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An Invitation to Linear Algebra and Analytic Geometry

Second Edition

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Contents

PREFACE

The purpose of these lecture notes is to provide some important ideas of Linear Algebra and Analytic Geometry and the ability to use the specific language and the related techniques. The book is intended for students who will apply these theories in engineering.

Traditional notation and terminology are preserved; rigor is used as an aid rather than as an impediment to understanding. Some key theorems are explained and used without proof.

Starting with an introductory chapter dedicated to determinants and matrices, the lecture continues with two chapters related to geometry. The forth chapter introduces Linear spaces and their applications and is continued with the fifth chapter related to Inner product spaces and Linear manifolds. The sixth chapter treats the wide subject of Linear maps between vector spaces with applications to matrix functions. The last chapter briefly describe Conics and quadrics with their reduction to canonical form and other applications. Each chapter is followed by related exercises and their given solutions. The proposed exercises serve to develop some mathematical skills and to strengthen understanding. Few of them, if any, should present difficulties to a student who read the corresponding parts of the theory. Of course, a mathematical text must be read slowly and, if possible, with pencil in hand. The reader should verify the calculations and supply the omitted steps.

As an invitation to Linear Algebra and Analytic Geometry, this book has an introductory character. It is intended to open the way to advanced books like some listed in the Bibliography.

CHAPTER 1

Determinants and matrices

1.1 Laplace's Theorem

We shall use the basic notions of linear algebra and the specific language of the literature ([1], [2], [7]). Let us consider a determinant D of order n. Let k be an integer, $1 \leq k \leq n$. Consider the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$. By deleting the other rows and columns we obtain a determinant of order k, called a *minor* of D and denoted by $M^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$.

Now, let us delete the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$; we obtain a determinant of order $n-k$. It is called the *complementary minor* of $M^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$ and is denoted by $\widetilde{M}^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$. Finally, let us denote $A^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} = (-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k}\widetilde{M}^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$.
$A^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$ is called the *cofactor* of $M^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$.

Using this notation we shall state (without proof) Laplace's Theorem:

Theorem 1.1 *$D = \sum M^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} A^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$, where:*

1) The indices $i_1, \ldots, i_k$ are fixed

2) The indices $j_1, \ldots, j_k$ take on all the possible values, such that $1 \le j_1 < j_2 < \cdots < j_k \le n$.

Remark 1.2 a) For $k = 1$, the above formula is the well-known expansion of a determinant using a fixed row.

b) In Theorem 1.1 we have used k fixed rows; a similar result obviously holds by using k fixed columns.

Theorem 1.3 *Let* $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{n1} & \ldots & b_{nn} \end{pmatrix}$ *where a_{ij} and b_{ij} are real or complex numbers. Then* $\det(A \cdot B) = \det A \cdot \det B$.

1.2 Vandermonde's determinant

The following determinant of order n:

$$V(a_1, \ldots, a_n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^{n-1} \end{vmatrix}$$

is called the Vandermonde's determinant of the (real or complex) numbers $a_1, \ldots, a_n$.

By induction it can be proved that:

$$V(a_1, \ldots, a_n) = \prod_{1 \le i < j \le n} (a_j - a_i)$$

1.3 Circulants

The following determinant is called a circulant:

$$C(a_0, a_1, \ldots, a_{n-1}) = \begin{vmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{vmatrix}$$

Let $\epsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1$. We have $\epsilon_k^n = 1$, $k = 0, 1, \ldots, n-1$. Let us denote $f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$.

Theorem 1.4 $C(a_0, a_1, \ldots, a_{n-1}) = f(\epsilon_0)f(\epsilon_1)\ldots f(\epsilon_{n-1})$.

Proof.

$$\begin{aligned} &C(a_0, a_1, \ldots, a_{n-1}) \cdot V(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}) = \\ &= \begin{vmatrix} f(\epsilon_0) & f(\epsilon_1) & \ldots & f(\epsilon_{n-1}) \\ \epsilon_0 f(\epsilon_0) & \epsilon_1 f(\epsilon_1) & \ldots & \epsilon_{n-1} f(\epsilon_{n-1}) \\ \ldots & \ldots & \ldots & \ldots \\ \epsilon_0^{n-1} f(\epsilon_0) & \epsilon_1^{n-1} f(\epsilon_1) & \ldots & \epsilon_{n-1}^{n-1} f(\epsilon_{n-1}) \end{vmatrix} = \\ &= f(\epsilon_0)f(\epsilon_1)\ldots f(\epsilon_{n-1})V(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}). \end{aligned}$$

