch of the two sheet hyperboloid is as the next figure shows.

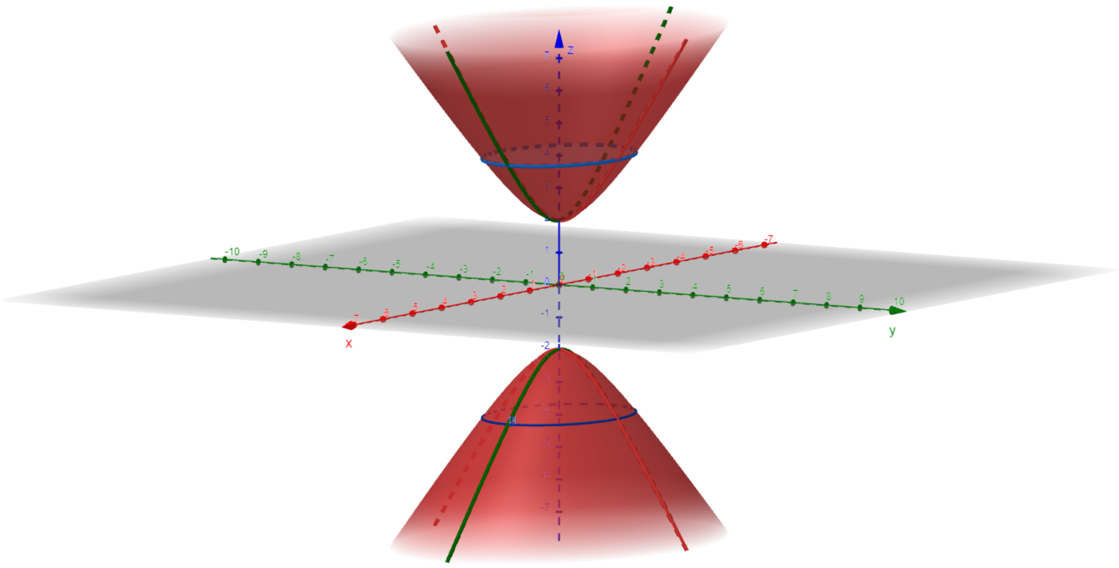


Figure 6.33: (S_3) - two sheet hyperboloid

d) $(S_4) : x^2 + 4y^2 = 4$ is a cylinder along Oz axis. The trace in xOy plane is $\frac{x^2}{4} + y^2 = 1$ an ellipse having the major axis on Ox. The trace is represented in the next figures.

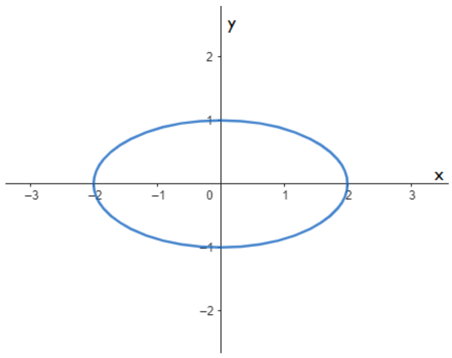


Figure 6.34: xy trace

The sketch of the cylinder is as the next figure shows.

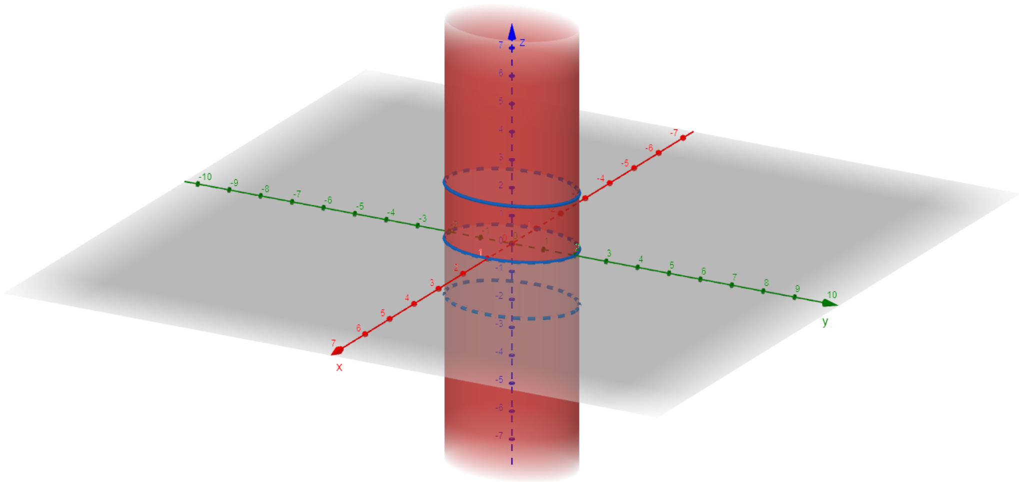


Figure 6.35: (S_4) - cylinder

e) $(S_5) : z = x^2 + 4y^2 - 4 \Longleftrightarrow z + 4 = x^2 + 4y^2$ is an elliptic paraboloid along Oz axis.

xy trace: $z = 0 \Longrightarrow x^2 + 4y^2 - 4 = 0 \Longleftrightarrow \dfrac{x^2}{4} + y^2 = 1$ which is an ellipse with the major axis on Ox.

If $z = -4 \Longrightarrow x^2 + 4y^2 = 0 \Longrightarrow x = y = 0 \Longrightarrow V(0, 0, -4)$ is the vertex of the elliptical paraboloid.

If $z < -4 \Longrightarrow x^2 + 4y^2 < 0 \Longrightarrow \varnothing$.

If $z > -4 \Longrightarrow x^2 + 4y^2 = z + 4$ which is the equation of an ellipse.

xz trace: $y = 0 \Longrightarrow z + 4 = x^2$ a parabola along the positive direction of Oz axis and the vertex at $V(0, 0, -4)$.

yz trace: $x = 0 \Longrightarrow z + 4 = 4y^2$ a parabola along the positive direction of Oz axis and the vertex at $V(0, 0, -4)$.

The traces are represented in the next figures.

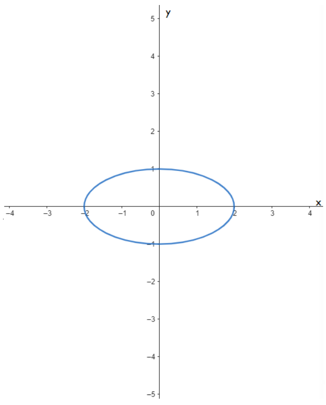


Figure 6.36: xy trace

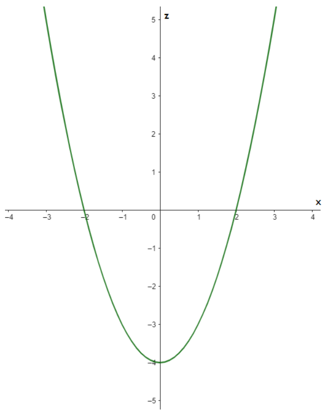


Figure 6.37: xz trace

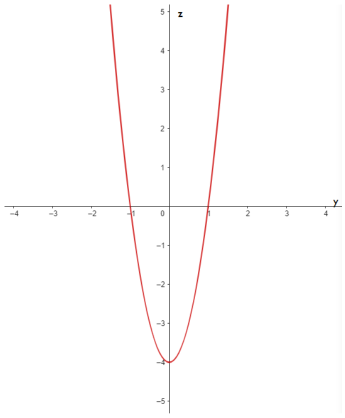


Figure 6.38: yz trace

The sketch of the elliptic paraboloid is as the next figure shows.

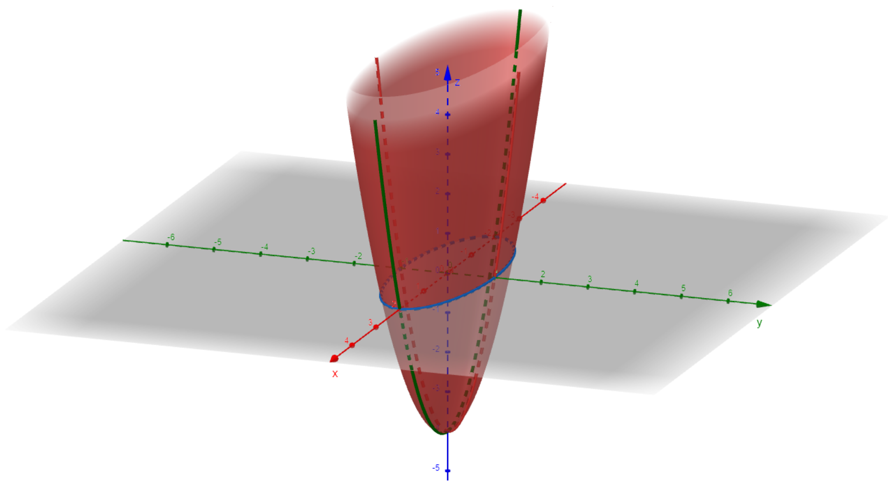


Figure 6.39: (S_5) - elliptic paraboloid

f) $(S_6): 9x^2 - 4y^2 + 9z^2 = 36$. We divide by 36 and we get:

$(S_6): \dfrac{x^2}{4} - \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$ which is the equation of a one sheet hyperboloid along Oy axis.

xy trace: $z = 0 \Longrightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1$ which is a hyperbola having the transverse axis on Ox and the vertex at $V_{1,2}(\pm 2, 0, 0)$.

xz trace: $y = 0 \Longrightarrow \frac{x^2}{4} + \frac{z^2}{4} = 1$ a circle with the radius $r = 2$.

If $y = \pm 6 \Longrightarrow \frac{x^2}{4} + \frac{z^2}{4} = 5$ a circle with the radius $r = 2\sqrt{5}$.

yz trace: $x = 0 \Longrightarrow -\frac{y^2}{9} + \frac{z^2}{4} = 1$ which is a hyperbola having the transverse axis on Oz and the vertex at $W_{1,2}(0, 0, \pm 2)$.

The traces are represented in the next figures.

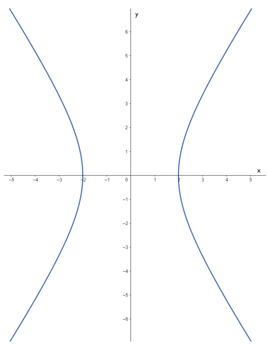


Figure 6.40: xy trace

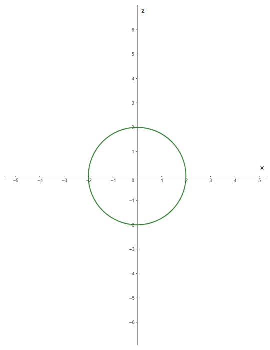


Figure 6.41: xz trace

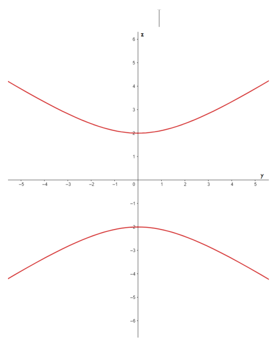


Figure 6.42: yz trace

The sketch of the one sheet hyperboloid is as the next figure shows.

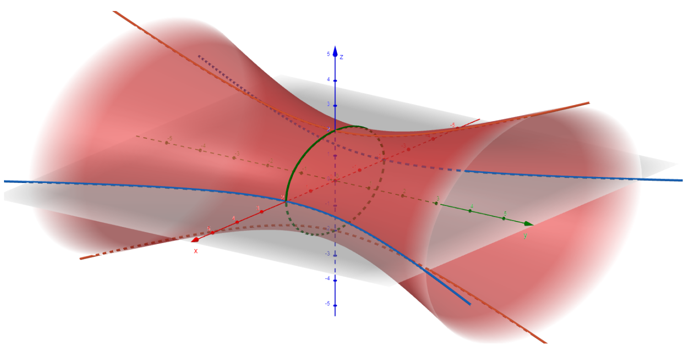


Figure 6.43: (S_6) - one sheet hyperboloid

g) $(S_7): x^2 - y^2 + 4z - 4 = 0$. We divide the equation by 4 and we obtain:

$(S_7): z - 1 = \dfrac{y^2}{4} - \dfrac{x^2}{4}$ which is the equation of a hyperbolic paraboloid.

xy trace: $z = 0 \Longrightarrow \dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$ which is a hyperbola having the transverse axis on Ox and the vertex at $V_{1,2}(\pm 2, 0, 0)$.

xz trace: $y = 0 \Longrightarrow x^2 = 4z - 4$ a parabola along the positive direction of Oz and the vertex $W(0, 0, 1)$.

yz trace: $x = 0 \Longrightarrow y^2 = -4z + 4$ a parabola along the negative direction of Oz axis and the vertex $W(0, 0, 1)$.

The traces are represented in the next figures.

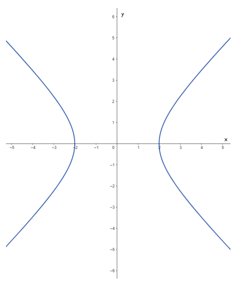


Figure 6.44: xy trace

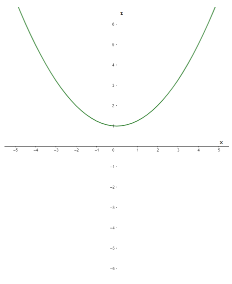


Figure 6.45: xz trace

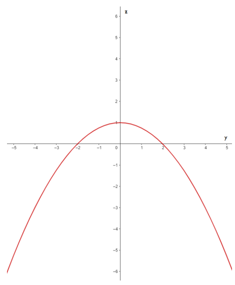


Figure 6.46: yz trace

The sketch of the hyperbolic paraboloid is as the next figure shows.

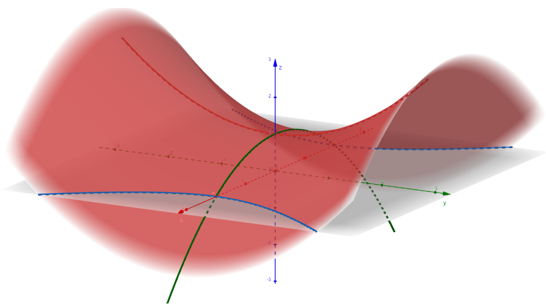


Figure 6.47: (S_7) - hyperbolic paraboloid

6.5 Problems

Problem 6.13. Write the equation of the circle such that :

a) The center is at the origin and the radius is 3.

b) $A(2,-1)$ is the center and the radius is 4.

c) The center is at $B(2,5)$ and passes through $C(-1,2)$.

d) The center is at $D(1,-1)$ and the line $d : 5x - 12y + 9 = 0$ is tangent to the circle.

Problem 6.14. Write the equation of the circle having the diameter AB, where $A(2,3)$, $B(4,-1)$.

Problem 6.15. Determine the radius and the center of the circle

$$(\mathcal{C}) : x^2 + y^2 - 4x + 6y + 5 = 0.$$

Write the equation of the tangent to the circle at $A(1,-2)$.

Problem 6.16. Write the equation of the ellipse with the center at the origin such that:

a) the length of the axes are 5 and 2 and is horizontally oriented;

b) the length of the major axis is 10, the focal distance is $2c = 8$ and is horizontally oriented;

c) the major axis is 24, the focal distance is $2c = 10$ and is vertically oriented;

d) $2c = 6$, the eccentricity is $\epsilon = \dfrac{3}{5}$ and is horizontally oriented;

e) the length of the minor axis is 20, the eccentricity is $\epsilon = \dfrac{12}{13}$ and is vertically oriented.

Problem 6.17. Write the equation of the ellipse (E) with the center at the origin if:

a) $M(-2\sqrt{5}, 2) \in (E)$, $b = 3$ and is horizontally oriented;

b) $M(2, -2) \in (E)$, $a = 4$ and is horizontally oriented;

c) the ellipse (E) passes through $M_1(4, -\sqrt{3})$ and $M_2(2\sqrt{2}, 3)$;

d) $M(\sqrt{15}, -1) \in (E)$, $2c = 8$ and is horizontally oriented;

e) $M(2, -\frac{5}{3}) \in (E)$, the eccentricity is $\epsilon = \dfrac{2}{3}$ and is horizontally oriented.

Problem 6.18. Determine the major axis, the minor axis, the vertices, the foci, the eccentricity for

a) $(E_1) : \dfrac{x^2}{16} + \dfrac{y^2}{9} - 1 = 0$;

b) $(E_2) : x^2 + 16y^2 - 16 = 0$;

c) $(E_3) : x^2 + 4y^2 - 1 = 0$;

then graph them.

Problem 6.19. Determine the relative position of the ellipse $(E) : 2x^2 + 5y^2 - 88 = 0$ and the line $d : 3x - 5y + 14 = 0$

Problem 6.20. Write the tangent lines to the ellipse $(E) : \dfrac{x^2}{10} + \dfrac{2y^2}{5} - 1 = 0$ parallel to the line $3x + 2y + 7 = 0$.

Problem 6.21. Write the equation of the tangent line at $A(2, 0)$ to the ellipse $\dfrac{x^2}{4} + y^2 - 1 = 0$.

Problem 6.22. Write the equation of the hyperbola with the center at the origin and:

a) $a = 5$, $b = 4$ and the transverse axis is on Ox;

b) $c = 5$, $a = 4$ and the transverse axis is on Oy;

c) $c = 3$, eccentricity is $\epsilon = \dfrac{3}{2}$ and the transverse axis is on Ox;

d) $a = 8$ eccentricity is $\epsilon = \dfrac{5}{4}$ and the transverse axis is on Ox;

e) the asymptotes are $y = \pm\dfrac{4}{3}x$, $c = 13$ and the transverse axis is on Ox.

Problem 6.23. Write the equation of the hyperbola (H) having the center at the origin and:

a) $M(10, -\sqrt{5}) \in (H)$, $a = \sqrt{20}$, and the transverse axis is on Oy;

b) the hyperbola passes through $M_1(5, \frac{15}{4})$ and $M_2(-4\sqrt{2}, 5)$;

c) $M(4, -\frac{4\sqrt{7}}{3}) \in (H)$, $c = 5$ and the transverse axis is on Ox;

d) $M(\frac{9}{2}, -1) \in (H)$, the equations of the asymptotes are $y = \pm\dfrac{2}{3}x$, and the transverse axis is on Ox.

Problem 6.24. Determine the transverse axis, the conjugate axis, the vertices, the foci, the eccentricity and the equations of the asymptotes for each of the hyperbolas

a) $(H_1) : \dfrac{x^2}{9} - \dfrac{y^2}{4} - 1 = 0$,

b) $(H_2) : x^2 - y^2 - 1 = 0$,

c) $(H_3) : y^2 - 4x^2 - 16 = 0$,

d) $(H_4) : 16x^2 - 9y^2 - 1 = 0$,

then graph them.

Problem 6.25. Determine the relative position of the line $d : x - y - 4 = 0$ and the hyperbola $(H) : \dfrac{x^2}{12} - \dfrac{y^2}{3} - 1 = 0$.

Problem 6.26. Write the equation of the tangent line of $(H) : \dfrac{x^2}{4} - y^2 - 1 = 0$ at the point $A(-6, 2\sqrt{2})$.

Problem 6.27. Write the equation of the parabola such that

a) the parabola opens to the right, the vertex is the origin $O(0,0)$ and the foci is $F(2,0)$;

b) the parabola opens to the left, the vertex is $O(0,0)$ and the directrix is $x = 5$;

c) the parabola opens down, the vertex is $A(2,-3)$ and the foci is $F(2,-6)$;

d) the parabola opens up, the foci is $F(-2,5)$ and the directrix is $y = 1$.

Problem 6.28. Determine the vertex, the foci, the directrix for the parabolas

a) $(P_1) : (y+1)^2 = 6(x-2)$,

b) $(P_2) : x^2 = 4(y-2)$,

c) $(P_3) : x^2 = -y$,

d) $(P_4) : (y-2)^2 = -4x$,

e) $(P_5) : y^2 - 6y - 8x + 1 = 0$,

f) $(P_6) : x^2 + 8x + 4y + 20 = 0$,

then graph each one of them.

Problem 6.29. Determine the equation of the parabola if:

a) is symmetric about Ox, passes through $A(9, 6)$ and the vertex is at the origin;

b) is symmetric about Oy, passes through $B(1, 1)$ and the vertex is at the origin.