Since $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}$ are pairwise distinct, we have $V(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}) \neq 0$ and hence $C(a_0, a_1, \ldots, a_{n-1}) = f(\epsilon_0)f(\epsilon_1)\ldots f(\epsilon_{n-1})$. $\square$

1.4 Rank. Elementary transformations.

Let K be the field of real numbers or the field of complex numbers. By $\mathcal{M}_{n,m}(K)$ we shall denote the set of all matrices with n rows, m columns and having entries from K. The number $r \in \mathbb{N}$ is called the rank of the matrix $A \in \mathcal{M}_{n,m}(K)$ if

1) There exists a square submatrix M of A, with r rows and columns, such that $\det M \neq 0$.

2) If $p > r$, for every submatrix N of A having p rows and columns we have $\det N = 0$.

We shall denote the rank of A by r_A. It can be proved that if $A \in \mathcal{M}_{n,m}(K)$ and $B \in \mathcal{M}_{m,p}(K)$, then

$$r_A + r_B - m \leq r_{AB} \leq \min\{r_A, r_B\}. \tag{1.1}$$

Theorem 1.5 *Let $A, B \in \mathcal{M}_{n,n}(K)$, $\det A \neq 0$. Then $r_{AB} = r_B$.*

Proof. Clearly $r_A = n$. By using (1.1) with $m = p = n$ we obtain $r_B \leq r_{AB} \leq r_B$. Hence $r_{AB} = r_B$. $\square$

Definition 1.6 The following operations are called *elementary row transformations on the matrix A:*

1) The interchange of any two rows;

2) The multiplication of a row by any non-zero number;

3) The addition of one row to another.

Similarly we can define the elementary column transformations.

Consider an arbitrary determinant. If it is nonzero, it will be nonzero after performing any elementary transformation; if it is equal to zero, it will remain equal to zero.

We conclude that *the rank of a matrix does not change if we perform any elementary transformation on the matrix.* So we can use elementary transformations in order to compute the rank of a matrix. Namely, given a matrix $A \in \mathcal{M}_{n,m}(K)$, we transform it - by an appropriate succession of elementary transformations - into a matrix B such that

(i) the diagonal entries of B are either 0 or 1, all the $1'$s preceding all the $0'$s on the diagonal,

(ii) all the other entries of B are equal to 0.

Since the rank is invariant under elementary transformations, we have $r_A = r_B$; but r_B is obviously equal to the number of $1'$s on the diagonal. The following example illustrates this method.

$$A = \begin{pmatrix} -2 & -1 & 0 & -5 & -1 \\ 1 & 2 & 6 & -2 & -1 \\ 3 & 1 & -1 & 8 & 1 \\ -1 & 0 & 2 & -4 & -1 \\ -1 & -2 & -7 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & -5 & -1 \\ -2 & 1 & 6 & -2 & -1 \\ -1 & 3 & -1 & 8 & 1 \\ 0 & -1 & 2 & -4 & -1 \\ 2 & -1 & -7 & 3 & 2 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & -3 & 6 & -12 & -3 \\ -1 & 1 & -1 & 3 & 0 \\ 0 & -1 & 2 & -4 & -1 \\ 2 & 3 & -7 & 13 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 2 & -4 & -1 \\ 0 & 3 & -7 & 13 & 4 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ It follows that $r_A = 3$.

The following theorem offers a procedure to compute the inverse of a matrix (if this inverse exists).

Theorem 1.7 *If a square matrix is reduced to the identity matrix by a sequence of elementary row operations, the same sequence of elementary row transformations performed on the identity matrix produces the inverse of the given matrix.*

Example 1.4.1 Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 6 \\ -1 & 2 & 0 \end{pmatrix}$.

We write the given matrix and the identity:

1	1	1	1	0	0
6	7	6	0	1	0
-1	2	0	0	0	1

Now we perform a succession of elementary *row* transformations in order to transform A into the identity; the same transformations are performed on the identity.

1	1	1	1	0	0
0	1	0	-6	1	0
0	3	1	1	0	1
1	1	1	1	0	0
0	1	0	-6	1	0
0	0	1	19	-3	1
1	0	0	-12	2	-1
0	1	0	-6	1	0
0	0	1	19	-3	1

It follows that $A^{-1} = \begin{pmatrix} -12 & 2 & -1 \\ -6 & 1 & 0 \\ 19 & -3 & 1 \end{pmatrix}$

Exercises

1.1 Evaluate the following n^{th} order determinants by reduction to triangular form:

$$\text{a)} \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ .. & .. & \dots & .. & .. \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix}; \qquad \text{b)} \begin{vmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ .. & .. & \dots & .. & .. \\ b & b & b & \dots & a \end{vmatrix};$$

c) $\begin{vmatrix} x & y & 0 & \dots & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 \\ 0 & 0 & x & \dots & 0 & 0 \\ .. & .. & ... & .. & .. \\ 0 & 0 & 0 & \dots & x & y \\ y & 0 & 0 & \dots & 0 & x \end{vmatrix}$; d) $\begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & n \\ 3 & 4 & 5 & \dots & n & n & n \\ .. & .. & .. & \dots & .. & .. & .. \\ n & n & n & \dots & n & n & n \end{vmatrix}$.

1.2 Calculate the determinant $C(1, 2, \dots, n)$.

1.3 Calculate the determinant $C(C^0_{n-1}, C^1_{n-1}, \dots, C^{n-1}_{n-1})$.

1.4 Calculate the n^{th} order determinant $C(a, b, b, \dots, b)$, with $a, b \in \mathbb{R}$.

1.5 For $a_1, a_2, \dots, a_n \in \mathbf{C}$, $k = 1, \dots, n$, calculate the determinant

$$V_k(a_1, a_2, \dots, a_n) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_1^{k-1} & a_2^{k-1} & \dots & a_n^{k-1} \\ a_1^{k+1} & a_2^{k+1} & \dots & a_n^{k+1} \\ \dots & \dots & \dots & \dots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix},$$

called the lacunary Vandermonde.

1.6 Prove the following identities without expanding the determinants:

a) $\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}$; b) $\begin{vmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \gamma \end{vmatrix} = \begin{vmatrix} a & -b & c \\ -x & y & -z \\ \alpha & -\beta & \gamma \end{vmatrix}$;

c) $\begin{vmatrix} a & b & c \\ p & q & r \\ a\alpha & b\beta & c\gamma \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ bcp & acq & abr \\ \alpha & \beta & \gamma \end{vmatrix}$.

1.7 Compute the determinants by using Laplace's Rule:

a) $\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix}$; b) $\begin{vmatrix} 2 & 3 & 0 & 0 & 1 & -1 \\ 9 & 4 & 0 & 0 & 3 & 7 \\ 4 & 5 & 1 & -1 & 2 & 4 \\ 3 & 8 & 3 & 7 & 6 & 9 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 & 0 \end{vmatrix}$; c) $\begin{vmatrix} 3 & 2 & 0 & 0 & 0 \\ 4 & 3 & 2 & 0 & 0 \\ 5 & 4 & 3 & 2 & 0 \\ 6 & 5 & 4 & 3 & 2 \\ 7 & 6 & 5 & 4 & 3 \end{vmatrix}$.

1.8 Calculate the determinants:

a) $D_{2n} = \begin{vmatrix} a & 0 & \dots & 0 & b \\ 0 & a & \dots & b & 0 \\ .. & .. & \dots & .. & .. \\ 0 & b & \dots & a & 0 \\ b & 0 & \dots & 0 & a \end{vmatrix}$, of order $2n$; b) $E_{n+2} = \begin{vmatrix} a & 0 & 0 & \dots & 0 & b \\ x_1 & u & v & \dots & v & y_1 \\ x_2 & v & u & \dots & v & y_2 \\ .. & .. & .. & \dots & .. & .. \\ x_n & v & v & \dots & u & y_n \\ b & 0 & 0 & \dots & 0 & a \end{vmatrix}$,

of order $n+2$.

1.9 Find the inverse of the matrix of order n:

a) $A = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$; b) $B = \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ .. & .. & \dots & .. & .. \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}$;

c) $B = \begin{pmatrix} 1 & -5 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -5 & \dots & 0 & 0 & 0 \\ \dots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & -5 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & -5 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$.

1.10 Find the inverse of the matrix $A = \begin{pmatrix} \hat{2} & \hat{3} & \hat{1} \\ \hat{0} & \hat{1} & \hat{4} \\ \hat{5} & \hat{6} & \hat{2} \end{pmatrix}$ in $\mathbb{Z}_7$.