Problem 6.30. Determine the relative position of the line and the parabola if:

a) $d : x - y + 2 = 0,\ (P) : y^2 = 8x;$

b) $d : 8x + 3y - 15 = 0,\ (P) : x^2 = -3x;$

c) $d : 5x - y - 15 = 0,\ (P) : y^2 = -5x.$

Problem 6.31. Determine the center and the radius of the circle at the intersection of the sphere $(S) : x^2+y^2-4x-2y-6z+1 = 0$ and the plane $(P) : x+2y-z-3 = 0$.

Problem 6.32. Determine the intersection of the line $d : x - 3 = y - 1 = \dfrac{z-6}{3}$ with the hyperboloid $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{9} - 1 = 0$.

Problem 6.33. Determine the center and the radius of the sphere

$$(S) :\ x^2 + y^2 + z^2 - x + 3y - 4z - 1 = 0.$$

Problem 6.34. Let $(S) : x^2 + y^2 + z^2 = 1$ be the sphere and $M(1, 0, 0)$ be a point that belongs to (S). Determine a point $N \in (P)$, $(P) : z = 5$ such that MN is tangent to the sphere (S).

Problem 6.35. Determine the intersection of the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{3} + \dfrac{z^2}{9} - 1 = 0$ with the line $x = y = z$. Write the equation of the tangent plane to the ellipsoid at these points.

Problem 6.36. Determine the intersection of the hyperboloid $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} - 1 = 0$ with the coordinates planes and identify the conics.

Problem 6.37. Determine the equations of the tangent planes to the surface of

a) the elliptic paraboloid $\frac{x^2}{5} + \frac{y^2}{3} = z$;

b) the hyperbolic paraboloid $x^2 - \frac{y^2}{4} = 3z$;

which are parallel to the plane $(P) : x - 3y + 2z - 1 = 0$.

Problem 6.38. Determine the equation of the sphere having the center at the point $C(3, -5, -2)$ and is tangent to the plane $(P) : 2x - y - 3z + 11 = 0$.

Problem 6.39. Sketch the appropriate traces then sketch and identify the surface

a) $2x^2 + y^2 - 3z^2 = 0$;

b) $x^2 + y^2 + 4z^2 + 4y = 0$;

c) $-x^2 + y^2 - 2z^2 = 1$;

d) $2x^2 + 3y^2 - z^2 = 1$;

e) $x^2 + 2y^2 = 1 - z$;

f) $2x^2 - y^2 + 3z^2 = 1$.

7

Generation of surfaces

7.1 Cylindrical surfaces

The surface formed by the motion of a line called *the generator* of the surface moving parallel to itself and intersecting a given fixed curve called *the directrix* of the surface is called **cylindrical surface**.

Let d be the generator line $d : \begin{cases} (P_1) : a_1x + b_1y + c_1z + d_1 = 0 \\ (P_2) : a_2x + b_2y + c_2z + d_2 = 0 \end{cases}$ and the directrix $(\Gamma) : \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0. \end{cases}$

A straight line parallel to d is $g : \begin{cases} (P_1) = \alpha \\ (P_2) = \beta \end{cases}$, $\alpha, \beta \in \mathbb{R}$.

The line g intersects the directrix if $\varphi(\alpha, \beta) = 0$.

The equation of the cylindrical surface is

$$(S) : \varphi((P_1), (P_2)) = 0.$$

7.2 Conical surfaces

A surface which is the union of all lines that pass through a fixed point called the *vertex* or *apex* and intersect a fixed curve called the *directrix* that does not contain the apex, is a **conical surface**. Each of the lines is called a *generator* for the conical surface.

Let V be the vertex at the intersection of the planes

$$\begin{cases} (P_1): a_1x + b_1y + c_1z + d_1 = 0 \\ (P_2): a_2x + b_2y + c_2z + d_2 = 0 \\ (P_3): a_3x + b_3y + c_3z + d_3 = 0. \end{cases}$$

The generators are $g: \begin{cases} (P_1) = \alpha(P_2) \\ (P_2) = \beta(P_3) \end{cases}$.

The compatibility condition is $\varphi(\alpha, \beta) = 0$, so the equation of the conical surface is

$$(S): \varphi\left(\frac{(P_1)}{(P_2)}, \frac{(P_2)}{(P_3)}\right) = 0.$$

Remark 7.1. *If the vertex is given by its coordinates* $V(x_0, y_0, z_0)$ *and the directrix is* $(\Gamma): \begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$ *then, the equations of the generator are* $g: \dfrac{x - x_0}{\alpha} = \dfrac{y - y_0}{\beta} = \dfrac{z - z_0}{\gamma}$, $\alpha, \beta, \gamma \in \mathbb{R}$.

The generator intersect the curve (Γ) *if* $\varphi(\alpha, \beta, \gamma) = 0$, *hence the equation of the surface is*

$$(S): \varphi(x - x_0, y - y_0, z - z_0) = 0.$$

7.3 Conoid

A **conoid** is a ruled surface, whose rulings fulfill the conditions:

- All rulings are parallel to a plane, called the *directrix* plane.
- All rulings intersect a fixed line, the axis.
- The rulings intersect a curve.

The conoid is a right conoid if its axis is perpendicular to its directrix plane. Therefore all rulings are perpendicular to the axis.

Let $(P) : ax + by + cz + d = 0$, be the directrix plane and the axis
$d : \begin{cases} (P_1) : a_1x + b_1y + c_1z + d_1 = 0 \\ (P_2) : a_2x + b_2y + c_2z + d_2 = 0. \end{cases}$
The generators are of equations:

$$g : \begin{cases} (P) = \alpha \\ (P_1) = \beta(P_2). \end{cases}$$

The compatibility condition is $\varphi(\alpha, \beta) = 0$, and the equation of the conoid surface is

$$(S) : \varphi\left((P), \frac{(P_1)}{(P_2)}\right) = 0.$$

7.4 Surfaces of revolution

A **surface of revolution** is a surface created by rotating a curve called *the generatrix* around an axis of rotation (*axis of revolution*).

- Sections of a surface of revolution perpendicular to this axis are circles, called parallel circles or simply parallels.

- Sections containing the axis are meridian sections, or simply meridians.

Let $(\Gamma) : \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$ be the generatrix, and the axis of revolution the line $d : \dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$.

A surface of revolution can also be generated by a circle $\mathcal{C}$ moving always perpendicular to a fixed line d with its center on the fixed line and expanding or contracting so as to continually pass through a curve (Γ) which always lies in a plane with the straight line.

$$\mathcal{C} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \alpha^2 \\ lx + my + nz = \beta. \end{cases}$$

The circle $\mathcal{C}$ intersect the curve (Γ) if $\varphi(\alpha^2, \beta) = 0$.

The equation of the surface of revolution is

$$(S) : \varphi((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, lx + my + nz) = 0.$$

7.5 Solved problems

Problem 7.1. Determine the equation of the cylindrical surface having as generators lines parallel to the direction $\overrightarrow{v} = 5\overrightarrow{i} + 3\overrightarrow{j} + 2\overrightarrow{k}$ and as directrix the curve $(\Gamma) : \begin{cases} x^2 + y^2 - 2 = 0 \\ z = 0. \end{cases}$

Solution:

Let us consider the line having as director vector $\overrightarrow{v}$:

$$d : \frac{x}{5} = \frac{y}{3} = \frac{z}{2} \Longleftrightarrow \begin{cases} 2x - 5z = \lambda \\ 2y - 3z = \mu \end{cases}.$$

$$(\Gamma)\cap d \Longleftrightarrow \begin{cases} 2x-5z=\lambda \\ 2y-3z=\mu \\ x^2+y^2-2=0 \\ z=0 \end{cases} \Longleftrightarrow \begin{cases} x=\dfrac{\lambda}{2} \\ y=\dfrac{\mu}{2} \\ z=0 \\ \dfrac{\lambda^2}{4}+\dfrac{\mu^2}{4}-2=0 \end{cases} \Longleftrightarrow \lambda^2+\mu^2-8=0.$$

The equation of the cylindrical surface is:

$$(2x-5z)^2+(2y-3z)^2-8=0.$$

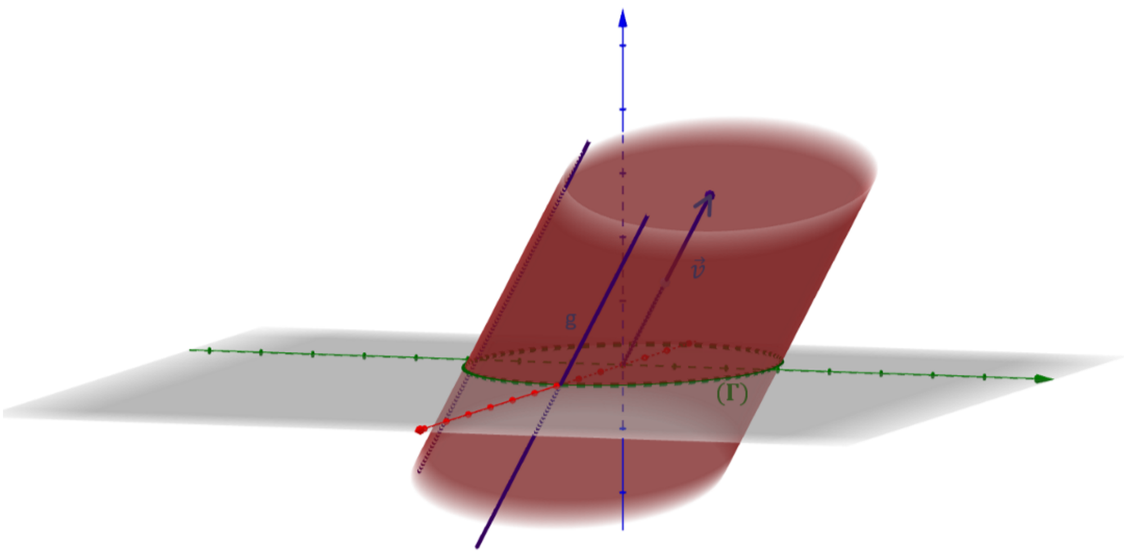


Figure 7.1: Cylindrical surface

Problem 7.2. Write the equation of the conical surface having the apex at $(1,1,1)$ and its directrix the curve $(\Gamma):\begin{cases} y^2+z^2-1=0 \\ x+y+z=0. \end{cases}$

Solution:

The vertex is at $V(1,1,1) \Longrightarrow \begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases} \Longleftrightarrow g : \begin{cases} x - 1 = \lambda(z-1) \\ y - 1 = \mu(z-1) \end{cases}$.

$$g \cap (\Gamma) \Longleftrightarrow \begin{cases} x - 1 = \lambda(z-1) \\ y - 1 = \mu(z-1) \\ y^2 + z^2 - 1 = 0 \\ x + y + z = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 1 + \lambda(z-1) \\ y = 1 + \mu(z-1) \\ x + y - 2 = (z-1)(\lambda+\mu) \\ x + y = -z \\ y^2 + z^2 - 1 = 0 \end{cases} \Longleftrightarrow$$

$$\begin{cases} x = 1 + \lambda(z-1) \\ y = 1 + \mu(z-1) \\ (z-1)(\lambda+\mu+1) = -3 \\ y^2 + z^2 - 1 = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 1 + \lambda(z-1) \\ y = 1 + \mu(z-1) \\ z = 1 - \dfrac{3}{\lambda+\mu+1} \\ y^2 + z^2 - 1 = 0 \end{cases} \Longleftrightarrow$$

$$\begin{cases} y = 1 + \mu(1 - \dfrac{3}{\lambda+\mu+1} - 1) \\ z = 1 - \dfrac{3}{\lambda+\mu+1} \\ y^2 + z^2 - 1 = 0 \end{cases} \Longleftrightarrow \begin{cases} y = 1 - \mu\dfrac{3}{\lambda+\mu+1} \\ z = 1 - \dfrac{3}{\lambda+\mu+1} \\ y^2 + z^2 - 1 = 0 \end{cases} \Longrightarrow$$

$$(1 - \mu\frac{3}{\lambda+\mu+1})^2 + (1 - \frac{3}{\lambda+\mu+1})^2 - 1 = 0.$$

The last condition can be rewritten as:

$$\left(\frac{\lambda+\mu+1-3\mu}{\lambda+\mu+1}\right)^2 + \left(\frac{\lambda+\mu+1-3}{\lambda+\mu+1}\right)^2 = 1 \Longleftrightarrow$$

$$(\lambda - 2\mu + 1)^2 + (\lambda + \mu - 2)^2 = (\lambda + \mu + 1)^2 \Longleftrightarrow$$

$$\lambda^2 + 4\mu^2 - 4\lambda\mu - 4\lambda - 10\mu + 4 = 0 \Longleftrightarrow$$

$$(\lambda - 2\mu)^2 - 4\lambda - 10\mu + 4 = 0.$$

We know that $\begin{cases} \lambda = \dfrac{x-1}{z-1} \\ \mu = \dfrac{y-1}{z-1} \end{cases}$, therefore the equation of the conic surface is:

$$(S) : \left(\frac{x-1}{z-1} - 2\frac{y-1}{z-1}\right)^2 - 4\frac{x-1}{z-1} - 10\frac{y-1}{z-1} + 4 = 0 \Longleftrightarrow$$

$$(S) : (x-2y+1)^2 - 4(x-1)(z-1) - 10(y-1)(z-1) + 4(z-1)^2 = 0$$

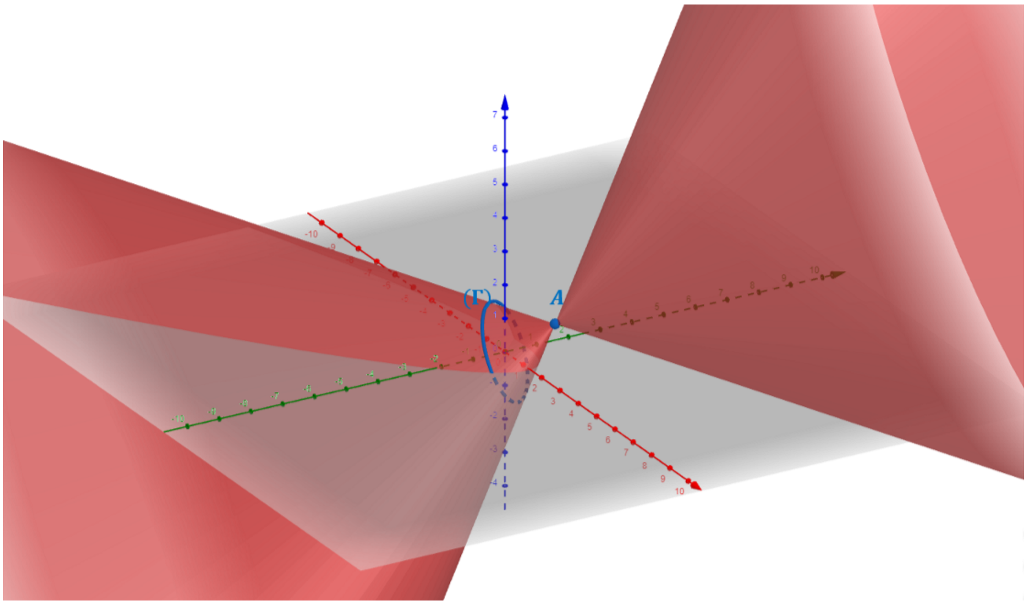


Figure 7.2: Conical surface

Problem 7.3. Write the equation of the surface generated by a line passing through $A(1,0,0)$ and the distance between the line and $B(1,2,3)$ is 2.

Solution:

We have a conical surface having the vertex at $A(1,0,0)$. If the distance is constant, then the lines are tangent to a sphere centered at B and having the radius 2, the generators of the surface are the tangent lines to the sphere.

The equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 4$.

The point $A(1,0,0)$ can be written as the intersection of the planes

$$A: \begin{cases} x = 1 \\ y = 0 \\ z = 0 \end{cases} \Longrightarrow g: \begin{cases} x - 1 = \lambda z \\ y = \mu z. \end{cases}$$

$$g \cap (\Gamma) \Longleftrightarrow \begin{cases} x - 1 = \lambda z \\ y = \mu z \\ (x-1)^2 + (y-2)^2 + (z-3)^2 = 4 \end{cases} \Longleftrightarrow$$

$\lambda^2 z^2 + (\mu z - 2)^2 + (z-3)^2 = 4 \Longleftrightarrow z^2(\lambda^2 + \mu^2 + 1) + z(-4\mu - 6) + 9 = 0.$

g are tangent to the sphere if the equation has a unique solution which is equivalent to $\Delta = 16\mu^2 + 48\mu + 36 - 36(\lambda^2 + \mu^2 + 1) = 0 \Longleftrightarrow 9\lambda^2 + 5\mu^2 - 12\mu = 0.$

The equation of the surface is:

$(S): 9\left(\frac{x-1}{z}\right)^2 + 5\frac{y^2}{z^2} - 12\frac{y}{z} = 0 \Longleftrightarrow$

$(S): 9(x-1)^2 + 5y^2 - 12yz = 0.$

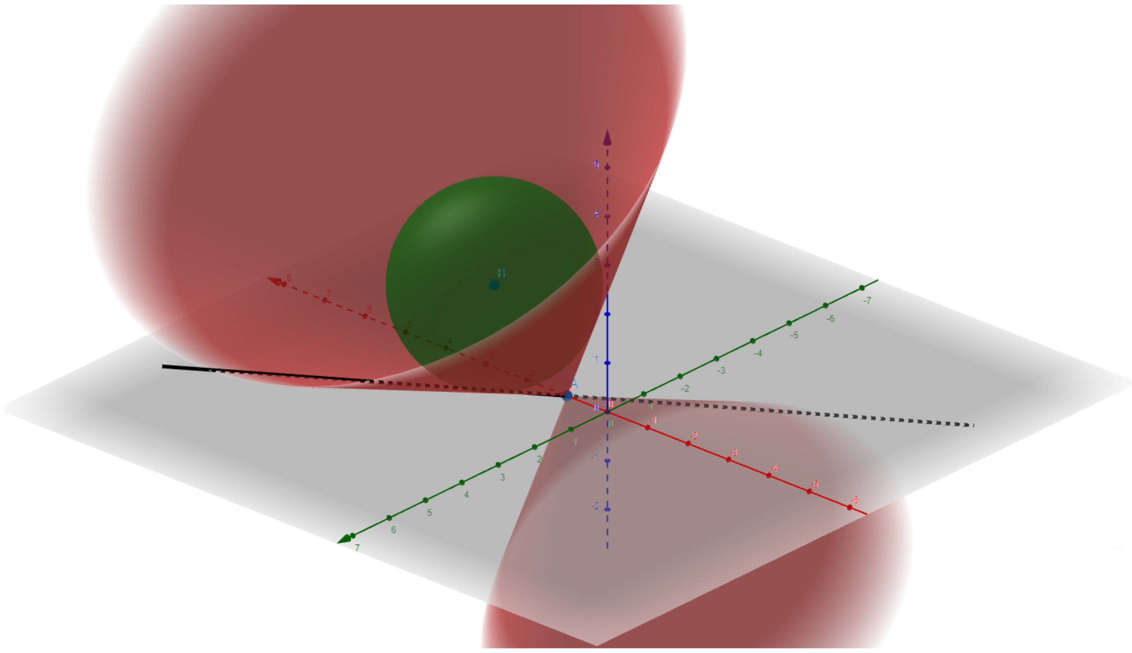


Figure 7.3: Conical surface

Problem 7.4. Write the equation of the conoid generate by a line which is parallel to the plane xOy, intersects the line $d : \begin{cases} x = 2 \\ y = 0 \end{cases}$ and the hyperbola

$$(\Gamma) : \begin{cases} \dfrac{x^2}{4} - \dfrac{z^2}{9} = 1 \\ y = 2 \end{cases}.$$

Solution: The equation of xOy plane is $z = 0$.

The generators are of equations $g : \begin{cases} z = \lambda \\ y = \mu(x-2) \end{cases}$.

$$g \cap (\Gamma) \iff \begin{cases} z = \lambda \\ y = \mu(x-2) \\ \dfrac{x^2}{4} - \dfrac{z^2}{9} = 1 \\ y = 2 \end{cases} \iff \begin{cases} z = \lambda \\ x = 2 + \dfrac{2}{\mu} \\ \dfrac{x^2}{4} - \dfrac{z^2}{9} = 1 \end{cases} \iff \frac{\left(2 + \frac{2}{\mu}\right)^2}{4} - \frac{\lambda^2}{9} = 1 \iff 9(1 + \frac{1}{\mu})^2 - \lambda^2 = 9.$$

The equation of the conoid is:

$(S) : 9\left(1 + \dfrac{x-2}{y}\right)^2 - z^2 = 9 \iff$

$(S) : 9(x + y - 2)^2 - y^2(z^2 + 9) = 0.$

Problem 7.5. Determine the equation of the graph of $(\Gamma) : \begin{cases} z = \sqrt{x} \\ y = 0 \end{cases}$ revolved about Ox axis and then about Oz axis. Graph the equation of the surface of revolution in each case.