Solutions

$\boxed{\mathbf{1.1}}$ a) $n!$; b) $b_1(b_2 - a_{12})(b_3 - a_{23}) \cdots (b_n - a_{n-1,n})$; c) $1 + 2n$; d) $(-1)^{n-1}n$.

$\boxed{\mathbf{1.2}}$ $C(1, 2, \dots, n) = \prod\limits_{k=0}^{n} P(\varepsilon_k)$, where $\varepsilon_k^n = 1$ and $P(X) = 1 + 2X + 3X^2 + \cdots + nX^{n-1}$. For $\varepsilon_k \neq 1$, we get $P(\varepsilon_k) = \dfrac{n}{\varepsilon_k - 1}$ and $P(1) = \dfrac{n(n+1)}{2}$. $C(1, 2, \dots, n) = \dfrac{n^n(n+1)}{2} \prod\limits_{k=1}^{n-1} \dfrac{1}{\varepsilon_k - 1}$. The values ε_k, $k = 1, \dots, n-1$ are the roots of the equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$, so $\prod_{k=1}^{n-1}(z - \varepsilon_k) = z^{n-1} + z^{n-2} + \cdots + z + 1$. Taking $z = 1$, we obtain $\prod\limits_{k=1}^{n-1}(\varepsilon_k - 1) = (-1)^{n-1}n$, so $C(1, 2, \dots, n) = (-1)^{n-1}\dfrac{n^{n-1}(n+1)}{2}$.

$\boxed{\mathbf{1.3}}$ $P(X) = C_{n-1}^0 + C_{n-1}^1 X + C_{n-1}^2 X^2 + \cdots + C_{n-1}^{n-1} X^{n-1} = (1 + X)^{n-1}$. The determinant has then the value $\prod\limits_{k=1}^{n-1}(1 + \varepsilon_k)^{n-1} = [(-1)^n((-1)^n - 1)]^{n+1}$.

$\boxed{\mathbf{1.4}}$ $P(X) = a + bX + bX^2 + \cdots + X^{n-1} = a + b\dfrac{X^n - X}{X - 1}$, for $X \neq 1$, and $P(1) = a + b(n-1)$. $C(a, b, \dots, b) = [a + (n-1)b](a - b)^{n-1}$. The same result can be obtained also directly, using the properties of determinants.

$\boxed{\mathbf{1.5}}$ Consider another Vandermonde determinant:

$$V(a_1,\ldots,a_n,X) = V(a_1,\ldots,a_n)\prod_{k=1}^{n}(X-a_k) =$$
$$= V(a_1,\ldots,a_n)(X^n - S_1X^{n-1} + \cdots + (-1)^{n-k}S_{n-k}X^k +$$
$$+ \cdots + (-1)^n S_n),$$

where S_k are the Viéte sums corresponding to the polynomial with the roots $a_1,\ldots,a_n$. On the other hand, expanding the same determinant by the last column we get: $V(a_1,\ldots,a_n,X) = (-1)^{n+2}V_0(a_1,\ldots,a_n) + \cdots + (-1)^{n+2+k}X^kV_k(a_1,\ldots,a_n) + \cdots + (-1)^{2n+2}X^nV_n(a_1,\ldots,a_n)$. From the two expressions we obtain $V_k(a_1,\ldots,a_n) = V(a_1,\ldots,a_n)S_{n-k}$.

$\boxed{\mathbf{1.6}}$ a) Multiply the second column of the determinant in the left-hand member of the identity by bc, the third column by ac and the fourth by ab. b) Multiply the second column and the second row by (-1). c) Multiply the second row of the determinant by abc then divide the first column by a, the second by b and the third by c.

$\boxed{\mathbf{1.7}}$ a) 9; b) For example, we expand after the last two rows: 1000.

$\boxed{\mathbf{1.8}}$ Using Laplace's formula with rows n and $n+1$ we get the recurrence relationship $D_{2n} = \begin{vmatrix} a & b \\ b & a \end{vmatrix}(-1)^{n+n+1+n+n+1}D_{2n-2} = (a^2-b^2)D_{2n-2}$, and by induction $D_{2n} = (a^2-b^2)^n$.

$\boxed{\mathbf{1.9}}$ a) Subtracting each row from the row above it, follows:

$$\begin{array}{|ccccc|cccccc|}
\hline
1 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 1 & 1 & 0 & 1 & 0 & \ldots & 0 & 0 \\
.. & .. & \ldots & .. & .. & .. & .. & .. & \ldots & .. & .. \\
0 & 0 & \ldots & 1 & 1 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 1 \\
\hline
1 & 0 & \ldots & 0 & 0 & 1 & -1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & 1 & -1 & \ldots & 0 & 0 \\
.. & .. & \ldots & .. & .. & .. & .. & .. & \ldots & .. & .. \\
0 & 0 & \ldots & 1 & 0 & 0 & 0 & 0 & \ldots & 1 & -1 \\
0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 1 \\
\hline
\end{array}$$

b) We can apply the following succession of elementary transformations: add all rows to the first one, multiply row one by $\dfrac{1}{n-1}$, subtract row one from all he other rows, add again all the rows to the first one and finally multiply all the rows (except the first) by -1. The inverse matrix is

$$B^{-1} = \frac{1}{n-1}\begin{pmatrix} 2-n & 1 & \ldots & 1 & 1 \\ 1 & 2-n & \ldots & 1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 2-n & 1 \\ 1 & 1 & \ldots & 1 & 2-n \end{pmatrix}.$$

$\boxed{\mathbf{1.10}}$ $A^{-1} = \begin{pmatrix} \hat{5} & \hat{0} & \hat{1} \\ \hat{5} & \hat{5} & \hat{5} \\ \hat{4} & \hat{6} & \hat{4} \end{pmatrix}.$

CHAPTER 2

Vectors

2.1 Vectors

A vector $\overrightarrow{v}$ in space is characterized by *magnitude* (denoted by $||\overrightarrow{v}||$), *direction* and *sense*. The vectors are added by either the *triangle law* or the *parallelogram law*.

Vector addition obeys the following postulates:

1. $\overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w}$ (associative law);
2. $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$ (commutative law);
3. There is a unique vector called the *null vector*, denoted by $\overrightarrow{0}$, such that $\overrightarrow{0} + \overrightarrow{v} = \overrightarrow{v}$ for all $\overrightarrow{v}$;
4. For every vector $\overrightarrow{v}$ there is a unique vector called its *negative* and denoted by $-\overrightarrow{v}$, such that $\overrightarrow{v} + (-\overrightarrow{v}) = \overrightarrow{0}$.

Thus, if we denote by $\mathcal{V}_3$ the set of all vectors in the space, then $(\mathcal{V}_3, +)$ is a *commutative group*. The following postulates hold for the multiplication of vectors by numbers:

5. $1\overrightarrow{a} = \overrightarrow{a}$
6. $s(t\overrightarrow{a}) = (st)\overrightarrow{a}$
7. $(s+t)\overrightarrow{a} = s\overrightarrow{a} + t\overrightarrow{a}$
8. $s(\overrightarrow{a} + \overrightarrow{b}) = s\overrightarrow{a} + s\overrightarrow{b}$

for all $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}_3$ and all $s, t \in \mathbb{R}$.

In talking about vectors, numbers are often called *scalars.* The vector $t\overrightarrow{v}$ is called a scalar multiple of the vector $\overrightarrow{v}$.

Consider now the axes Ox, Oy, Oz, mutually perpendicular, forming a right-handed rectangular Cartesian co-ordinate frame. Let $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ be the unit vectors for this system. Every vector $\overrightarrow{v}$ can be written, uniquely, in the form $\overrightarrow{v} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$, where a, b, c are scalars (called the *components* of $\overrightarrow{v}$). Other important formulas are $||a\overrightarrow{v}|| = |a|||\overrightarrow{v}||$ and $||\overrightarrow{u} + \overrightarrow{v}|| \leq ||\overrightarrow{u}|| + ||\overrightarrow{v}||$ for all $\overrightarrow{u}, \overrightarrow{v} \in \mathcal{V}_3$ and $a \in \mathbb{R}$.