Solution:

When we rotate the parabola about Ox axis the generator circle is

$$(G) : \begin{cases} x^2 + y^2 + z^2 = \alpha^2 \\ x = \beta \end{cases}.$$

$$(G) \cap (\Gamma) \iff \begin{cases} x^2 + y^2 + z^2 = \alpha^2 \\ x = \beta \\ z = \sqrt{x} \\ y = 0 \end{cases} \iff \beta^2 + \beta = \alpha^2.$$

The equation of the surface of revolution is

$(S) : x^2 + x = x^2 + y^2 + z^2 \iff$

$(S) : y^2 + z^2 = x$ which is an elliptic paraboloid.

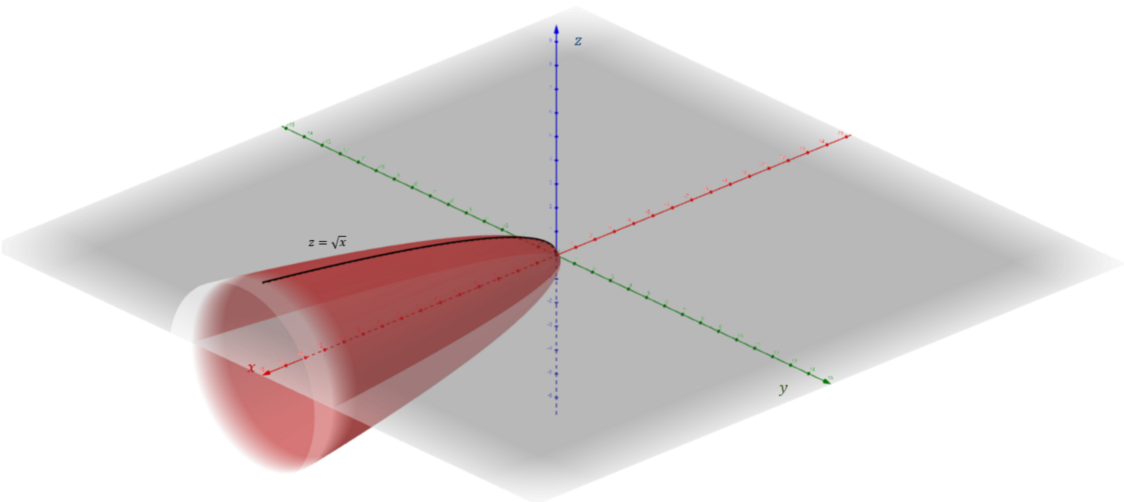


Figure 7.4: Parabola rotated about Ox axis

When we rotate the parabola about Oz axis the generator circle is

$$(G) : \begin{cases} x^2 + y^2 + z^2 = \alpha^2 \\ z = \beta \end{cases}.$$

$$(G) \cap (\Gamma) \iff \begin{cases} x^2 + y^2 + z^2 = \alpha^2 \\ z = \beta \\ z = \sqrt{x} \\ y = 0 \end{cases} \iff \beta^4 + \beta^2 = \alpha^2.$$

The equation of the surface of revolution is

$(S) : z^4 + z^2 = x^2 + y^2 + z^2 \iff$

$(S) : z^4 = x^2 + y^2.$

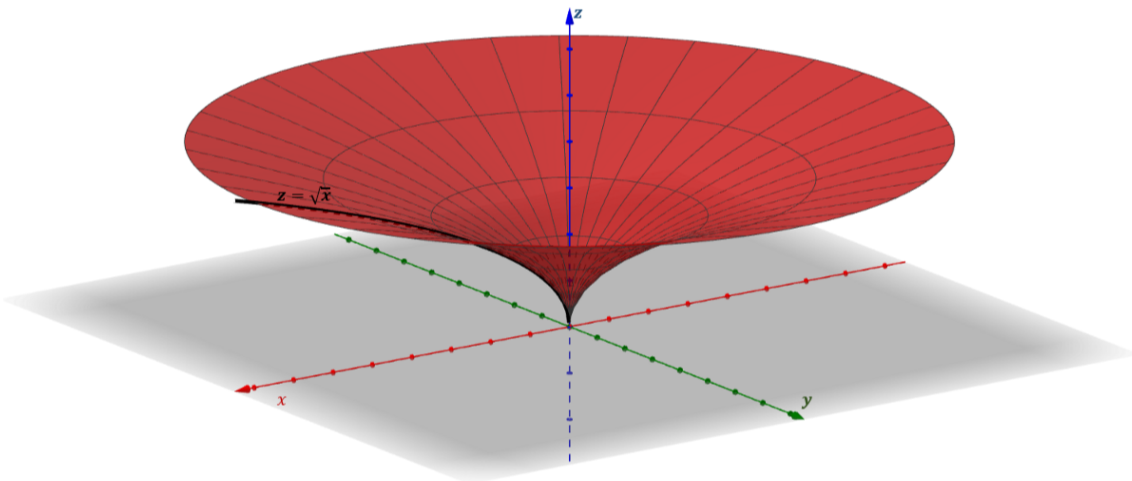


Figure 7.5: Parabola rotated about Oz axis

7.6 Problems

Problem 7.6. Determine the equation of the cylindric surface having as generator the line $d : \begin{cases} x - y - 3 = 0 \\ y - z + 2 = 0 \end{cases}$ and as directrix the curve $(\Gamma) : \begin{cases} xy = 4 \\ x = 0 \end{cases}$.

Problem 7.7. Determine the equation of the cylindrical surface having the generators parallel to the direction $\overrightarrow{v} = 2\overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}$ and the directrix the curve $(\Gamma) : \begin{cases} x^2 + 4z - 4 = 0 \\ y = 0. \end{cases}$

Problem 7.8. The curve $(\Gamma) : \begin{cases} x = y^2 + z^2 \\ x = 2z \end{cases}$ is the directrix of the cylindical surface (S). Its generators are perpendicular to the plane of the curve (Γ). Determine the equation of the surface (S).

Problem 7.9. Determine the equation of the conical surface having the vertex at the origin $O(0, 0, 0)$ and the directrix $(\Gamma) : \begin{cases} x^2 + y^2 - 1 = 0 \\ x + y + z - 1 = 0 \end{cases}$.

Problem 7.10. Write the equation of the conical surface having the apex at $(1, 1, 1)$ and its directrix the curve $(\Gamma) : \begin{cases} y^2 + z^2 - 1 = 0 \\ x + y + z = 0 \end{cases}$.

Problem 7.11. Determine the equation of the conoid generated by a line passing through the line $d : \begin{cases} x = 0 \\ y = 0 \end{cases}$, is parallel to xOy plane and intersect the hyperbola $(H) : \begin{cases} \dfrac{x^2}{4} - \dfrac{z^2}{9} - 1 = 0 \\ y = 2 \end{cases}$.

Problem 7.12. Determine the conoid generated by a line that mets the line $d :$ $x = y = z$, the curve $(\Gamma) : \begin{cases} x^4 + y^4 - 16 = 0 \\ z = 0 \end{cases}$ and is parallel to the plane $(P) :$ $x + y + z - 1 = 0$.

Problem 7.13. Consider the space curves:

a) $(\Gamma): \begin{cases} y^2 - 6z = 0 \\ x = 0 \end{cases}$;

b) $(\Gamma): \begin{cases} y^2 + z^2 = 9 \\ x = 0 \end{cases}$;

c) $(\Gamma): \begin{cases} \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \\ x = 0 \end{cases}$;

d) $(\Gamma): \begin{cases} \frac{x^2}{16} - y^2 - 1 = 0 \\ x = 0 \end{cases}$.

Determine the equation of the graph of (Γ) revolved about Oz axis. Graph the equation of the surface of revolution.

8

Plane curves

A plane curve is a curve that lies in a single plane.

8.1 Analytic representation of plane curves

A plane curve can be represented by:

- the **explicit** equation $y = y(x)$, $x \in I \subset \mathbb{R}$.
- the **implicit** equation $F(x, y) = 0$.
- the **parametric** equations $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t \in I \subset \mathbb{R}$.
- the **vector** equation $\overrightarrow{r} = \overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j}$, $t \in I \subset \mathbb{R}$.

All the conic sections presented in the previous chapter are plane curves.

Example 8.1. *The ellipse* $(E) : \dfrac{x^2}{4} + y^2 = 1$ *is given in the implicit form.*

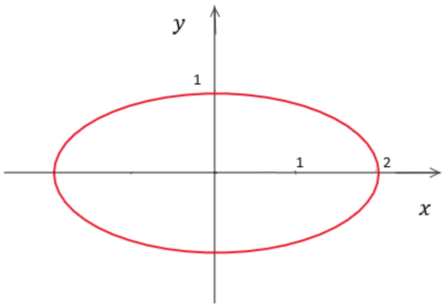


Figure 8.1: The ellipse $(E) : \dfrac{x^2}{4} + y^2 = 1$

The explicit equations are $y = \pm\sqrt{1 - \dfrac{x^2}{4}}$, $x \in [-2, 2]$.

The parametric equations are $\begin{cases} x = 2\cos t \\ y = \sin t \end{cases}, \quad t \in [0, 2\pi]$.

The vector equation is $\overrightarrow{r} = \overrightarrow{r}(t) = 2\cos t\,\overrightarrow{i} + \sin t\,\overrightarrow{j}$, $t \in [0, 2\pi]$.

Examples of plane curves

1. The **astroid** or **hypocicloid** is the locus described by a point on a circle of radius $\dfrac{a}{4}$ as it rolls inside of a fixed circle of radius a.

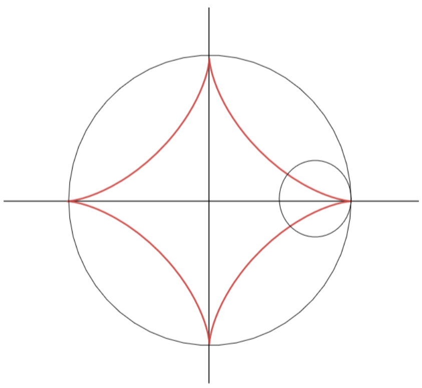


Figure 8.2: The astroid

The implicit equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

The parametric equations of the astroid are $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}, \quad t \in \mathbb{R}.$

2. The **cycloid** is the locus of a point on the rim of a circle of radius a rolling along a straight line.

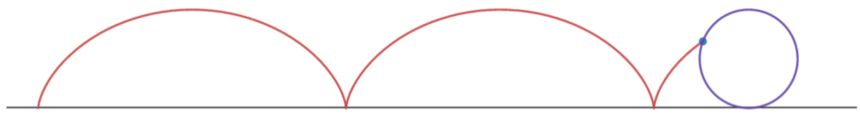


Figure 8.3: The cycloid

The parametric equations of the cycloid are $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \quad t \in \mathbb{R}.$

3. The **cardioid** is the plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius.

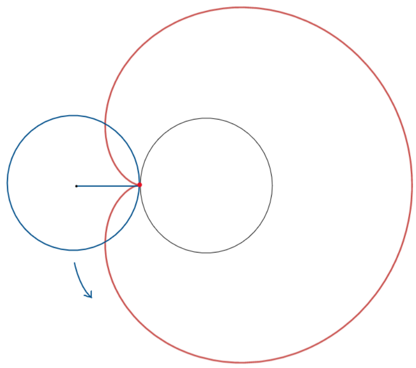


Figure 8.4: The cardioid

The implicit equation of the cardioid is $(x^2+y^2-2ax)^2 = 4a^2(x^2+y^2)$.

The parametric equations of the cardioid are $\begin{cases} x = 2a(1-\cos t)\cos t \\ y = 2a(1-\cos t)\sin t \end{cases}$, $t \in [0, 2\pi]$.

8.2 The tangent to a plane curve

In what's follow, differential calculus is involved, so we shall make some hypothesis on the functions involved in their analytic representation regarding:

- continuity of the functions;
- existence of and continuity of partial derivatives of certain order;
- regularity conditions.

If the plane curve is in the implicit form, $(\Gamma) : F(x,y) = 0$, $M(x_0, y_0) \in (\Gamma)$ is a **regular point** if $F'_x(x_0,y_0) \neq 0$ or $F'_y(x_0,y_0) \neq 0$. Every other point is called **singular** point.

For the plane curve given in the parametric form, $(\Gamma) : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $t \in [a,b] \subset \mathbb{R}$, $x(t)$ and $y(t)$ must have continuous derivatives on $[a,b]$ and $x'^2(t) + y'^2(t) \neq 0$.

Definition 8.2. *The* ***tangent line*** *to a regular curve* (Γ) *at a point* $M_0(x_0,y_0) \in (\Gamma)$ *is defined as the limit of the secant* MM_0 *when the point* M *approaches* M_0 *on the curve* (Γ).

The line passing through M_0 *and is perpendicular to the tangent is called the* ***normal*** *to the curve* (Γ) *at the point* M_0.

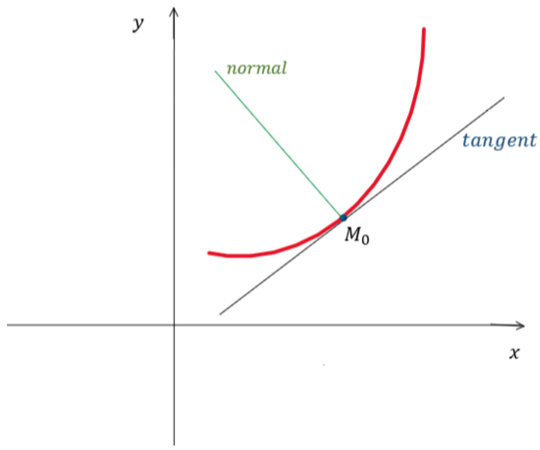


Figure 8.5: Tangent and normal to a curve at M_0

Remark 8.3. *Is is well know from analytic geometry in $\mathbb{R}^2$ that if two lines are perpendicular then the product of their slopes is -1. So, if we have the equation of the tangent line $tg : y - y_0 = m(x - x_0)$, is easy to write the equation of the normal line as $n : y - y_0 = -\dfrac{1}{m}(x - x_0)$.*

In what follows we give the equations of the tangent line and the normal line to a regular curve (Γ) at a point M_0 if the curve is given in one of the following analytical expression.

1. If the curve is given in the **explicit** form, $(\Gamma) : y = f(x)$.

   - $tg : y - y_0 = y'(x_0)(x - x_0)$
   - $n : y - y_0 = -\dfrac{1}{y'(x_0)}(x - x_0)$

2. If the curve is given in the **parametric** form $(\Gamma) : \begin{cases} x = x(t) \\ y = y(t) \end{cases}, t \in I$.

- $tg : \dfrac{x - x(t_0)}{x'(t_0)} = \dfrac{y - y(t_0)}{y'(t_0)}$ or
- $n : y - y(t_0) = -\dfrac{x'(t_0)}{y'(t_0)}(x - x(t_0))$

where t_0 is such that $x(t_0) = x_0$ and $y(t_0) = y_0$.

3. If the curve is given in the **implicit** form $(\Gamma) : F(x, y) = 0$.

- $tg : y - y_0 = -\dfrac{\frac{\partial F}{\partial x}(x_0, y_0)}{\frac{\partial F}{\partial y}(x_0, y_0)}(x - x_0)$
- $n : y - y_0 = \dfrac{\frac{\partial F}{\partial y}(x_0, y_0)}{\frac{\partial F}{\partial x}(x_0, y_0)}(x - x_0)$

8.3 The length of a plane curve

Let $(\Gamma) : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t \in I$ be a regular curve, and $A(x(a), y(a))$ and $B(x(b), y(b))$ two points on the curve (Γ).

The length of the arc $\overset{\frown}{AB}$ denoted by $L(\overset{\frown}{AB})$ is

$$L(\overset{\frown}{AB}) = \int_a^b \sqrt{x'^2(t) + y'^2(t)}dt = \int_a^b \|\overrightarrow{r}'(t)\|dt.$$

It is of interest to consider the length $s(t)$ of the curve from a fixed point $A(x(a), y(a))$ to a variable point $M(x(t), y(t))$.

Then

$$s = s(t) = L(\overset{\frown}{AM}) = \int_a^t \sqrt{x'^2(\tau) + y'^2(\tau)}d\tau$$

From the last equation we have that $\dfrac{ds}{dt} = \sqrt{x'^2(t) + y'^2(t)}$ and the **element of arc** (linear element) is

$$ds = \sqrt{x'^2(t) + y'^2(t)}dt.$$

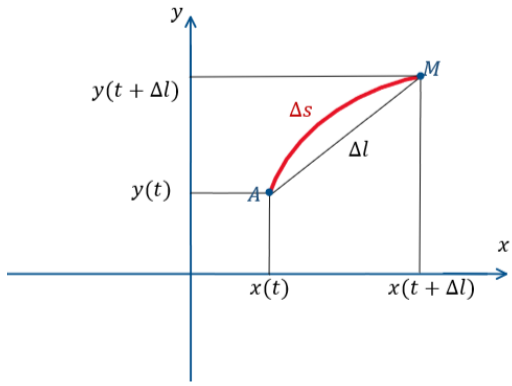


Figure 8.6: Element of arc

Remark 8.4.
- *If the curve is in its explicit form* $(\Gamma) : y = f(x)$*, then the element of arc is*

$$ds = \sqrt{1 + y'^2(x)}dx.$$

- *Sometimes is useful to use* s *as a* ***natural parameter*** *so we will obtain the natural parametrization of the curve* $\overrightarrow{r}(s) = x(s)\overrightarrow{i} + y(s)\overrightarrow{j}$*, so the magnitude of the tangent vector is a unit vector* $\|\overrightarrow{r'}(s)\| = 1$.

The length of the curve (Γ) when $t \in [a, b]$ is $L(\Gamma) = \int_a^b ds$.

8.4 The curvature of a plane curve

The **curvature, K,** of the curve can be defined as the ratio of the rotation angle of the tangent $\Delta\alpha$ to the traversed arc length $\Delta s = MM_1$.

Definition 8.5. *The* ***mean curvature*** *of the arc* MM_1 *is defined by:*

$$K_m = \frac{\Delta\alpha}{\Delta s}.$$

The curvature K at a point is defined by

$$K = \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta s}.$$

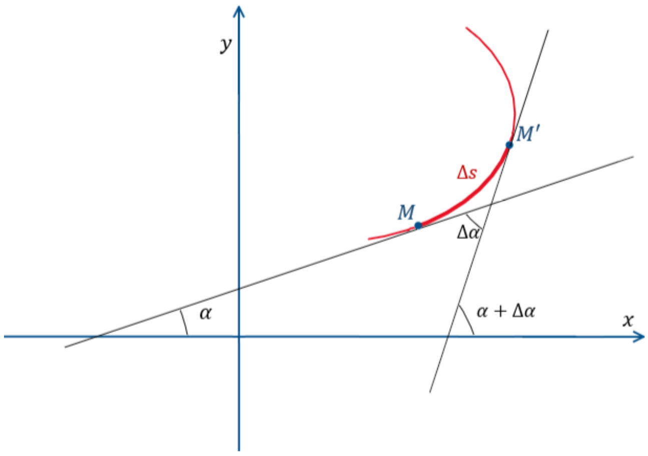


Figure 8.7: The curvature

1. The curvature of $(\Gamma) : \begin{cases} x = x(t) \\ y = y(t) \end{cases}, t \in \mathbb{R}$, at $M(x(t_0), y(t_0))$ is

$$K = \frac{y''(t_0)x'(t_0) - x''(t_0)y'(t_0)}{(x'^2(t_0) + y'^2(t_0))^{\frac{3}{2}}}.$$

2. The curvature of $(\Gamma) : F(x, y) = 0$, at $M(x_0, y_0)$ is

$$K = \left. \frac{F_y'^2 F_{xx}'' - 2F_x' F_y' F_{xy}'' + F_x'^2 F_{yy}''}{(F_x'^2 + F_y'^2)^{\frac{3}{2}}} \right|_{(x_0, y_0)}.$$

3. The curvature of $(\Gamma) : y = f(x)$ at $M(x_0, f(x_0))$ is

$$K = \frac{y''(x_0)}{(1 + y'^2(x_0))^{\frac{3}{2}}}.$$

Remark 8.6. *From the definition it follows that the curvature at a point of a curve characterises the speed of rotation of the tangent curve at this point (how quickly the curve turns).*

Definition 8.7. *The inverse of the curvature K at a point of the curve is called the* ***radius curvature****, $R = \dfrac{1}{|K|}$.*

Remark 8.8.

- *The radius curvature is the radius of the circular arc which best approximates the curve at the point.*
- *The* ***osculating circle*** *is the circle with the radius and the center located on the inner normal line and it will most closely approximate the plane curve at the given point.*

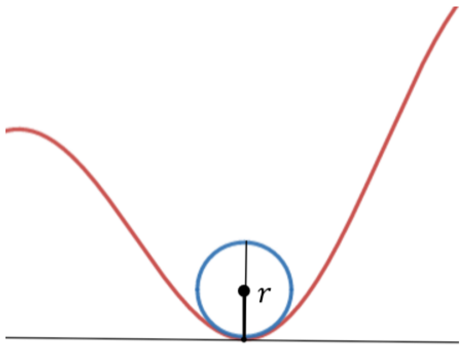


Figure 8.8: The osculating circle

- *The equation of the osculating circle is:*

$$(x-h)^2 + (y-k)^2 = r^2,$$

where:

$\diamond$ $r = \dfrac{1}{|K|} = \dfrac{(x'^2(t) + y'^2(t))^{\frac{3}{2}}}{|y''(t)x'(t) - x''(t)y'(t)|}$.

$\diamond$ $h = x(t) - y'(t)\dfrac{x'^2(t) + y'^2(t)}{y''(t)x'(t) - x''(t)y'(t)}$.

$\diamond$ $k = y(t) + x'(t)\dfrac{x'^2(t) + y'^2(t)}{y''(t)x'(t) - x''(t)y'(t)}$.

Remark 8.9. *1. The curvature of a circle having the radius r is $K = \dfrac{1}{r}$.*

2. The curvature of a straight line is 0.

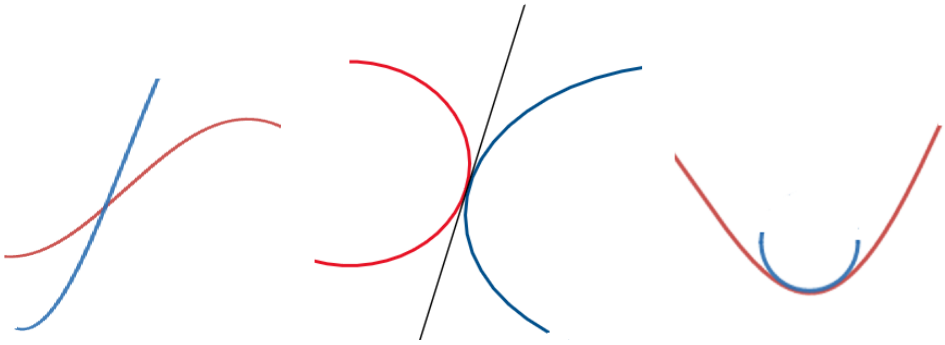
8.5 The contact of plane curves

Let $(\Gamma_1) : y = y_1(x)$ and $(\Gamma_2) : y = y_2(x)$ be two plane curves.
They have common points (or they intersect) if the equation $y_1(x) = y_2(x)$ has solutions.
If x_0 is a solution of the above equation, x_0 is:

- 0th-order contact if the curves have a simple crossing (not tangent).
- 1st-order contact if the two curves are tangent.
- 2nd-order contact if the curvatures of the curves are equal. Such curves are said to be osculating.

Definition 8.10. *The curves $(\Gamma_1) : y = y_1(x)$ and $(\Gamma_2) : y = y_2(x)$ have an k* ***order contact*** *at $M(x_0, y_0)$ if*

$y_1(x_0) = y_2(x_0)$,
$y_1'(x_0) = y_2'(x_0)$,
$y_1''(x_0) = y_2''(x_0)$,
$\ldots$
$y_1^{(k)}(x_0) = y_2^{(k)}(x_0)$,
$y_1^{(k+1)}(x_0) \neq y_2^{(k+1)}(x_0)$.



(a) (b) (c)

Figure 8.9: (a) 0th-order contact (b) 1th-order contact (c) 2th-order contact

Remark 8.11. *The osculating circle of the curve* (Γ) *at* M_0 *is the circle having two-point contact with* (Γ) *at* M_0.

8.6 Solved problems

Problem 8.1. Determine the element of arc, the arc length and the natural parameter of the cycloid $(\Gamma) : \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \quad t \in [0, 2\pi]$.

Solution:

The derivatives of the two components of the curve are: $\begin{cases} x' = a(1 - \cos t) \\ y' = a \sin t \end{cases}$.

$$\begin{aligned} ds &= \sqrt{a^2(1-2\cos t+\cos^2 t)+a^2\sin^2 t dt} \\ &= \sqrt{a^2(2-2\cos t)}dt \\ &= a\sqrt{4\sin^2\frac{t}{2}}dt \\ &= 2a|\sin\frac{t}{2}|dt \end{aligned}$$

We applied the trigonometric formulas $\sin^2 x+\cos^2 x = 1$ and $\sin^2 t = \dfrac{1-\cos 2t}{2}$.
Because $t \in [0, 2\pi] \Longrightarrow \dfrac{t}{2} \in [0, \pi] \Longrightarrow \sin\dfrac{t}{2} \geqslant 0 \quad \forall t \in [0, 2\pi]$.
$L(\Gamma) = \displaystyle\int_{(\Gamma)} ds = \int_0^{2\pi} 2a\sin\frac{t}{2}\, dt = -4a\cos\frac{t}{2}\Big|_0^{2\pi} = 8a.$
The natural parameter is

$$s = \int_0^t 2a\sin\frac{u}{2}du = -2a\cos\frac{t}{2} + 2a = 2a\left(1-\cos\frac{t}{2}\right).$$

So, $s = 2a\left(1-\cos\dfrac{t}{2}\right) \Longrightarrow \cos\dfrac{t}{2} = 1-\dfrac{s}{2a} \Longrightarrow t = 2\arccos\left(1-\dfrac{s}{2a}\right)$.
The natural parametrization of the cycloid is

$$\begin{cases} x = a\left(2\arccos\left(1-\frac{s}{2a}\right) - \sin\left(2\arccos\left(1-\frac{s}{2a}\right)\right)\right) \\ y = a\left(1-\cos\left(2\arccos\left(1-\frac{s}{2a}\right)\right)\right). \end{cases}.$$

Problem 8.2. Write the equation of the tangent line and the equation of the normal line to the curve $(\Gamma): x^3 - xy^2 + 2x + y - 3 = 0 = 0$ at its intersection with Ox axis.

Solution:

$(\Gamma) \cap Ox = A(a, 0) \Longrightarrow a^2 + 2a - 3 = 0 \Longrightarrow x = 1 \Longrightarrow A(1, 0)$.

We have the implicit form for the equation of the curve, so, the slope of the tangent is $m_{tg} = -\dfrac{F'_x(1,0)}{F'_y(1,0)}$, where $F(x,y) = x^3 - xy^2 + 2x + y - 3$.

$F'_x(x,y) = 3x^2 - y^2 + 2 \Longrightarrow F'_x(1,0) = 5$.

$F'_y(x, y) = -2xy + 1 \Longrightarrow F'_y(1, 0) = 1.$

$y'(1) = -\dfrac{5}{1} = -5 = m_{tg}.$

The equation of the tangent line at A is:

$tg : (y - 0) = -5(x - 1) \Longleftrightarrow$

$tg : 5x + y - 5 = 0.$

The slope of the normal is $m_n = -\dfrac{1}{m_{tg}} = \dfrac{1}{5}$. The equation of the normal line is:

$n : (y - 0) = \dfrac{1}{5}(x - 1) \Longleftrightarrow$

$n : -x + 5y + 1 = 0.$

Problem 8.3. Write the equation of the tangent line and the equation of the normal line to the curve $(\Gamma) : \begin{cases} x = t^2 + t - 2 \\ y = t^3 + 3t^2 - 4 \end{cases}, t \in \mathbb{R}$ at the point corresponding to $t = -1$.

Solution:

We have the parametric form of the equation of the curve, so, the slope of the tangent is $m_{tg} = \dfrac{y'(-1)}{x'(-1)}$.

The point corresponding to $t = -1$ is $A\,(x(-1), y(-1)) \Longrightarrow A(-2, -2)$.

$x'(t) = 2t + 1 \Longrightarrow x'(-1) = -1.$

$y'(t) = 3t^2 + 6t \Longrightarrow y'(-1) = -3.$

The slope of the tangent is $m_{tg} = 3$ while the slope of the normal is $m_n = -\dfrac{1}{3}$.

The equation of the tangent line at A is:

$tg : (y - (-2)) = 3(x - (-2)) \Longleftrightarrow$

$tg : 3x - y + 4 = 0.$

The equation of the normal line is

$n : (y - (-2)) = -\dfrac{1}{3}(x - (-2)) \Longleftrightarrow$

$n : x + 3y + 8 = 0.$

Problem 8.4. Write the equation of the tangent line to the curve

$$(\Gamma): \begin{cases} x = t^2 - 1 \\ y = t^3 + 1 \end{cases}, t \in \mathbb{R}$$

parallel to the line $d: 2x - y + 3 = 0$.

Solution:

Let $M(x_0, y_0)$ be the point on (Γ) such that at this point the tangent line is parallel to d, $M(t_0^2 - 1, t_0^3 + 1)$.

The slope of the tangent line is $m_{tg} = \dfrac{y'(t_0)}{x'(t_0)}$.

$x'(t) = 2t$, $y'(t) = 3t^2$, so $m_{tg} = \dfrac{3t_0^2}{2t_0} = \dfrac{3}{2}t_0$.

$tg \parallel d \Longleftrightarrow m_{tg} = m_d \Longrightarrow \dfrac{3}{2}t_0 = 2 \Longrightarrow t_0 = \dfrac{4}{3}$.

$x\left(\dfrac{4}{3}\right) = \dfrac{7}{9}$, $y\left(\dfrac{4}{3}\right) = \dfrac{91}{27}$.

The equation of the tangent line is:

$tg: y - \dfrac{91}{27} = 2\left(x - \dfrac{7}{9}\right) \Longleftrightarrow$

$tg: 2x - y + \dfrac{49}{27} = 0$.

Problem 8.5. Determine the curvature and the radius of curvature for the curve (Γ) at the given point.

1. $(\Gamma): \begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$ at $A(t = \pi)$.

2. $(\Gamma): y = x^4 - 4x^3 - x^2$ at $O(0, 0)$.

Solution:

1. $\begin{cases} x'(t) = 2 - 2\cos t \\ y'(t) = 2\sin t \end{cases} \Longrightarrow \begin{cases} x'(\pi) = 4 \\ y'(\pi) = 0 \end{cases}.$

$$\begin{cases} x''(t) = 2\sin t \\ y''(t) = 2\cos t \end{cases} \Longrightarrow \begin{cases} x''(\pi) = 0 \\ y'(\pi) = -2 \end{cases}.$$

$$K = \frac{x'(\pi)y''(\pi) - x''(\pi)y'(\pi)}{(x'(\pi)^2 + y'(\pi)^2)^{\frac{3}{2}}} = \frac{4\cdot(-2) - 0\cdot 0}{(4^2+0^2)^{\frac{3}{2}}} = \frac{-8}{4^3} = -\frac{1}{8}.$$

$$R = \frac{1}{|K|} = 8.$$

2. Since we have the curve in the explicit form, the formula of the curvature is $K = \dfrac{y''(0)}{(1+y'(0)^2)^{\frac{3}{2}}}$.

   $y'(x) = 4x^3 - 12x^2 - 2x \Longrightarrow y'(0) = 0.$

   $y''(x) = 12x^2 - 24x - 2 \Longrightarrow y''(0) = -2.$

   The curvature is $K = \dfrac{-2}{1} = -2 \Longrightarrow$ the radius of curvature is $R = \dfrac{1}{2}$.

Problem 8.6. Determine the order of contact for the curves

$$(\Gamma_1) : y_1(x) = e^x + x - 1$$

and

$$(\Gamma_2) : y_2(x) = 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + x^4$$

at $x = 0$.

Solution:

We determine the derivatives of y_1 and y_2 at $x = 0$.

$y_1(0) = 0$

$y_1'(x) = e^x + 1 \Longrightarrow y_1'(0) = 2$

$y_1''(x) = e^x \Longrightarrow y_1'(0) = 1$

$y_1'''(x) = e^x \Longrightarrow y_1'''(0) = 1$

$y_1^{(4)}(x) = e^x \Longrightarrow y_1^{(4)}(0) = 1$

$y_2(0) = 0$

$y_2'(x) = 2+x+\frac{1}{2}x^2+4x^3 \Longrightarrow y_2'(0) = 2$

$y_2''(x) = 1 + x + 12x^2 \Longrightarrow y_2''(0) = 1$

$y_2'''(x) = 1 + 24x \Longrightarrow y_2'''(0) = 1$

$y_2^{(4)}(x) = 24 \Longrightarrow y_2^{(4)}(0) = 24$

$y_1^{(n)}(0) = y_2^{(n)}(0), \forall n \in \{0,1,2,3\}$ and $y_1^{(4)}(0) \neq y_2^{(4)}(0)$, therefore, the order of contact of (Γ_1) and (Γ_2) is 3.

Problem 8.7. Determine the equation of the osculating circle of the curve (Γ) : $\overrightarrow{r}(t) = (t^2 - 2t)\overrightarrow{i} + (t^3 + t)\overrightarrow{j}$, $t \in \mathbb{R}$, at $t_0 = 1$.

Solution:

We can apply the formula for the equation of the osculating circle which is

$$(x-h)^2 + (y-k)^2 = r^2,$$

where:

$$r = \frac{1}{|K|} = \frac{(x'^2(1) + y'^2(1))^{\frac{3}{2}}}{|y''(1)x'(1) - x''(1)y'(1)|}$$

$$h = x(1) - y'(1)\frac{x'^2(1) + y'^2(1)}{y''(1)x'(1) - x''(1)y'(1)}$$

$$k = y(1) + x'(1)\frac{x'^2(1) + y'^2(1)}{y''(1)x'(1) - x''(1)y'(1)}$$

We calculate the derivatives of x and y at 1.

$$\begin{cases} x(t) = t^2 - 2t \\ y(t) = t^3 + t \end{cases} \Longrightarrow \begin{cases} x(1) = -1 \\ y(1) = 2 \end{cases}.$$

$$\begin{cases} x'(t) = 2t - 2 \\ y'(t) = 3t^2 + 1 \end{cases} \Longrightarrow \begin{cases} x'(1) = 0 \\ y'(1) = 4 \end{cases}.$$

$$\begin{cases} x''(t) = 2 \\ y'(t) = 6t \end{cases} \Longrightarrow \begin{cases} x''(1) = 2 \\ y''(1) = 6 \end{cases}.$$

$$r = \frac{(0^2 + 4^2)^{\frac{3}{2}}}{|6 \cdot 0 - 2 \cdot 4|} = 8.$$

$$h = -1 - 4\frac{0^2 + 4^2}{6 \cdot 0 - 2 \cdot 4} = -1 + 8 = 7.$$

$$k = 2 + 0 \cdot \frac{0^2 + 4^2}{6 \cdot 0 - 2 \cdot 4} = 2.$$

The equation of the osculating circle is $(x-7)^2 + (y-2)^2 = 64$.

Remark. We can apply also the fact that the osculating circle and the curve at the given point have at least a second order contact.

Let us denote the function

$F(x(t), y(t)) = (x(t) - h)^2 + (y(t) - k)^2 - r^2$,

where $x(t) = t^2 - 2t$ and $y(t) = t^3 + t$.

The osculating circle and the curve at the given point have at least a second order contact if and only if $\begin{cases} F(x(1), y(1)) = 0 \\ F'(x(1), y(1)) = 0 \\ F''(x(1), y(1)) = 0 \end{cases}$.

$F(x(t), y(t)) = (t^2 - 2t - h)^2 + (t^3 + t - k)^2 - r^2$.

$F'(x(t), y(t)) = 2(t^2 - 2t - h)(2t - 2) + 2(t^3 + t - k)(3t^2 + 1)$.

$F''(x(t), y(t)) = 2(2t-2)(2t-2) + 2(t^2-2t-h)2 + 2(3t^2+1)(3t^2+1) + 2(t^3+t-k)6t$.

$$\begin{cases} F(x(1), y(1)) = 0 \\ F'(x(1), y(1)) = 0 \\ F''(x(1), y(1)) = 0 \end{cases} \iff \begin{cases} (1 - 2 - h)^2 + (1 + 1 - k)^2 - r^2 = 0 \\ 2(1 - 2 - h)(2 - 2) + 2(2 - k)4 = 0 \\ 2 \cdot 0 \cdot 0 + 4(-1 - h) + 2 \cdot 4^2 + 12(2 - k) = 0 \end{cases} \iff$$

$$\begin{cases} (-1 - h)^2 + (2 - k)^2 - r^2 = 0 \\ 8(2 - k) = 0 \\ 4(-1 - h) + 32 + 12(2 - k) = 0 \end{cases} \iff \begin{cases} r = 8 \\ h = 7 \\ k = 2 \end{cases}.$$

We obtain the same equation of the osculating circle i.e. $(x-7)^2 + (y-2)^2 = 64$.

8.7 Problems

Problem 8.8. Write the equation of the tangent and the normal line to the curve $(\Gamma) : x^2 + xy^3 + y^2 + 2x - 4y - 4 = 0$ at its intersection with Oy axis.

Problem 8.9. Write the tangent line and the normal line to the curve

$$(\Gamma) : y = x \ln |x| + 1$$

at $x = 1$.

Problem 8.10. Write the equation of the tangent and the normal line to the curve $(\Gamma) : y + \sin x + x \cos y - \dfrac{\pi}{2} = 0$ at its intersection with Ox axis.

Problem 8.11. Write the tangent line to the curve

$$(\Gamma) : \begin{cases} x = t^2 + t - 2 \\ y = t^3 + 3t^2 - 4 \end{cases}, t \in \mathbb{R}$$

which is parallel to the line $d : 3x - y + 3 = 0$.

Problem 8.12. Write the tangent line and the normal line to the curve

$$(\Gamma) : \begin{cases} x = 2t^3 + t^2 + t \\ y = t^2 + t - 1 \end{cases}, t \in \mathbb{R}$$

which passes through $A(1, 0)$.

Problem 8.13. Determine the curvature of the curve at the given point for:

a) $(\Gamma_1) : \begin{cases} x = \sin t \\ y = t \cos t \end{cases}$, at $A(t = \pi)$.

b) $(\Gamma_2) : y = x^3 - x^2 + 2x - 2$, at $x = 0$.

c) $(\Gamma_3) : \dfrac{x^2}{4} + y^2 - 1 = 0$, at $A(0, 1)$.

Problem 8.14. Determine the length of the curve $(\Gamma) : \begin{cases} x = 8t^3 \\ y = 3(2t^2 - t^4) \end{cases}, t \in \mathbb{R}$ on the interval $[0, \sqrt{2}]$.

Problem 8.15. Determine the element of arc of the curve

$$(\Gamma) : \begin{cases} x = \ln(t + \sqrt{1 + t^2}) \\ y = \sqrt{1 + t^2} \end{cases}$$

between the points corresponding to $t_1 = 0$ and $t_2 = 1$.

Problem 8.16. Determine the osculating circle of the curve $(\Gamma) : \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ at $B(0, 3)$.

Problem 8.17. Determine the osculating circle of the curve

$$(\Gamma) : \begin{cases} x = \sin t \\ y = \cos(2t) \end{cases}, t \in [0, 2\pi],$$

at $A(t = \dfrac{\pi}{6})$.

Problem 8.18. Determine the osculating circle of the curve

$$(\Gamma) : \begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$$

at the point $A(t = \dfrac{\pi}{4})$.

Problem 8.19. Determine the osculating circle of the ellipse

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, t \in [0, 2\pi].$$

Problem 8.20. Determine the order of contact for the curves at $O(0, 0)$ if:

a) $(\Gamma_1) : y = e^x$ and $(\Gamma_2) : y = 1 + x + \dfrac{x^2}{2}$.

b) $(\Gamma_1) : y = x^3$ and $(\Gamma_2) : y = x\sin^2 x$.

c) $(\Gamma_1) : y = x^4$ and $(\Gamma_2) : y = x^2\sin^2 x$.