2.2 Scalar product and vector product

One associates with any two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ a number called their *scalar product* (or inner product) and denoted by $\overrightarrow{a} \cdot \overrightarrow{b}$. The definition reads:

$$\overrightarrow{a} \cdot \overrightarrow{b} = ||\overrightarrow{a}||||\overrightarrow{b}|| \cos\theta, \quad \theta = \text{angle between } \overrightarrow{a} \text{ and } \overrightarrow{b}$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = 0 \text{ if either } \overrightarrow{a} = \overrightarrow{0} \text{ or } \overrightarrow{b} = \overrightarrow{0}.$$

For all $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}_3$ and $s \in \mathbb{R}$ we have

1) $\overrightarrow{a}\,\overrightarrow{b} = \overrightarrow{b}\,\overrightarrow{a}$

2) $\overrightarrow{a}(\overrightarrow{b}+\overrightarrow{c})=\overrightarrow{a}\,\overrightarrow{b}+\overrightarrow{a}\,\overrightarrow{c}$

3) $(s\overrightarrow{a})\overrightarrow{b}=s(\overrightarrow{a}\,\overrightarrow{b})$

4) $\overrightarrow{a}\cdot\overrightarrow{a}\geq 0$; $\overrightarrow{a}\cdot\overrightarrow{a}=0 \Longleftrightarrow \overrightarrow{a}=\overrightarrow{0}$.

Let us note that $\overrightarrow{a}\cdot\overrightarrow{a}=||\overrightarrow{a}||^2$ and $\cos\theta=\dfrac{\overrightarrow{a}\cdot\overrightarrow{b}}{||\overrightarrow{a}||\cdot||\overrightarrow{b}||}$. In particular, $\overrightarrow{a}\cdot\overrightarrow{b}=0 \Longleftrightarrow \overrightarrow{a}\perp\overrightarrow{b}$.

On the other hand, $\overrightarrow{i}\cdot\overrightarrow{i}=\overrightarrow{j}\cdot\overrightarrow{j}=\overrightarrow{k}\cdot\overrightarrow{k}=1$, $\overrightarrow{i}\cdot\overrightarrow{j}=\overrightarrow{j}\cdot\overrightarrow{k}=\overrightarrow{k}\cdot\overrightarrow{i}=0$. Let $\overrightarrow{a},\overrightarrow{b}\in\mathcal{V}_3$, $\overrightarrow{a}=a_1\overrightarrow{i}+a_2\overrightarrow{j}+a_3\overrightarrow{k}$, $\overrightarrow{b}=b_1\overrightarrow{i}+b_2\overrightarrow{j}+b_3\overrightarrow{k}$.

By using the properties of the scalar product, mentioned above, we deduce $\overrightarrow{a}\cdot\overrightarrow{b}=(a_1\overrightarrow{i}+a_2\overrightarrow{j}+a_3\overrightarrow{k})\cdot(b_1\overrightarrow{i}+b_2\overrightarrow{j}+b_3\overrightarrow{k})=a_1b_1\overrightarrow{i}\,\overrightarrow{i}+a_2b_1\overrightarrow{j}\,\overrightarrow{i}+a_3b_1\overrightarrow{k}\,\overrightarrow{i}+a_1b_2\overrightarrow{i}\,\overrightarrow{j}+a_2b_2\overrightarrow{j}\,\overrightarrow{j}+a_3b_2\overrightarrow{k}\,\overrightarrow{j}+a_1b_3\overrightarrow{i}\,\overrightarrow{k}+a_2b_3\overrightarrow{j}\,\overrightarrow{k}+a_3b_3\overrightarrow{k}\,\overrightarrow{k}$.

Thus we have the following important formula:

$$\overrightarrow{a}\cdot\overrightarrow{b}=a_1b_1+a_2b_2+a_3b_3.$$

Combining the previous results we can write:

$$||\overrightarrow{a}||=\sqrt{a_1^2+a_2^2+a_3^2}$$

$$\cos\theta=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

$$\overrightarrow{a}\perp\overrightarrow{b}\Longleftrightarrow a_1b_1+a_2b_2+a_3b_3=0.$$

The vector product of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is the vector, denoted by $\overrightarrow{a}\times\overrightarrow{b}$, characterized by:

1) $||\overrightarrow{a}\times\overrightarrow{b}||=||\overrightarrow{a}||||\overrightarrow{b}||\sin\theta$

2) $\overrightarrow{a} \times \overrightarrow{b}$ is perpendicular to both $\overrightarrow{a}$ and $\overrightarrow{b}$

3) The triad of vectors $\{\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{a} \times \overrightarrow{b}\}$ is oriented like the triad $\{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\}$.

For all $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}_3$ and $s \in \mathbb{R}$ we have

I) $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$

II) $(s\overrightarrow{a}) \times \overrightarrow{b} = \overrightarrow{a} \times (s\overrightarrow{b}) = s(\overrightarrow{a} \times \overrightarrow{b})$

III) $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}$

IV) $\overrightarrow{a} \times \overrightarrow{0} = \overrightarrow{0}$, $\overrightarrow{a} \times \overrightarrow{a} = \overrightarrow{0}$

V) $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0} \iff \overrightarrow{a} || \overrightarrow{b}$

VI) $||\overrightarrow{a} \times \overrightarrow{b}||$ equals the numerical value of the area of the parallelogram constructed on $\overrightarrow{a}$ and $\overrightarrow{b}$.