9

Space curves

9.1 Analytic representation of space curves

In $\mathbb{R}^3$ a single equation in x, y, z represents a surface. Two equations are needed to specify a curve.

A space curve can be represented by:

- the intersection of two surfaces which are the **implicit** equations of the curve

$$(\Gamma) : \begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases} .$$

- the **parametric** equations of the curve are $(\Gamma) : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} , \ t \in I \subset \mathbb{R}.$

- the **vector** equation of the curve is

$$(\Gamma) : \overrightarrow{r} = \overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}, \ t \in I \subset \mathbb{R}.$$

If the curve (Γ) is represented by the implicit equations, the point $M_0(x_0, y_0, z_0) \in (\Gamma)$ is called a **regular** point if the rank of the matrix

$$\begin{pmatrix} F'_x(x_0,y_0,z_0) & F'_y(x_0,y_0,z_0) & F'_z(x_0,y_0,z_0) \\ G'_x(x_0,y_0,z_0) & G'_y(x_0,y_0,z_0) & G'_z(x_0,y_0,z_0) \end{pmatrix}$$

is 2.

If the curve is represented by the parametric equations or by the vector equation, then the functions x, y, z are differentiable on I. The point $M_0(x(t_0), y(t_0), z(t_0)) \in (\Gamma)$ is called **singular** if $x'(t_0) = y'(t_0) = z'(t_0) = 0$. The point M_0 is **regular** if $x'(t_0)$, $y'(t_0)$ and $z'(t_0)$ do not vanish simultaneously. If $x'(t)$, $y'(t)$ and $z'(t)$ never vanish simultaneously on I then the curve is a **regular curve**.

Definition 9.1. *The derivative of a vector valued function $\overrightarrow{r}(t)$ at t_0 is*

$$\overrightarrow{r}(t_0) = \lim_{h\to 0} \frac{\overrightarrow{r}(t_0+h) - \overrightarrow{r}(t_0)}{h}.$$

Theorem 9.2. *A vector function $\overrightarrow{r}'(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$ is differentiable at t_0 iff each of its component functions are differentiable at t_0 and*

$$\overrightarrow{r}'(t_0) = x'(t_0)\overrightarrow{i} + y'(t_0)\overrightarrow{j} + z'(t_0)\overrightarrow{k}.$$

Examples of space curves

1. The **circular helix** sometimes also called a coil, is a curve for which the tangent makes a constant angle with a fixed line. The shortest path between two points on a cylinder (one not directly above the other) is a fractional turn of a helix, as can be seen by cutting the cylinder along one of its sides, flattening it out, and noting that a straight line connecting the points becomes helical upon re-wrapping. It is for this reason that squirrels chasing one another up and around tree trunks follow helical paths.

The circular helix has the parametric equations $(\Gamma) : \begin{cases} x = r\cos t \\ y = r\sin t \\ z = ct \end{cases}$.

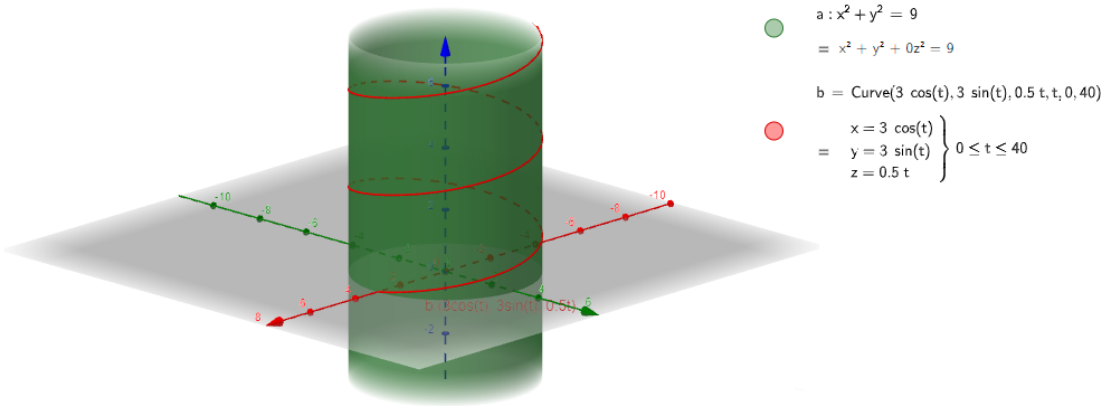


Figure 9.1: The circular helix

2. The **conical helix** (or conical spiral) is a space curve on a right circular cone, whose floor plan is a plane spiral.

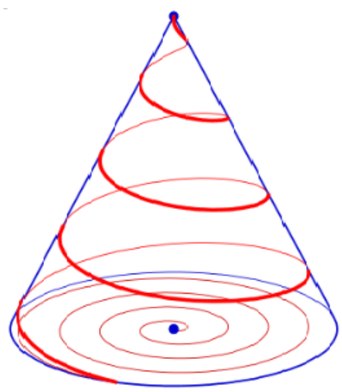


Figure 9.2: The conic helix

The conical helix has the parametric equations: $(\Gamma): \begin{cases} x = at\cos t \\ y = at\sin t \\ z = bt \end{cases}$.

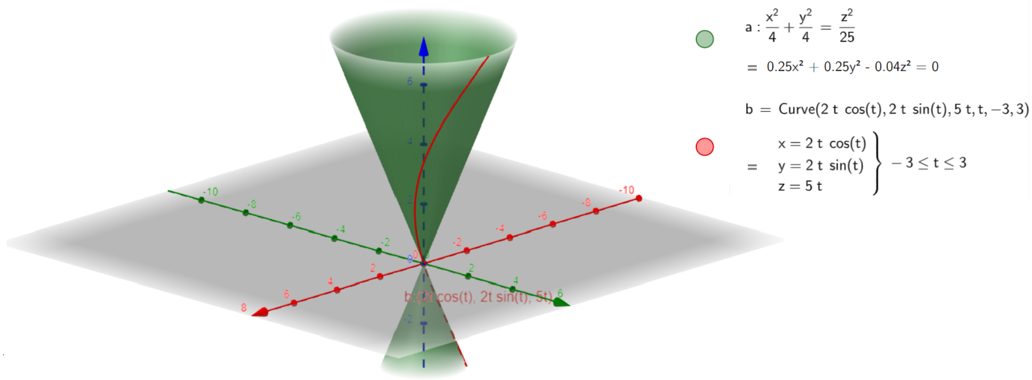


Figure 9.3: The conic helix

3. The **Viviani's curve** is the intersection of a sphere with a cylinder that is tangent to the sphere and passes through two poles (a diameter) of the sphere. The equations of the curve are $(\Gamma): \begin{cases} x^2 + y^2 + z^2 = r^2 \\ x^2 + y^2 = rx \end{cases}$.

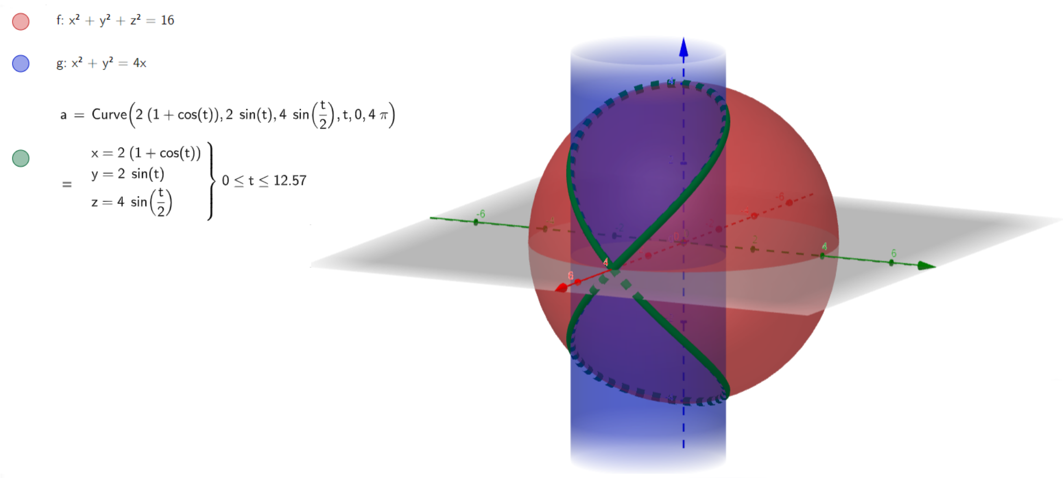


Figure 9.4: The Viviani's curve

4. The curve at the intersection of the cylinder $x^2 + y^2 = 9$ and the parabolic hyperboloid $9z = x^2 - y^2$. The curve it's also called the 'Pringle' curve.

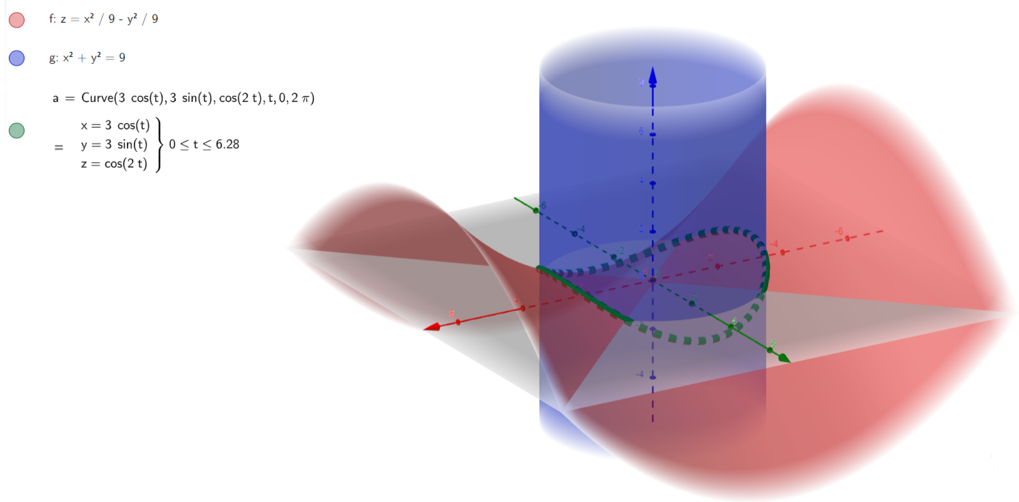


Figure 9.5: The 'Pringle' curve

5. The **toroidal spiral** - it is a space curve that lies on a torus.

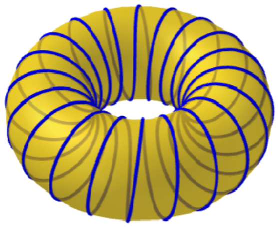


Figure 9.6: A spiral on a torus

$$\text{The parametric equations are } (\Gamma) : \begin{cases} x(t) = (a + \sin bt)\cos t \\ y(t) = (a + \sin bt)\sin t \\ z(t) = \cos bt \end{cases} .$$

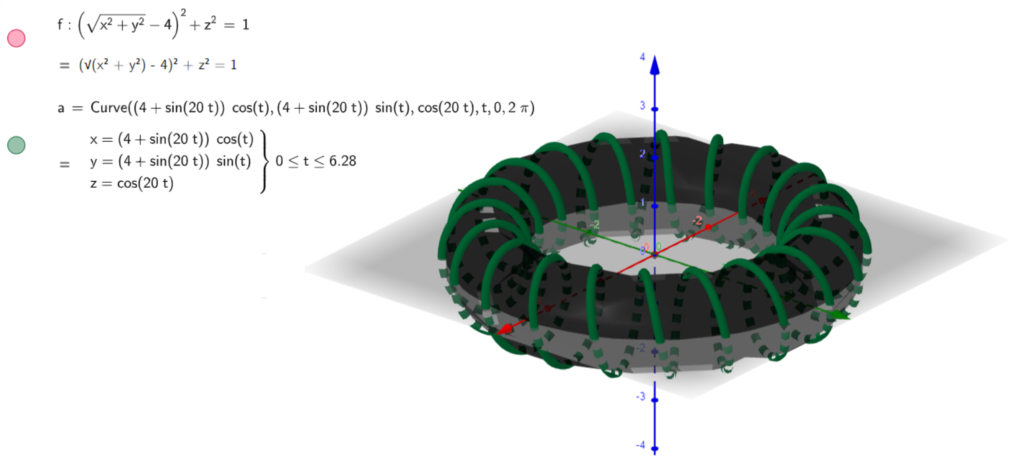


Figure 9.7: The toroidal spiral

9.2 The length of a space curve

Let (Γ) be a regular curve $(\Gamma):\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$, $t \in I \subset \mathbb{R}$, and $A(x(a), y(a), z(a))$ and $B(x(b), y(b), z(b)) \in (\Gamma)$.

The length of the arc $\overset{\frown}{AB}$ is

$$L(\overset{\frown}{AB}) = \int_a^b \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\, dt.$$

The length of the arc of the curve from initial point A to a variable point $M(x(t), y(t), z(t))$ is

$$s = s(t) = L(\overset{\frown}{AM}) = \int_a^t \sqrt{x'^2(\tau) + y'^2(\tau) + z'^2(\tau)}\, d\tau.$$

The element of arc is $ds = s'(t)\, dt$

$$ds = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\, dt.$$

Remark 9.3. *The arc length "s" of a regular curve can always be chosen as parameter since $x'^2 + y'^2 + z'^2 \neq 0$. When s is chose as parameter then, the tangent vector is a unit vector, namely*

$$s'(t) = \frac{ds}{dt} = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} = \|\overrightarrow{r}'(t)\| = 1.$$

9.3 The tangent line and the normal plane

Let $(\Gamma) : \overrightarrow{r} = \overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$, $t \in [a, b]$, be a regular curve and M_0 and M two neighbouring points on the curve.

We consider the unit vector

$$\lim_{t\to t_0} \frac{\overrightarrow{r}(t)-\overrightarrow{r}(t_0)}{\|\overrightarrow{r}(t)-\overrightarrow{r}(t_0)\|} = \frac{\overrightarrow{r}'(t_0)}{\|\overrightarrow{r}'(t_0)\|}.$$

This is called the **unit tangent vector** to (Γ) at M_0 and is denoted by $\overrightarrow{\tau}$.

$$\overrightarrow{\tau} = \frac{\overrightarrow{r}'(t_0)}{\|\overrightarrow{r}'(t_0)\|}.$$

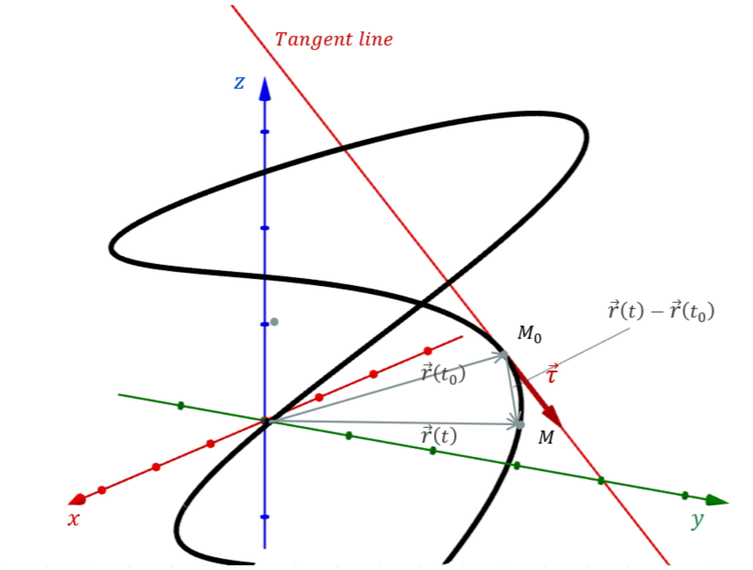


Figure 9.8: The unit tangent vector and the tangent line to a space curve

Definition 9.4. *The line passing through the point $M_0(x(t_0), y(t_0), z(t_0))$ and having as director vector $\overrightarrow{\tau}$ is called the* ***tangent line****,*

$$tg : \frac{x-x(t_0)}{x'(t_0)} = \frac{y-y(t_0)}{y'(t_0)} = \frac{z-z(t_0)}{z'(t_0)}.$$

The plane passing through $M_0(x(t_0), y(t_0), z(t_0))$ perpendicular to the tangent line is called the ***normal plane****,*

$$(P_N) : x'(t_0)(x-x(t_0)) + y'(t_0)(y-y(t_0)) + z'(t_0)(z-z(t_0)) = 0.$$

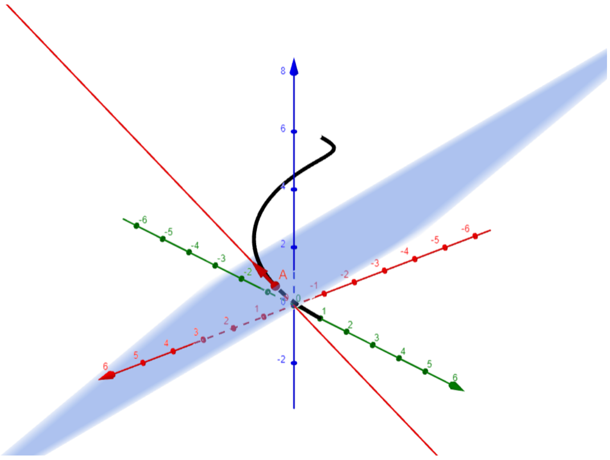


Figure 9.9: The tangent line and the normal plane at a point on a space curve

If the curve (Γ) is given as the intersection of two curves $(\Gamma): \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$, then the tangent line of (Γ) at $M_0(x_0,y_0,z_0)$ is

$$tg: \frac{x-x_0}{\left.\frac{D(F,G)}{D(y,z)}\right|_{M_0}} = \frac{y-y_0}{\left.\frac{D(F,G)}{D(z,x)}\right|_{M_0}} = \frac{z-z_0}{\left.\frac{D(F,G)}{D(x,y)}\right|_{M_0}},$$

where $\left.\frac{D(F,G)}{D(y,z)}\right|_{M_0} = \begin{vmatrix} F'_y(x_0,y_0,z_0) & F'_z(x_0,y_0,z_0) \\ G'_y(x_0,y_0,z_0) & G'_z(x_0,y_0,z_0) \end{vmatrix}$,

$$\left.\frac{D(F,G)}{D(z,x)}\right|_{M_0} = \begin{vmatrix} F'_z(x_0,y_0,z_0) & F'_x(x_0,y_0,z_0) \\ G'_z(x_0,y_0,z_0) & G'_x(x_0,y_0,z_0) \end{vmatrix},$$

$$\left.\frac{D(F,G)}{D(x,y)}\right|_{M_0} = \begin{vmatrix} F'_x(x_0,y_0,z_0) & F'_y(x_0,y_0,z_0) \\ G'_x(x_0,y_0,z_0) & G'_y(x_0,y_0,z_0) \end{vmatrix}.$$

The equation of the normal plane can be put in the form:

$$(P_N): \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ F'_x(x_0,y_0,z_0) & F'_y(x_0,y_0,z_0) & F'_z(x_0,y_0,z_0) \\ G'_x(x_0,y_0,z_0) & G'_y(x_0,y_0,z_0) & G'_z(x_0,y_0,z_0) \end{vmatrix} = 0.$$

9.4 The moving trihedron. TNB (Frenet-Serret) Frame

TNB Frames describe the motion of a particle traveling along a curve or how a particle on a space curve is heading, turning and twisting.

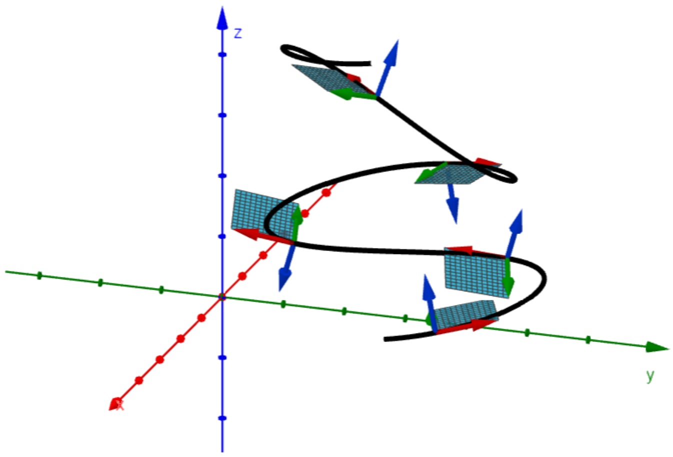


Figure 9.10: The moving trihedron

Let $(\Gamma) : \overrightarrow{r} = \overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$, $t \in I \subseteq \mathbb{R}$ be a regular curve of second order in space, that means that exists $\overrightarrow{r}'(t_0)$ and $\overrightarrow{r}''(t_0)$ for all $t_0 \in I$. Moreover, let us suppose that the vectors $\overrightarrow{r}'(t_0)$ and $\overrightarrow{r}''(t_0)$ are not parallel, so $\overrightarrow{r}'(t_0) \times \overrightarrow{r}''(t_0) \neq 0$.

At every point $M_0(x_0, y_0, z_0) \in (\Gamma)$, $(\Gamma) : \overrightarrow{r} = \overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$, $t \in I \subseteq \mathbb{R}$, the TNB frame gives us the unit vectors:

- the **tangent** unit vector, $\overrightarrow{\tau}$ (the direction in which the curve is going);
- the **normal** unit vector, $\overrightarrow{n}$ (how the curve is turning);
- the **binormal** unit vecotr, $\overrightarrow{b}$ (how the curve is twisting).

The unit vectors $(\overrightarrow{\tau}, \overrightarrow{n}, \overrightarrow{b})$ are mutually orthogonal like $(\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ and they can be calculated by the following formulas.

- ♦ $\overrightarrow{\tau} = \dfrac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|}$
- ♦ $\overrightarrow{n} = \dfrac{\overrightarrow{\tau}'(t)}{\|\overrightarrow{\tau}'(t)\|} = \overrightarrow{b} \times \overrightarrow{\tau}$
- ♦ $\overrightarrow{b} = \overrightarrow{\tau} \times \overrightarrow{n} = \dfrac{\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)}{\|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\|}$

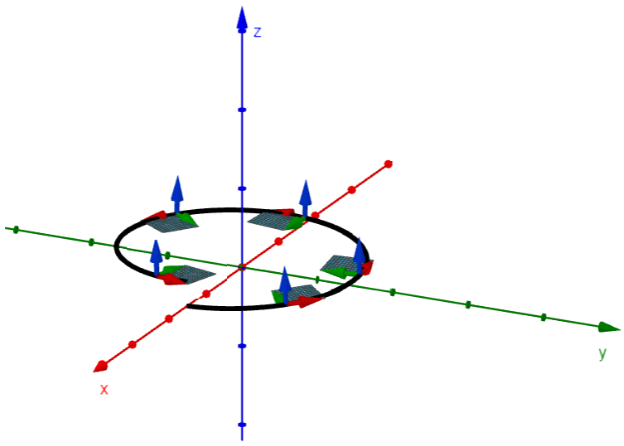


Figure 9.11: The moving trihedron

The faces of the Frenet-Serret Frame are:

- the **osculating plane** - the plane spanned by $\overrightarrow{\tau}$ and $\overrightarrow{n}$ (has as its normal $\overrightarrow{b}$). The osculating plane can be defined in the following way. Let (Γ) be a space curve and A and B be two neighboring points on (Γ). The limiting position of the plane that contains the tangent line at A and passes through the point B as $B \longrightarrow A$ is defined as the osculating plane at A.

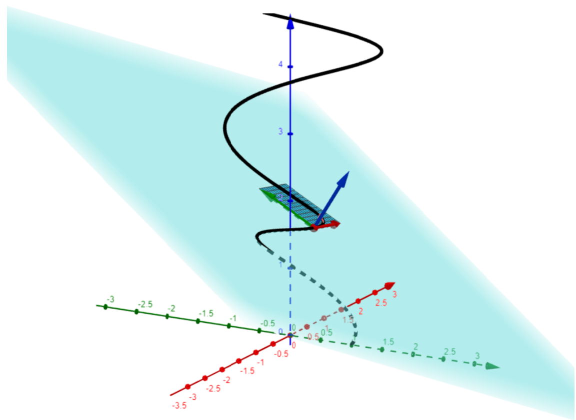


Figure 9.12: Osculating plane at a point of a space curve

- the **normal plane** - the plane spanned by $\overrightarrow{n}$ and $\overrightarrow{b}$ (has as its normal the vector $\overrightarrow{\tau}$);
- the **rectifying plane** - the plane spanned by $\overrightarrow{\tau}$ and $\overrightarrow{b}$ (has as its normal the vector $\overrightarrow{n}$).

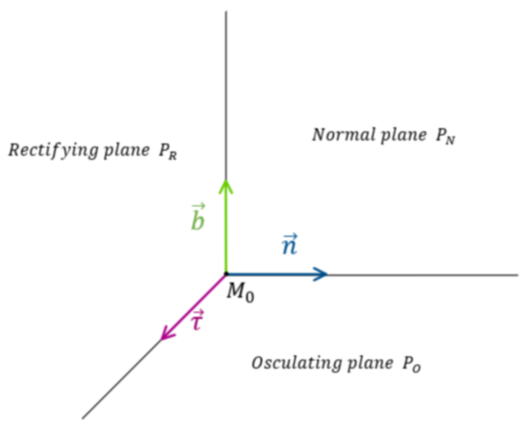


Figure 9.13: TNB frame

The equation of the osculating plane can be put in the following form:

$$(P_O): \begin{vmatrix} x-x(t_0) & y-y(t_0) & z-z(t_0) \\ x'(t_0) & y'(t_0) & z'(t_0) \\ x''(t_0) & y''(t_0) & z''(t_0) \end{vmatrix} = 0.$$

If we denote $A = \begin{vmatrix} y'(t_0) & z'(t_0) \\ y''(t_0) & z''(t_0) \end{vmatrix}$, $B = \begin{vmatrix} z'(t_0) & x'(t_0) \\ z''(t_0) & x''(t_0) \end{vmatrix}$, $C = \begin{vmatrix} x'(t_0) & y'(t_0) \\ x''(t_0) & y''(t_0) \end{vmatrix}$, we can write:

- the equation of the binormal line:

$$M_0B: \frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C}.$$

- the equation of the rectifying plane:

$$(P_R): \begin{vmatrix} x-x(t_0) & y-y(t_0) & z-z(t_0) \\ x'(t_0) & y'(t_0) & z'(t_0) \\ A & B & C \end{vmatrix} = 0.$$

- the equation of the normal line:

$$M_0N: \frac{x-x(t_0)}{\begin{vmatrix} y'(t_0) & z'(t_0) \\ B & C \end{vmatrix}} = \frac{y-y(t_0)}{\begin{vmatrix} z'(t_0) & x'(t_0) \\ C & A \end{vmatrix}} = \frac{z-z(t_0)}{\begin{vmatrix} x'(t_0) & y'(t_0) \\ A & B \end{vmatrix}}.$$

9.5 The curvature and the torsion

The curvature of a space curve shows how points "curve" in the osculating plane.

Definition 9.5. *For $\overrightarrow{\tau}$, the unit tangent vector to a regular curve, the* ***curvature*** *is defined as*

$$K = \|\frac{d\overrightarrow{\tau}}{ds}\|.$$

$\dfrac{d\overrightarrow{\tau}}{ds}$ is the derivative of the tangent unit vector with respect to the arc length s. The parametrization is quite difficult to compute, so if we want to write an expression of the curvature with respect to a parameter "t" we can write:

$$\|\frac{d\overrightarrow{\tau}}{ds}\| = \|\frac{d\overrightarrow{\tau}}{dt}\cdot\frac{dt}{ds}\| = \|\frac{\frac{d\overrightarrow{\tau}}{dt}}{\frac{ds}{dt}}\| = \frac{\|\overrightarrow{\tau}'(t)\|}{\|\overrightarrow{r}'(t)\|}.$$

The analytical expression of the curvature of the curve (Γ) at $M(x(t), y(t), z(t)) \in (\Gamma)$ is

$$K = \frac{\|\overrightarrow{r}'(t)\times\overrightarrow{r}''(t)\|}{\|\overrightarrow{r}'(t)\|^3}.$$

The **radius curvature** of (Γ) at M is $R = \dfrac{1}{K}$.

The next graph represents the curvature of a curve. The sharper the turn in the curve, the greater the curvature, and the smaller the radius of the inscribed circle.

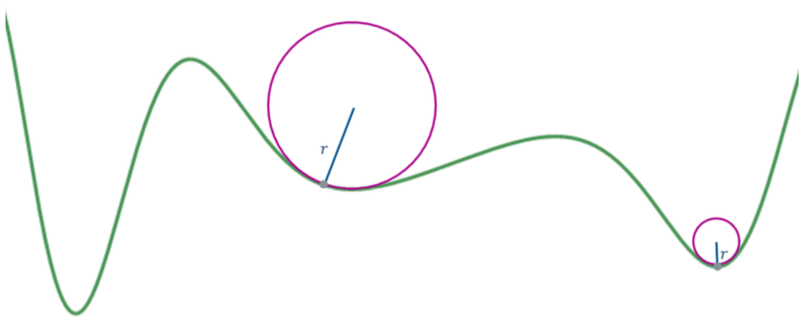


Figure 9.14: Representation of curvature and the radius curvature for a curve

The **torsion** of a curve at a point is telling us how the curve is twisting, actually how the osculating plane twists.

Definition 9.6. *The torsion is defined as*

$$T = -\frac{d\overrightarrow{b}}{ds}\cdot\overrightarrow{n}.$$

The analytical expression of the torsion of the curve (Γ) at $M(x(t), y(t), z(t)) \in (\Gamma)$ is

$$T = \frac{(\overrightarrow{r}'(t), \overrightarrow{r}''(t), \overrightarrow{r}'''(t))}{\|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\|^2}.$$

Remark 9.7. *A space curve lies in a plane if and only if the torsion is null. That means that the curve lies in the osculating plane.*

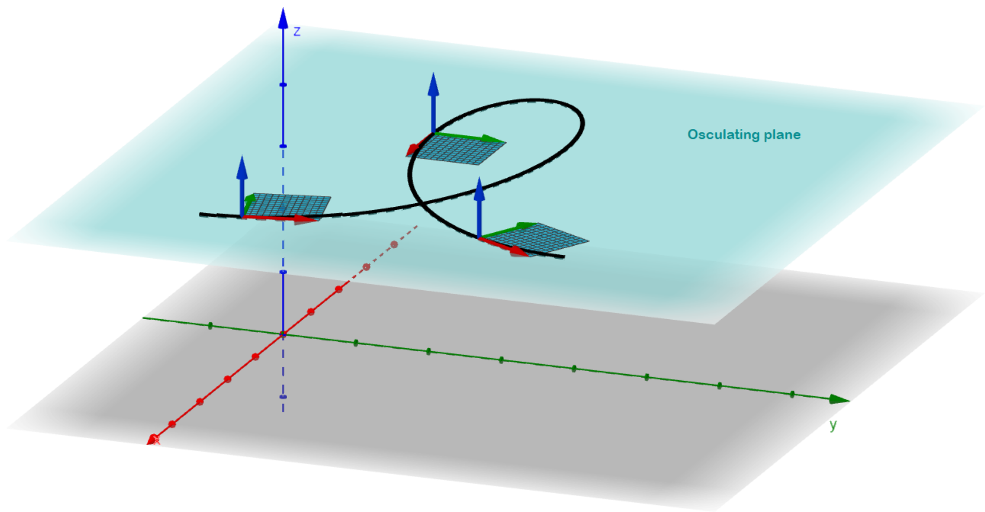


Figure 9.15: A space curve in a plane

9.6 The Frenet formulas

The unit vectors $\overrightarrow{\tau}$, $\overrightarrow{n}$, $\overrightarrow{b}$ defined previously can be express in the following equations called the Frenet formulas:

$$\begin{cases} \dfrac{d\overrightarrow{\tau}}{ds} = K \cdot \overrightarrow{n} \\ \dfrac{d\overrightarrow{n}}{ds} = -K \cdot \overrightarrow{\tau} + T \cdot \overrightarrow{b} \\ \dfrac{d\overrightarrow{b}}{ds} = -T \cdot \overrightarrow{n} \end{cases}$$

where K is the curvature and T is the torsion.

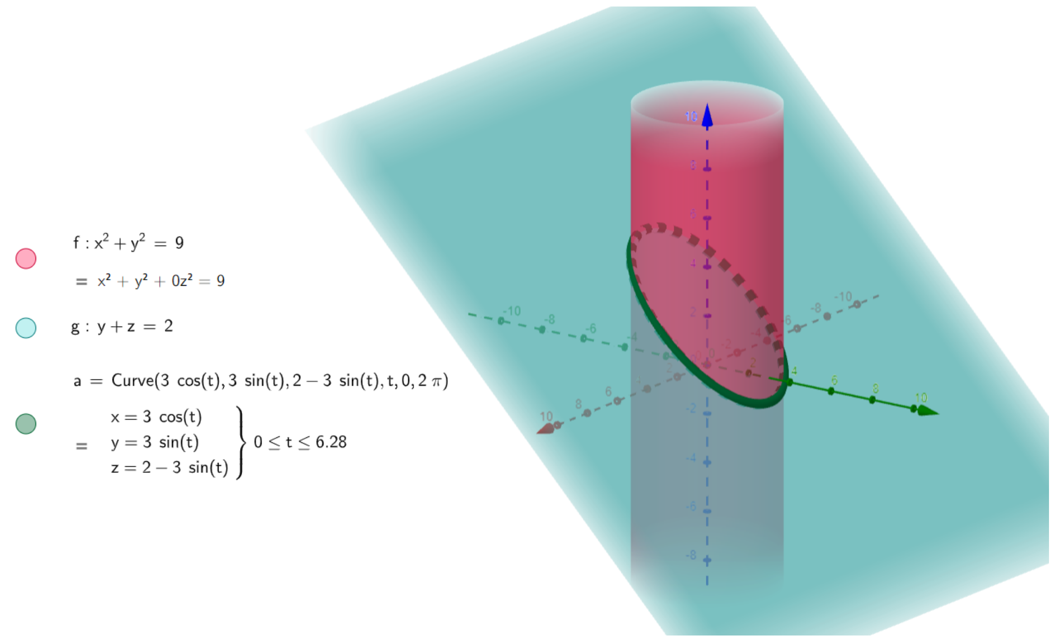
The Frenet-Serret formulas are also known as Frenet-Serret theorem, and can be stated more concisely using matrix notation:

$$\begin{pmatrix} \vec{\tau}' \\ \vec{n}' \\ \vec{b}' \end{pmatrix} = \begin{pmatrix} 0 & K & 0 \\ -K & 0 & T \\ 0 & -T & 0 \end{pmatrix} \cdot \begin{pmatrix} \vec{\tau} \\ \vec{n} \\ \vec{b} \end{pmatrix}.$$

9.7 Solved Problems

Problem 9.1. Find a vector function for the curve of intersection of $x^2 + y^2 = 9$ and $y + z = 2$.

Solution: The first surface is a cylinder having the xy-trace a circle and the second one is a plane.



The parametrization of the circle is $\begin{cases} x(t) = 3\cos t \\ y(t) = 3\sin t \end{cases}$, $t \in [0, 2\pi]$. From the equation of the plane we can calculate $z = 2 - y = 2 - 3\cos t$. So, the parametrization of the curve at the intersection of the two surfaces is

$$(\Gamma): \begin{cases} x(t) = 3\cos t \\ y(t) = 3\sin t \\ z(t) = 2 - 3\sin t \end{cases}, t \in [0, 2\pi].$$

Problem 9.2. For the points $A(2, 2, -3)$ and $B(-2, 5, -1)$ give the parametric equations for the line segment connecting A and B.

Solution: We can write the equation of the line AB passing through A and having as its director vector the vector $\overrightarrow{AB}$.

$AB : \dfrac{x-2}{-4} = \dfrac{y-2}{3} = \dfrac{z+3}{2}$. So, the parametric equations of the line are

$$AB: \begin{cases} x = -4t + 2 \\ y = 3t + 2 \\ z = 2t - 3 \end{cases}, \ t \in \mathbb{R}.$$

For the line segment $[AB]$ we will choose t in the interval $[0, 1]$ (it's obviously that if we plug in $t = 0$ we obtain the coordinates of A and for $t = 1$ we obtain the coordinates of B). So, the parametric equations of the line segment $[AB]$ are

$$[AB]: \begin{cases} x = -4t + 2 \\ y = 3t + 2 \\ z = 2t - 3 \end{cases}, \ t \in [0, 1].$$

Problem 9.3. Let $(\Gamma) : \overrightarrow{r}(t) = (\cos t, \sin t, t)$ be a helix. Determine the distance from the point $A(t = 0)$ to $B\left(t = \dfrac{\pi}{2}\right)$ on the helix.

Solution:

The derivatives of the components of the curve are: $\begin{cases} x'(t) = -\sin t \\ y'(t) = \cos t \\ z'(t) = 1 \end{cases}$.

$ds = \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \sqrt{2}\, dt.$

For $t = 0$ we obtain $A(1, 0, 0)$ and for $t = \dfrac{\pi}{2}$ we have $B\left(0, 1, \dfrac{\pi}{2}\right)$.

$$L(\Gamma) = \int_\Gamma ds = \int_0^{\frac{\pi}{2}} ds = \int_0^{\frac{\pi}{2}} \sqrt{2}\, dt = \sqrt{2}t\Big|_0^{\frac{\pi}{2}} = \frac{\sqrt{2}\pi}{2}.$$

Problem 9.4. Prove that the curve

$$(\Gamma) : \begin{cases} x = 2t^2 - 3t + 1 \\ y = t + 2 \\ z = t^2 + 3t \end{cases}, t \in \mathbb{R}$$

is a plane curve. Write the equation of this plane.

Solution:

A space curve lies in a plane if the torsion is 0. We can write the vector expression of the curve (Γ), $\overrightarrow{r}(t)$, and then we will compute the derivatives of $\overrightarrow{r}(t)$.

$\overrightarrow{r}(t) = (2t^2 - 3t + 1)\overrightarrow{i} + (t + 2)\overrightarrow{j} + (t^2 + 3t)\overrightarrow{k}$.

$\overrightarrow{r}'(t) = (4t - 3)\overrightarrow{i} + \overrightarrow{j} + (2t + 3)\overrightarrow{k}$.

$\overrightarrow{r}''(t) = 4\overrightarrow{i} + 2\overrightarrow{k}$.

$\overrightarrow{r}'''(t) = \overrightarrow{0}$.

$$T = \frac{(\overrightarrow{r}'(t), \overrightarrow{r}''(t), \overrightarrow{r}'''(t))}{\|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\|^2} = \frac{\begin{vmatrix} 4t-3 & 1 & 2t+3 \\ 4 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix}}{\|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\|^2} = 0,$$ so the curve lies in the osculating plane.

$$(P_O): \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ 4t-3 & 1 & 2t+3 \\ 4 & 0 & 2 \end{vmatrix} = 0.$$ If we choose $t=0$ we have $M_0(1,2,0) \in (\Gamma)$

and

$$(P_O): \begin{vmatrix} x-1 & y-2 & z \\ -3 & 1 & 3 \\ 4 & 0 & 2 \end{vmatrix} = 0 \Longleftrightarrow$$

$(P_O): x + 9y - 2z - 19 = 0.$

Problem 9.5. Suppose

$$(\Gamma): \begin{cases} xy = 1 \\ 2y^2 - z - 1 = 0 \end{cases}$$

is a space curve. Determine the points on (Γ) such that the binormal lines are perpendicular to the line

$$d: \begin{cases} x + y = 0 \\ -4x - z + 6 = 0 \end{cases}.$$

Write the equations of the binormal and the equation of the osculating plane at each point previously determined.

Solution:

A parametrization of the curve (Γ) can be

$$(\Gamma): \begin{cases} x = \dfrac{1}{t} \\ y = t \\ z = 2t^2 - 1 \end{cases}, \quad t \in \mathbb{R}^*.$$

The vector expression of (Γ) is:

$\overrightarrow{r}(t) = \frac{1}{t}\overrightarrow{i} + t\overrightarrow{j} + (2t^2-1)\overrightarrow{k}.$

$\overrightarrow{r}'(t) = -\dfrac{1}{t^2}\overrightarrow{i} + \overrightarrow{j} + 4t\overrightarrow{k}.$

$\overrightarrow{r}''(t) = \dfrac{2}{t^3}\overrightarrow{i} + 4\overrightarrow{k}.$

The binormal line has as director vector

$$\overrightarrow{v_b} = \overrightarrow{r}' \times \overrightarrow{r}'' = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -\dfrac{1}{t^2} & 1 & 4t \\ \dfrac{2}{t^3} & 0 & 4 \end{vmatrix} = 4\overrightarrow{i} + \frac{12}{t^2}\overrightarrow{j} - \frac{2}{t^3}\overrightarrow{k}.$$

The binormal line is perpendicular to the line d if $\overrightarrow{v_b} \perp \overrightarrow{v_d} \Longleftrightarrow \overrightarrow{v_b} \cdot \overrightarrow{v_d} = 0.$

$$\overrightarrow{v_d} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & 0 \\ -4 & 0 & -1 \end{vmatrix} = -\overrightarrow{i} + \overrightarrow{j} + 4\overrightarrow{k}.$$

$\overrightarrow{v_b} \cdot \overrightarrow{v_d} = 0 \Longleftrightarrow -4 + \dfrac{12}{t^2} - \dfrac{8}{t^3} = 0 \Longleftrightarrow t^3 - 3t + 2 = 0.$

The last equation has the solutions $t_{1,2} = 1$ and $t_3 = -2$.