It is easy to construct the following table:

$\times$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overrightarrow{k}$
$\overrightarrow{i}$	$\overrightarrow{0}$	$\overrightarrow{k}$	$-\overrightarrow{j}$
$\overrightarrow{j}$	$-\overrightarrow{k}$	$\overrightarrow{0}$	$\overrightarrow{i}$
$\overrightarrow{k}$	$\overrightarrow{j}$	$-\overrightarrow{i}$	$\overrightarrow{0}$

Let $\overrightarrow{a} = a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k}$, $\overrightarrow{b} = b_1\overrightarrow{i} + b_2\overrightarrow{j} + b_3\overrightarrow{k}$.
Then we can write:

$$\overrightarrow{a} \times \overrightarrow{b} = (a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k}) \times (b_1\overrightarrow{i} + b_2\overrightarrow{j} + b_3\overrightarrow{k}) =$$
$$= a_1b_1\overrightarrow{i} \times \overrightarrow{i} + a_2b_1\overrightarrow{j} \times \overrightarrow{i} + a_3b_1\overrightarrow{k} \times \overrightarrow{i} + a_2b_2\overrightarrow{j} \times \overrightarrow{j} +$$
$$+ a_3b_2\overrightarrow{k} \times \overrightarrow{j} + a_1b_3\overrightarrow{i} \times \overrightarrow{k} + a_2b_3\overrightarrow{j} \times \overrightarrow{k} + a_3b_3\overrightarrow{k} \times \overrightarrow{k} =$$
$$= (a_2b_3 - a_3b_2)\overrightarrow{i} + (a_3b_1 - a_1b_3)\overrightarrow{j} + (a_1b_2 - a_2b_1)\overrightarrow{k}.$$

Finally we have

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This is a remarkable formula! Its simplicity enables us to compute easily the vector product.

2.3 Triple vector product

The vector $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$ is called the triple vector product of the vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$. It has no important geometrical meaning but is expressed by a formula which is of use for applications. To deduce this formula let us choose the Cartesian axes in such a way that the x-axis is directed along the vector $\overrightarrow{b}$ and the y-axis lies in the plane of vectors $\overrightarrow{b}$ and $\overrightarrow{c}$. Clearly we have $\overrightarrow{b} = b_1 \overrightarrow{i}$, $\overrightarrow{c} = c_1 \overrightarrow{i} + c_2 \overrightarrow{j}$, $\overrightarrow{a} = a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{k}$.

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ b_1 & 0 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} = b_1 c_2 \overrightarrow{k}$$

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & b_1 c_2 \end{vmatrix} = a_2 b_1 c_2 \overrightarrow{i} - a_1 b_1 c_2 \overrightarrow{j} =$$

$$= (a_1 c_1 + a_2 c_2) b_1 \overrightarrow{i} - a_1 b_1 (c_1 \overrightarrow{i} + c_2 \overrightarrow{j}) =$$

$= (\overrightarrow{a}\,\overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a}\,\overrightarrow{b})\overrightarrow{c}$ (check up these formulas!).

Thus we have

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a}\,\overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a}\,\overrightarrow{b})\overrightarrow{c}.$$

This final formula no longer contains any components and therefore does not depend on the particular choice of the axes.

2.4 Triple scalar product

The triple scalar product of the vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ is denoted by $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ and is defined by $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = \overrightarrow{a}(\overrightarrow{b} \times \overrightarrow{c})$.
Clearly we have

$$(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = (a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k}) \cdot \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

Finally we have

$$(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Taking into account this formula, it is easy to prove that

1) $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = (\overrightarrow{c}, \overrightarrow{a}, \overrightarrow{b}) = (\overrightarrow{b}, \overrightarrow{c}, \overrightarrow{a})$
2) $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = -(\overrightarrow{a}, \overrightarrow{c}, \overrightarrow{b})$
3) $(s\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = s(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$
4) $(\overrightarrow{u} + \overrightarrow{v}, \overrightarrow{b}, \overrightarrow{c}) = (\overrightarrow{u}, \overrightarrow{b}, \overrightarrow{c}) + (\overrightarrow{v}, \overrightarrow{b}, \overrightarrow{c})$

We have also $|(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})| = |(\overrightarrow{a}(\overrightarrow{b} \times \overrightarrow{c})|$ = volume of the parallelepiped constructed on $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$.
In particular
$(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = 0 \Longleftrightarrow \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are parallel to the same plane.