So, we have two points for which the binormal is perpendicular to the given line.

- For $t = 1$ we obtain the point $M_1 = (1, 1, 1)$ and $\overrightarrow{v_b} = (4, 12, -2)$. So, the equation of the binormal is

  $b : \dfrac{x-1}{4} = \dfrac{y-1}{12} = \dfrac{z-1}{-2}.$

  The osculating plane has as its normal vector $\overrightarrow{v_b} = (4, 12, -2)$, so the equation of the osculating plane is

  $(P_O) : 4(x-1) + 12(y-1) - 2(z-1) = 0 \Longleftrightarrow$

  $(P_O) : 2x + 6y - z - 7 = 0.$

- For $t = -2$ we obtain the point $M_2\left(-\dfrac{1}{2}, -2, 7\right)$ and $\overrightarrow{v_b} = \left(4, 3, \dfrac{1}{4}\right)$. So, the equation of the binormal is

  $b : \dfrac{x+\frac{1}{2}}{4} = \dfrac{y+2}{3} = \dfrac{z-7}{\frac{1}{4}}.$

The osculating plane has as its normal vector $\overrightarrow{v_b} = (4, 3, \frac{1}{4})$, so the equation of the osculating plane is

$(P_O): 4\left(x + \frac{1}{2}\right) + 3(y + 2) + \frac{1}{4}(z - 7) = 0$

$(P_O): 16x + 12y + z + 25 = 0.$

Problem 9.6. Let $(\Gamma): \overrightarrow{r} = t\overrightarrow{i} + (1 - t^2)\overrightarrow{j} + \frac{2}{3}t^3\overrightarrow{k}$ be a space curve.

a) Determine the unit vectors of the Frenet frame at $t = 1$.

b) Write the equation of the normal at an arbitrary point of the curve.

c) Determine the curvature and the torsion of the curve at $t = 1$.

Solution:

a) $\overrightarrow{\tau} = \dfrac{\overrightarrow{r}'}{\|\overrightarrow{r}'\|}$

$\overrightarrow{b} = \dfrac{\overrightarrow{r}' \times \overrightarrow{r}''}{\|\overrightarrow{r}' \times \overrightarrow{r}''\|}$

$\overrightarrow{n} = \overrightarrow{b} \times \overrightarrow{\tau}$

The derivatives of $\overrightarrow{r}$ are:

$\overrightarrow{r}'(t) = \overrightarrow{i} - 2t\overrightarrow{j} + 2t^2\overrightarrow{k}$

$\overrightarrow{r}''(t) = -2\overrightarrow{j} + 4t\overrightarrow{k}.$

At $t = 1$ we obtain

$\overrightarrow{r}'(1) = \overrightarrow{i} - 2\overrightarrow{j} + 2\overrightarrow{k}$

$\overrightarrow{r}''(1) = -2\overrightarrow{j} + 4\overrightarrow{k}.$

$\overrightarrow{\tau} = \dfrac{\overrightarrow{r}'(1)}{\|\overrightarrow{r}'(1)\|} = \dfrac{\overrightarrow{i} - 2\overrightarrow{j} + 2\overrightarrow{k}}{\sqrt{1 + 4 + 4}} = \dfrac{1}{3}(\overrightarrow{i} - 2\overrightarrow{j} + 2\overrightarrow{k})$

$$\overrightarrow{r}' \times \overrightarrow{r}'' = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -2 & 2 \\ 0 & -2 & 4 \end{vmatrix} = -4\overrightarrow{i} - 4\overrightarrow{j} - 2\overrightarrow{k}.$$

$$\overrightarrow{b} = -\frac{4\overrightarrow{i} + 4\overrightarrow{j} + 2\overrightarrow{k}}{\sqrt{16+16+4}} = -\frac{1}{3}(2\overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k}).$$

$$\overrightarrow{n} = \overrightarrow{b} \times \overrightarrow{\tau} = -\frac{1}{9}\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = -\frac{1}{9}(6\overrightarrow{i} - 3\overrightarrow{j} - 6\overrightarrow{k}) = \frac{1}{3}(2\overrightarrow{i} - \overrightarrow{j} - 2\overrightarrow{k}).$$

b) The corresponding point on the curve (Γ) at $t = 1$ is $A\left(1, 0, \frac{2}{3}\right)$, therefore the normal line has the equations

$$n : \frac{x-1}{-\frac{2}{3}} = \frac{y}{\frac{1}{3}} = \frac{z-\frac{2}{3}}{\frac{2}{3}} \Longleftrightarrow n : \frac{x-1}{-2} = y = \frac{z-\frac{2}{3}}{2}.$$

The rectifying plane has as its normal the direction of $\overrightarrow{n}$, so the equation of the rectifying plane is:

$$(P_R) : -\frac{2}{3}(x-1) + \frac{1}{3}y + \frac{2}{3}(z - \frac{2}{3}) = 0 \Longleftrightarrow$$

$$(P_R) : -6x + 3y + 6z + 2 = 0.$$

c) The curvature at $t = 1$ is $K = \left.\frac{\|\overrightarrow{r}' \times \overrightarrow{r}''\|}{\|\overrightarrow{r}'\|^3}\right|_{t=1} = \frac{6}{3^3} = \frac{2}{9}$.

The torsion at $t = 1$ is $T = \left.\frac{(\overrightarrow{r}', \overrightarrow{r}'', \overrightarrow{r}''')}{\|\overrightarrow{r}' \times \overrightarrow{r}''\|^2}\right|_{t=1}$.

$\overrightarrow{r}'''(t) = 4\overrightarrow{k} \Longrightarrow \overrightarrow{r}'''(1) = 4\overrightarrow{k}$.

$$(\overrightarrow{r}', \overrightarrow{r}'', \overrightarrow{r}''')|_{t=1} = \begin{vmatrix} 1 & -2 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & 4 \end{vmatrix} = -8 \Longrightarrow T = \frac{-8}{6^2} = -\frac{2}{9}.$$

9.8 Problems

Problem 9.7. Write the equation of the normal plane and the equation of the tangent of (Γ) at the given point:

a) $(\Gamma) : \overrightarrow{r} = 2t\overrightarrow{i} + \frac{2}{t}\overrightarrow{j} + t^2\overrightarrow{k}$, $t = 2$;

b) $(\Gamma) : \begin{cases} x = 3t \\ y = 2t^3 \\ z = -t^2 \end{cases}$, $M_0(6, 16, -4)$.

Problem 9.8. Determine the length of the arc curve (Γ):

a) $(\Gamma) : \begin{cases} x = at \\ y = \sqrt{3ab}t^2 \\ z = 2bt^3, \end{cases}$, $0 \leqslant t \leqslant 1$;

b) $(\Gamma) : \overrightarrow{r} = a\cos t\,\overrightarrow{i} + a\sin t\,\overrightarrow{j} + bt\overrightarrow{k}$, $0 \leqslant t \leqslant 2$.

Problem 9.9. Determine the equation of the tangent line at $A(m, m, 2m^2)$ and the equation of the normal plane at an arbitrary point of $(\Gamma) : \begin{cases} z = x^2 + y^2 \\ x = y \end{cases}$.

Problem 9.10. Write the equation of the osculating plane of the space curve

$$(\Gamma) : \begin{cases} y^2 = x \\ x^2 = z \end{cases}$$

at $M(1, 1, 1)$.

Problem 9.11. Let $(\Gamma) : \overrightarrow{r} = t\overrightarrow{i} + (1 - t^2)\overrightarrow{j} + \frac{2}{3}t^3\overrightarrow{k}$ be a space curve.

a) Determine the unit vectors of the Frenet frame at $t = 1$.

b) Write the equation of the normal at an arbitrary point of the curve.

c) Determine the curvature and the torsion of the curve at $t = 1$.

Problem 9.12. Determine the curvature and the torsion of the curve

$$(\Gamma) : \begin{cases} x = \cos t \\ y = \sin t \\ z = \cos 2t, \end{cases}$$

at $M\left(t = \frac{\pi}{2}\right)$.

Problem 9.13. Let $(\Gamma) : \overrightarrow{r} = t\overrightarrow{i} + \frac{1}{2}t^2\overrightarrow{j} + \frac{1}{6}t^3\overrightarrow{k}$ be a space curve.

a) Determine the element of arc.

b) Determine the unit vectors of the tangent, normal and binormal at $t = 1$.

c) Write the equations of the rectifying plane and osculating plane at $t = 1$.

d) Determine the curvature and the torsion of (Γ) at $t = 1$.

Problem 9.14. Determine the points of the curve

$$(\Gamma) : \overrightarrow{r} = (2t - 1)\overrightarrow{i} + t^3\overrightarrow{j} + (1 - t^2)\overrightarrow{k}$$

such that the osculating plane of the curve at these points is perpendicular to the plane $(P) : 7x - 12y + 5z - 4 = 0$.

Problem 9.15. For each of the following curves determine the unit vectors of the moving trihedron, the curvature, the torsion, the equations of the osculating plane, the normal plane, the equations of the normal line and the tangent line at the given point:

a) $(\Gamma_1) : \overrightarrow{r} = (3t^2 - 2)\overrightarrow{i} + t^3\overrightarrow{j} + (1-t)\overrightarrow{k},\ \ M(t = 2).$

b) $(\Gamma_2) : \begin{cases} x = t^3 - 2t^2 \\ y = 3t + 2 \\ z = t^2 - 5 \end{cases},\ \ M(-1, 5, -4).$

c) $(\Gamma_3) : \overrightarrow{r} = 4\cos t\,\overrightarrow{i} + 2\sin t\,\overrightarrow{j} + 2t\,\overrightarrow{k},\ \ M\left(t = \frac{\pi}{3}\right).$

Problem 9.16. Determine the length of the arc curve (Γ):

a) $(\Gamma) : \begin{cases} x = e^t \cos t \\ y = e^t \sin t \\ z = e^t, \end{cases},\ t \in [0, \frac{\pi}{2}];$

b) $(\Gamma) : \begin{cases} y = \dfrac{x^2}{2} \\ z = \dfrac{x^3}{6}, \end{cases},\ t \in [0, 6].$

Problem 9.17. Determine the points of the curve

$$(\Gamma) : \begin{cases} xz = 1 \\ y = \ln z \end{cases}$$

such that the principal normal of the curve at these points is parallel to the plane $(P) : 5x + 2y - 5z = 1$.

Problem 9.18. Determine the curvature and the torsion of the curve

$$(\Gamma) : \begin{cases} x = 2t \\ y = \ln t \\ z = t^2, \end{cases},\ \ t > 0$$

at $t = 1$.

10

Surfaces

10.1 Analytic representation of surfaces

In $\mathbb{R}^3$ a single equation in x, y, z represents a surface.

A surface can be represented by:

- The **implicit** equation of the surface

$$(S) : F(x, y, z) = 0.$$

- The **explicit** equation of the surface

$$(S) : z = z(x, y).$$

- The **parametric** equations of the surface

$$(S) : \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} , \ (u, v) \in D \subset \mathbb{R}^2.$$

- The **vector** equation of the surface

$$(S) : \overrightarrow{r} = \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k},\ (u,v) \in D \subset \mathbb{R}^2$$

If the surface (S) is represented by the parametric equations then the point $M_0(u_0, v_0) \in (S)$ is called an **ordinary** point if the rank of the matrix

$$\left.\begin{pmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix}\right|_{(u_0,v_0)}$$

is 2 or, equivalently, $\overrightarrow{r}'_u(u_0, v_0)$ and $\overrightarrow{r}'_v(u_0, v_0)$ are linearly independent.

If all the points of the surface (S) are ordinary points, then the surface is called a **regular** surface.

10.2 Curves on a surface

Let $(S) : \overrightarrow{r} = \overrightarrow{r}(u,v),\ (u,v) \in \mathbb{R}^2$ be a surface.

If $u, v : I \to \mathbb{R}$ are single valued functions $u = u(t),\ v = v(t)$, then $\overrightarrow{r} = \overrightarrow{r}(u(t), v(t))$ is a curve lying on the surface (S).

Examples of curves on a surface

1. The **circular helix** lies on the cylinder $(S) : \begin{cases} x = r\cos v \\ y = r\sin v \\ z = u \end{cases}$, r the radius been a positive real constant.

   Let us make $\begin{cases} v = t \\ u = ct \end{cases}$, c a constant.

The circular helix has the parametric equations $(\Gamma):\begin{cases} x = r\cos t \\ y = r\sin t \\ z = ct \end{cases}$.

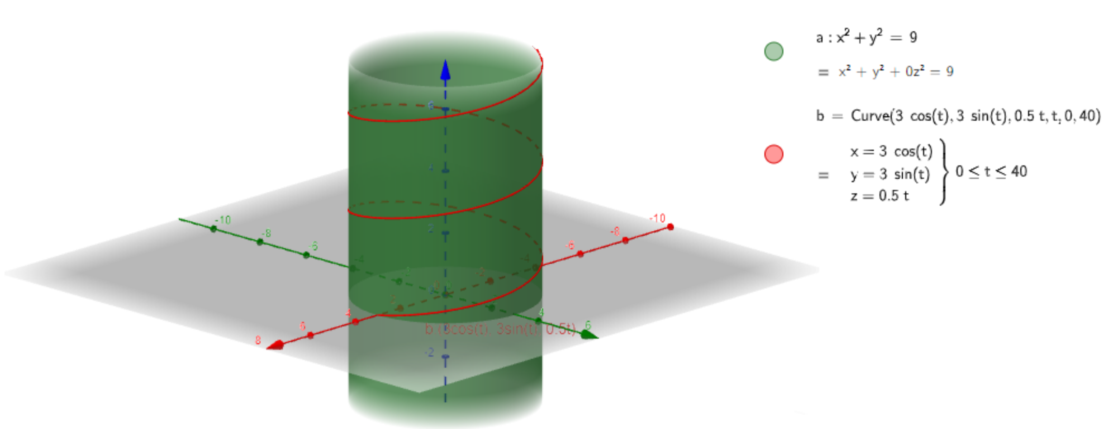


Figure 10.1: The circular helix

2. When consider $u = u_0$ and $v = v_0$, one obtain two curves on the surface, and (u_0, v_0) are called the **curvilinear coordinates** of the point M_0.

   Let $(S): x^2+y^2+z^2 = r^2$ be a sphere and $M_0(x_0, y_0, z_0) \in (S)$. The parametric equations of the sphere are:

   $$(S):\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases},\ \theta \in [0, 2\pi],\ \varphi \in [0, \pi].$$

   If we consider $\varphi =$ constant, the curves on the sphere, in geographic terms, are the parallels, while if we consider $\theta =$ constant, the curves represent the meridians.

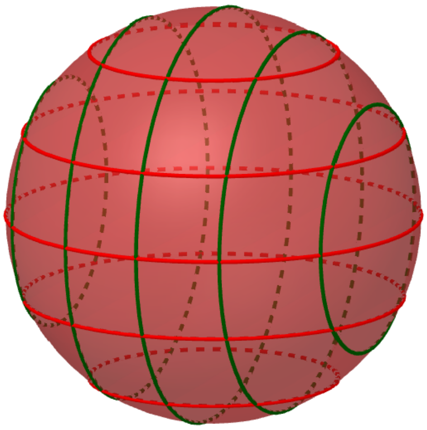


Figure 10.2: Meridians and parallels on a sphere

10.3 The tangent plane and the normal line to a surface

Let $(S) : \overrightarrow{r} = \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}$, $(u,v) \in D \subset \mathbb{R}^2$ be a regular surface.

The plane passing through $M(u_0,v_0) \in (S)$ and having as parallel directions $\overrightarrow{r}'_u(u_0,v_0)$ and $\overrightarrow{r}'_v(u_0,v_0)$ is called the **tangent plane** to the surface S at M_0.

The equation of the tangent plane is:

$$(P_{tg}) : \begin{vmatrix} x - x(u_0,v_0) & y - y(u_0,v_0) & z - z(u_0,v_0) \\ x'_u(u_0,v_0) & y'_u(u_0,v_0) & z'_u(u_0,v_0) \\ x'_v(u_0,v_0) & y'_v(u_0,v_0) & z'_v(u_0,v_0) \end{vmatrix} = 0.$$

Let us denote $A = \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix}$, $B = \begin{vmatrix} z'_u & x'_u \\ z'_v & x'_v \end{vmatrix}$, $C = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}$, evaluated at (u_0,v_0).

The **normal** to the surface (S) at M_0 is the line passing through M_0 perpendicular to the tangent plane. Its equation is:

$$n : \frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C},$$

where $x_0 = x(u_0, v_0)$, $y_0 = y(u_0, v_0)$, $z_0 = z(u_0, v_0)$.

Remark 10.1. *If the surface is given by the implicit equation* $(S) : F(x, y, z) = 0$ *then the tangent plane is:*

$$(P_{tg}) : F'_x(x_0, y_0, z_0)(x - x_0) + F'_y(x_0, y_0, z_0)(y - y_0) + F'_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The equation of the normal line is:

$$n : \frac{x - x_0}{F'_x(x_0, y_0, z_0)} = \frac{y - y_0}{F'_y(x_0, y_0, z_0)} = \frac{z - z_0}{F'_z(x_0, y_0, z_0)}$$

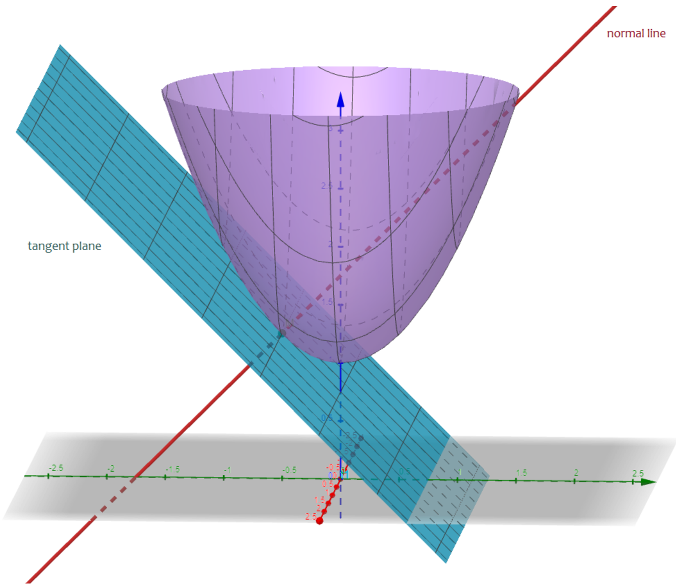


Figure 10.3: Tangent plane and normal line to a surface

10.4 The first fundamental quadratic form

Length of curves on a surface

Let $(S): \overrightarrow{r} = \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}$, $(u,v) \in D \subset \mathbb{R}^2$ be a regular surface and $(\Gamma): \overrightarrow{r} = \overrightarrow{r}(u(t), v(t))$ a regular curve on the surface (S).

We can define the length of an arc of this curve as:

$$s(t) = \int_{t_1}^{t_2} \sqrt{\overrightarrow{r}'^2(\tau)}\, d\tau.$$

$\overrightarrow{r}'(t) = \overrightarrow{r}'_u \cdot u'(t) + \overrightarrow{r}'_v \cdot v'(t)$.

Using Gauss's notations:

$E = \overrightarrow{r}'^2_u = x'^2_u + y'^2_u + z'^2_u$

$F = \overrightarrow{r}'_u \overrightarrow{r}'_v = x'_u x'_v + y'_u y'_v + z'_u z'_v$

$G = \overrightarrow{r}'^2_v = x'^2_v + y'^2_v + z'^2_v$

we obtain

$$ds = \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt$$

Definition 10.2. *The quadratic form*

$$Edu^2 + 2Fdudv + Gdv^2$$

is called ***the first fundamental quadratic form*** *of the surface.*

The length of the arc M_1M_2 of the curve (Γ) corresponding to the values t_1 and t_2 of the parameter t is:

$$l(\overset{\frown}{M_1M_2}) = \int_{t_1}^{t_2} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt.$$

Remark 10.3. *If the surface is given in the explicit form $(S): z = z(x,y)$, and $p = z'_x$ and $q = z'_y$, the the first fundamental quadratic form of the surface is*

$$ds^2 = (1+p^2)dx^2 + 2pqdxdy + (1+q^2)dy^2.$$

Angle measurement on Surfaces

Let $(S) : \overrightarrow{r} = \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}$, $(u,v) \in D \subset \mathbb{R}^2$ a regular surface and

$(\Gamma_1) : \overrightarrow{r_1} = \overrightarrow{r_1}(u(t), v(t))$

$(\Gamma_2) : \overrightarrow{r_2} = \overrightarrow{r_2}(u(t), v(t))$

two regular curves on the surface (S), and $\overrightarrow{\tau_1}$, $\overrightarrow{\tau_2}$ the tangent unit vectors of (Γ_1) and (Γ_2) respectively, at their common point M_0.

Let $d\overrightarrow{r}$, du and dv the differential along (Γ_1) and $\delta\overrightarrow{r}$, δu and δv the differential along (Γ_2).

Definition 10.4. *The angle between the tangent vectors* $\overrightarrow{\tau_1}$ *and* $\overrightarrow{\tau_2}$ *at the point* $M_0 \in (S)$ *is called the angle of the curves* (Γ_1) *and* (Γ_2)*.*

$$\cos\theta = \cos\sphericalangle((\Gamma_1),(\Gamma_2)) = \frac{Edu\delta u + F(du\delta v + \delta u dv) + Gdv\delta v}{\sqrt{Edu^2 + 2Fdudv + Gdv^2}\sqrt{E\delta u^2 + 2F\delta u\delta v + G\delta v^2}}.$$

Two curves are **orthogonal** if

$$Edu\delta u + F(du\delta v + \delta u dv) + Gdv\delta v = 0.$$

Area of a surface

Let $(S) : \overrightarrow{r} = \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}$, $(u,v) \in D \subset \mathbb{R}^2$ a regular surface.

The **element of area** of the surface (S) is

$$d\sigma = \sqrt{EG - F^2}dudv.$$

The area of the surface is

$$\iint_D d\sigma = \iint_D \sqrt{EG - F^2}dudv,$$

where D is the domain in which u and v vary.

Remark 10.5. • *If the surface is given in the explicit form $(S) : z = z(x, y)$, and $p = z'_x$ and $q = z'_y$, the element of the area is*

$$d\sigma = \sqrt{1 + p^2 + q^2}dxdy.$$

• *If the surface is given in the implicit form $(S) : F(x, y, z) = 0$, the element of the area is*

$$d\sigma = \frac{1}{|F'_z|}\sqrt{F'^2_x + F'^2_y + F'^2_z}dxdy.$$

10.5 Solved Problems

Problem 10.1. Let the surface $(S) : \begin{cases} x = u + v \\ y = u - v \\ z = uv \end{cases}, (u, v) \in \mathbb{R}^2.$

a) Determine the coordinates of the points $A(u = 2, v = 1)$, $B(u = 1, v = 2)$.

b) Check if $M(4, 2, 3)$ and $N(1, 4, -2)$ are on the surface (S).

c) Determine the cartesian equation of the surface.

Solution:

a) For $u = 2$ and $v = 1$ we obtain $x = 3$, $y = 1$, $z = 2 \Longrightarrow A(3, 1, 2)$.

For $u = 1$ and $v = 2$ we obtain $x = 3$, $y = -1$, $z = 2 \Longrightarrow B(3, -1, 2)$.

b) $M \in (S) \iff$ the system $\begin{cases} u+v=4 \\ u-v=2 \\ u \cdot v = 3. \end{cases}$ is consistent. Adding the first two equation we obtain $2u = 6 \Longrightarrow u = 3$. We can determine $v = 1$ and we verify in the third equation if $u \cdot v = 3 \iff 3 \cdot 1 = 3$ which is correct, so $M \in (S)$.

$N \in (S) \iff$ the system $\begin{cases} u+v=1 \\ u-v=4 \\ u \cdot v = -2. \end{cases}$ is consistent. Adding the first two equation we obtain $2u = 5 \Longrightarrow u = \frac{5}{2}$. We can easily determine $v = -\frac{3}{2}$. Let's verify the third equation $u \cdot v = \frac{5}{2} \cdot \left(-\frac{3}{2}\right) = -\frac{15}{4} \neq -2$, so $N \notin (S)$.

c) In order to obtain the explicit or the implicit equation of the surface we need to eliminate u and v form the parametric equations of the surface.

By adding and then subtracting the first two equations we obtain $x + y = 2u$ and $x - y = 2v$. Replacing $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$ in the third equation we obtain $z = \frac{(x+y)(x-y)}{4} \iff 4z = x^2 - y^2$ which is the equation of a parabolic hyperboloid. The explicit equation of the surface is

$$(S) : z(x, y) = \frac{1}{4}(x^2 - y^2).$$

Problem 10.2. Let $(S) : \begin{cases} x = ue^v \\ y = ue^{-v} \\ z = 4uv \end{cases}, (u, v) \in \mathbb{R}^2$ be a surface.

a) Determine the equation of the tangent plane of (S) at the point $M(u = 2, v = 0)$.

b) Write the equation of the normal line at the point M.

c) Determine the unit vector of the normal line.

Solution:

a) The coordinates of the point M are $x = 2$, $y = 2$, $x = 0$, $M(2, 2, 0)$. We calculate the partial derivatives of the functions:

$$\begin{cases} x'_u(u,v) = e^v \\ y'_u(u,v) = e^{-v} \\ z'_u(u,v) = 4v \end{cases} \Longrightarrow \begin{cases} x'_u(2,0) = 1 \\ y'_u(2,0) = 1 \\ z'_u(2,0) = 0 \end{cases}.$$

$$\begin{cases} x'_v(u,v) = ue^v \\ y'_v(u,v) = -ue^{-v} \\ z'_v(u,v) = 4u \end{cases} \Longrightarrow \begin{cases} x'_v(2,0) = 2 \\ y'_v(2,0) = -2 \\ z'_v(2,0) = 8 \end{cases}.$$

The equation of the tangent plane is

$$(P_{tg}) : \begin{vmatrix} x-2 & y-2 & z \\ 1 & 1 & 0 \\ 2 & -2 & 8 \end{vmatrix} = 0 \Longleftrightarrow$$

$(P_{tg}) : 2x - 2y - z = 0.$

b) The normal line is perpendicular to the tangent plane, so the direction of the normal line is $\overrightarrow{v} = \overrightarrow{n}_{P_{tg}} = 2\overrightarrow{i} - 2\overrightarrow{j} - \overrightarrow{k}$.

The equation of the normal line is:

$$n : \frac{x-2}{2} = \frac{y-2}{-2} = \frac{z}{-1}.$$

c) The unit vector of the normal line is $\overrightarrow{n} = \dfrac{2\overrightarrow{i} - 2\overrightarrow{j} - \overrightarrow{k}}{\|2\overrightarrow{i} - 2\overrightarrow{j} - \overrightarrow{k}\|} = \dfrac{1}{3}(2\overrightarrow{i} - 2\overrightarrow{j} - \overrightarrow{k})$.

Problem 10.3. Write the equation of the tangent plane at $M(x_0, y_0, z_0)$ to the surface $(S) : z = e^{\frac{y}{x}}$.

Solution:

The equation of the surface is given in the explicit form.

$p = z'_x(x, y) = -\dfrac{y}{x^2}e^{\frac{y}{x}}$.

$q = z'_y(x, y) = \dfrac{1}{x}e^{\frac{y}{x}}$.

The equation of the tangent plane is:

$(P_{tg}) : z - z_0 = -\dfrac{y_0}{x_0^2}e^{\frac{y_0}{x_0}}(x - x_0) + \dfrac{1}{x_0}e^{\frac{y_0}{x_0}}(y - y_0)$.

Multiplying by x_0^2 the equation and knowing that $z_0 = e^{\frac{y_0}{x_0}}$ we obtain:

$(P_{tg}) : x_0^2(z - z_0) = -y_0z_0(x - x_0) + x_0z_0(y - y_0)$.

Problem 10.4. Determine the length of the arc of the curve $u = 0$ on the surface $(S) : \overrightarrow{r}(u, v) = (u^2+v)\overrightarrow{i}+(u+v^2)\overrightarrow{j}+(u+v)\overrightarrow{k}$ between the points $M_1(u = 0, v = 0)$ and $M_2(u = 0, v = 1)$.

Solution:

We calculate the partial derivatives of the functions $x = x(u, v) = u^2 + v$, $y = y(u, v) = u + v^2$ and $z = z(u, v) = u + v$.

$$\begin{cases} x'_u(u, v) = 2u \\ y'_u(u, v) = 1 \\ z'_u(u, v) = 1 \end{cases} ; \begin{cases} x'_v(u, v) = 1 \\ y'_v(u, v) = 2v \\ z'_v(u, v) = 1 \end{cases} .$$

$E = 4u^2 + 1 + 1 = 4u^2 + 2$;

$F = 2u + 2v + 1$;

$G = 1 + 4v^2 + 1 = 4v^2 + 2$.

$ds^2 = 2(2u^2 + 1)du^2 + 2(2(u + v) + 1)dudv + 2(2v^2 + 1)dv^2$.

Since $u = 0 \Longrightarrow du = 0 \Longrightarrow ds^2 = 2(2v^2+1)dv^2$.

$$\begin{aligned} l(M_1M_2) = I &= \int_0^1 \sqrt{4v^2+2}dv = 2\int_0^1 \sqrt{v^2+\frac{1}{2}}dv = 2\int_0^1 v'\sqrt{v^2+\frac{1}{2}}dv \\ &= 2v\sqrt{v^2+\frac{1}{2}}\Bigg|_0^1 - 2\int_0^1 \frac{v^2+\frac{1}{2}-\frac{1}{2}}{\sqrt{v^2+\frac{1}{2}}}dv \\ &= 2\sqrt{\frac{3}{2}} - 2I + \int_0^1 \frac{1}{\sqrt{v^2+\frac{1}{2}}}dv \\ &= \sqrt{6} - 2I + \ln\left(v+\sqrt{v^2+\frac{1}{2}}\right)\Bigg|_0^1 \Longrightarrow \end{aligned}$$

$$3I = \sqrt{6} + \ln\left(1+\sqrt{\frac{3}{2}}\right) - \ln\sqrt{\frac{1}{2}} \Longrightarrow$$

$$l(M_1M_2) = I = \frac{\sqrt{6}}{3} + \frac{1}{3}\ln\left(\sqrt{2}+\sqrt{3}\right).$$

Problem 10.5. Determine the angle between the curves $v = 6u$ and $v = -6u$ which lie on the cylinder $(S): x^2+y^2 = 9$.

Solution:

The parametrisation of the cylinder is $(S): \begin{cases} x = 3\cos v \\ y = 3\sin v \\ z = u \end{cases}$, $v \in [0, 2\pi], u \in \mathbb{R}$.

The partial derivatives of x, y, z are $\begin{cases} x'_u(u,v) = 0 \\ y'_u(u,v) = 0 \\ z'_u(u,v) = 1 \end{cases}$; $\begin{cases} x'_v(u,v) = -3\sin v \\ y'_v(u,v) = 3\cos v \\ z'_v(u,v) = 0 \end{cases}$.

$E = 1;\ F = 0;\ G = 9\sin^2 v + 9\cos^2 v = 9$.

The first quadratic form is:

$ds^2 = Edu^2 + 2Fdudv + Gdv^2 = du^2 + 9dv^2$.

The intersection point of the curves is $\begin{cases} v = 6u \\ v = -6u \end{cases} \iff u = v = 0 \Longrightarrow M(3, 0, 0)$.

For $(\Gamma_1): \ v = 6u \Longrightarrow dv = 6du$.

For $(\Gamma_2): \ v = -6u \Longrightarrow \delta v = -6\delta u$.

$$\begin{aligned} \cos \sphericalangle((\Gamma_1), (\Gamma_2)) &= \frac{Edu\delta u + F(du\delta v + \delta u dv) + Gdv\delta v}{\sqrt{Edu^2 + 2Fdudv + Gdv^2}\sqrt{E\delta u^2 + 2F\delta u\delta v + G\delta v^2}} \\ &= \frac{1du\delta u + 0 + 9 \cdot 6 \cdot (-6)du\delta u}{\sqrt{du^2 + 9 \cdot 36du^2} \cdot \sqrt{\delta u^2 + 9 \cdot 36\delta u^2}} \\ &= -\frac{323du\delta u}{325du\delta u} = -\frac{323}{325} \Longrightarrow \end{aligned}$$

$$\sphericalangle((\Gamma_1), (\Gamma_2)) = \pi - \arccos \frac{323}{325}.$$

Problem 10.6. Prove that the curves $(\Gamma_1): u - e^v = 0$ and $(\Gamma_2): u^2 + u + 1 - e^{-v} = 0$ which lie on the surface $(S): \begin{cases} x = u \cos v \\ y = u \sin v \\ z = u + v \end{cases}$, are orthogonal.

Solution:

The partial derivatives of x, y, z are $\begin{cases} x'_u(u, v) = \cos v \\ y'_u(u, v) = \sin v \\ z'_u(u, v) = 1 \end{cases}$; $\begin{cases} x'_v(u, v) = -u \sin v \\ y'_v(u, v) = u \cos v \\ z'_v(u, v) = 1. \end{cases}$

$E = \cos^2 v + \sin^2 v + 1 = 2$.

$F = -u \cos v \sin v + u \sin v \cos v + 1 = 1$.

$G = u^2 \sin^2 v + u^2 \cos^2 v + 1 = u^2 + 1$.

The first quadratic form is:

$ds^2 = Edu^2 + 2Fdudv + Gdv^2 = 2du^2 + 2dudv + (u^2+1)dv^2.$

For $(\Gamma_1): \; u = e^v \Longrightarrow du = e^v dv.$

For $(\Gamma_2): \; u^2 + u + 1 = e^{-v} \Longrightarrow (2u+1)\delta u = -e^{-v}\delta v \Longrightarrow \delta u = -\dfrac{1}{e^v(2e^v+1)}\delta v,$

$G = u^2 + 1 = e^{-v} - u = e^{-v} - e^v.$

The curves are orthogonal if

$$Edu\delta u + F(du\delta v + \delta u dv) + Gdv\delta v = 0 \Longleftrightarrow$$
$$2e^v dv\left(-\frac{1}{e^v(2e^v+1)}\delta v\right) + e^v dv\delta v - \frac{1}{e^v(2e^v+1)}\delta v dv + (e^{-v} - e^v)dv\delta v = 0 \Longleftrightarrow$$
$$dv\delta v\left(-\frac{2}{2e^v+1} + e^v - \frac{1}{e^v(2e^v+1)} + e^{-v} - e^v\right) = 0 \Longleftrightarrow$$
$$\frac{-2e^v}{e^v(2e^v+1)} - \frac{1}{e^v(2e^v+1)} + \frac{2e^v+1}{e^v(2e^v+1)} = 0 \Longleftrightarrow$$
$$\frac{-2e^v - 1 + 2e^v + 1}{e^v(2e^v+1)} = 0$$
which is true, so the curves are orthogonal.

Problem 10.7. Let $(S): \begin{cases} x = u^2 + v^2 \\ y = u^2 - v^2 \\ z = uv \end{cases}$ be a surface.

a) Write the first fundamental quadratic form of (S).

b) Determine the element of arc for the curve $(\Gamma): v = 2u$ which lies on the surface.

c) Determine the length of the arc M_1M_2 on the curve (Γ) where $M_1(u=1)$, $M_2(u=2)$.

d) Determine the element of area for the surface (S).

Solution:

a) The partial derivatives of x, y, z are: $\begin{cases} x'_u(u,v) = 2u \\ y'_u(u,v) = 2u \\ z'_u(u,v) = v \end{cases}$; $\begin{cases} x'_v(u,v) = 2v \\ y'_v(u,v) = -2v \\ z'_v(u,v) = u \end{cases}$.

$E = 4u^2 + 4u^2 + v^2 = 8u^2 + v^2;$

$F = 4uv - 4uv + uv = uv;$

$G = 4v^2 + 4v^2 + u^2 = 8v^2 + u^2.$

The first quadratic form is:

$ds^2 = Edu^2 + 2Fdudv + Gdv^2 = (8u^2 + v^2)du^2 + 2uvdudv + (8v^2 + u^2)dv^2.$

b) $(\Gamma) : v = 2u \Longrightarrow dv = 2du.$

$ds^2 = (8u^2 + 4u^2)du^2 + 2u2udu2du + (8 \cdot 4u^2 + u^2)4du^2 \Longrightarrow ds^2 = 152u^2du \Longrightarrow$
$ds = \sqrt{152}udu.$

c) $l(M_1M_2) = \int_1^2 ds = \int_1^2 \sqrt{152}u\, du = \sqrt{38}u^2\Big|_1^2 = \sqrt{38}(4-1) = 3\sqrt{38}.$

d) The element of the area is

$$\begin{aligned} d\sigma = \sqrt{EG - F^2}dudv &= \sqrt{(8u^2 + v^2)(8v^2 + u^2) - u^2v^2}dudv \\ &= 2\sqrt{2}\sqrt{u^4 + 8u^2v^2 + v^4}dudv. \end{aligned}$$

10.6 Problems

Problem 10.8. Let $(S) : z = x^2 - y^2 + 2y - 4x + 5$ be a surface. Determine:

a) The tangent plane and the normal line to the surface at $M(1, -2, -6)$.

b) The first fundamental form of the surface (S).

c) The element of area of the surface (S).

Problem 10.9. Determine the length of the arc of the curve $v = \ln\left(u + \sqrt{u^2+9}\right)$ on the surface $(S) : \overrightarrow{r}(u,v) = u\cos v\,\overrightarrow{i} + u\sin v\,\overrightarrow{j} + 3v\,\overrightarrow{k}$ between the points $M_1(u=1, v=2)$ and $M_2(u=2, v=3)$.

Problem 10.10. Determine the element of area of the surface

$$(S) : \overrightarrow{r}(u,v) = u\,\overrightarrow{i} + v\,\overrightarrow{j} + uv\,\overrightarrow{k}.$$

Problem 10.11. Determine the area of the sphere.

Problem 10.12. Write the cartesian form of the surface

$$(S) : \overrightarrow{r}(u,v) = u^3\,\overrightarrow{i} + uv\,\overrightarrow{j} + (3u+v^2)\,\overrightarrow{k}.$$

Determine the first fundamental form of the surface. Write the equations of the tangent plane and normal line of the surface (S) at $M(1,0,3)$.

Problem 10.13. Let $(S) : \overrightarrow{r}(u,v) = (u-v)\,\overrightarrow{i} + (u+v)\,\overrightarrow{j} + \dfrac{u^2+v^2}{2}\,\overrightarrow{k}$ be a surface. Determine the first fundamental form of the surface. Write the integral which gives the length of the curve $(\Gamma) : v = 1$ on the surface from $u = 1$ and $u = 2$.

Problem 10.14. Let $(S) : \overrightarrow{r}(u,v) = (2+u^2)\cos v\,\overrightarrow{i} + (2+u^2)\sin v\,\overrightarrow{j} + u\,\overrightarrow{k}$ be a surface. Determine the first fundamental form of the surface. Write the integral which gives the length of the curve $(\Gamma) : v = 0$ on the surface from $u = -1$ to $u = 2$. Calculate the cosine of the angle between the curves $(\Gamma_1) : \begin{cases} u = 0 \\ v = t \end{cases}$ and $(\Gamma_2) : \begin{cases} u = 2t \\ v = t + \pi \end{cases}$ on the surface (S) at the point $M(u=0, v=\pi)$.

Problem 10.15. Compute the first fundamental form of the following surfaces:

a) elliptic paraboloid $(S) : \overrightarrow{r}(u,v) = au\cos v\,\overrightarrow{i} + bu\sin v\,\overrightarrow{j} + u^2\,\overrightarrow{k}$.

b) hyperbolic paraboloid $(S) : \overrightarrow{r}(u,v) = au\cosh v\,\overrightarrow{i} + bu\sinh v\,\overrightarrow{j} + u^2\,\overrightarrow{k}$.

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