Exercises

2.1 Consider a triangle ABC and the heights $AA_1 \perp BC$, $A_1 \in (BC)$, $BB_1 \perp AC$, $B_1 \in (AC)$ with the intersection point H. Prove that $CH \perp AB$.

2.2 Consider four points A, B, C and D in space.
a) Prove that $\overrightarrow{DA} \cdot \overrightarrow{BC} + \overrightarrow{DB} \cdot \overrightarrow{CA} + \overrightarrow{DC} \cdot \overrightarrow{AB} = 0$.
b) If $DA \perp BC$ and $DB \perp CA$ then $DC \perp AB$.

2.3 Let G be the weight center of the triangle ABC.
a) Prove that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = 0$.
b) If M is an arbitrary point then $3\overrightarrow{MG} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}$.

2.4 Let ABC and MNP be two triangles (in the same plane or different planes). Prove that, if $\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CP} = 0$, then the weight centers of the two triangles coincide.

2.5 If $\vec{a} = (3, -1, \alpha)$, $\vec{b} = (0, 1, -2)$ and $\vec{c} = (1, 0, -1)$, determine $\alpha \in \mathbb{R}$ such that the vector $\vec{a} \times (\vec{b} \times \vec{c})$ is parallel to the plane $y0z$.

2.6 Find the angle between
a) the vector $\vec{a} = \dfrac{\sqrt{3}}{2}\vec{i} + \dfrac{1}{2}\vec{k}$ and the axis Ox
b) $\vec{AB}$ and $\vec{AC}$ where $A(3, 1, -2)$, $B(2, 1, -1)$ and $C(3, 0, -1)$.

2.7 Let $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = \vec{j} - 2\vec{k}$. Determine the height of the parallelogram with the edges $\vec{a}$ and $\vec{b}$, considering $\vec{a}$ as the basis.

2.8 Determine the vector $\vec{w}$ such that $\|\vec{w}\|$=2, $\vec{w}$ is perpendicular on the axis Oz and makes a $45°$ angle with the positive direction of Ox.

2.9 Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j}$, $\vec{c} = \vec{j} + 3\vec{k}$. Determine the height of the parallelepiped with the edges $\vec{a}$, $\vec{b}$, $\vec{c}$ considering the parallelogram with edges $\vec{a}$, $\vec{b}$ as basis.

2.10 Prove the identity of Lagrange $\|\vec{a} \times \vec{b}\|^2 + (\vec{a} \cdot \vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2$, for any vectors $\vec{a}, \vec{b}$.

2.11 Prove that $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$, for any vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$.

2.12 Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors, making two by two, angles of measures α, β, γ. Prove that, if

$$\vec{a} \times (\vec{a} \times \vec{b}) + \vec{b} \times (\vec{b} \times \vec{c}) + \vec{c} \times (\vec{c} \times \vec{a}) = 0$$

then $\cos\alpha \cos\beta \cos\gamma = 1$.

Solutions

$\boxed{\textbf{2.1}}$ Use, for instance the fact that $\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{BH} \cdot \overrightarrow{AC} = 0$, $\overrightarrow{AH} = \overrightarrow{AC} + \overrightarrow{CH}$, $\overrightarrow{BH} = \overrightarrow{BC} + \overrightarrow{CH}$.

$\boxed{\textbf{2.2}}$ a) Using the triangle rule we get that $\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC} = \overrightarrow{DC} - \overrightarrow{DB}$, $\overrightarrow{CA} = \overrightarrow{DA} - \overrightarrow{DC}$, $\overrightarrow{AB} = \overrightarrow{DB} - \overrightarrow{DA}$ and the equality follows. b) Follows directly from a).

$\boxed{\textbf{2.3}}$ a) Let A_1 be the middle of (BC). Use the relations $\overrightarrow{BG} = \overrightarrow{BA_1} + \overrightarrow{A_1G}$, $\overrightarrow{CG} = \overrightarrow{CA_1} + \overrightarrow{A_1G}$, $\overrightarrow{AG} = 2\overrightarrow{GA_1}$.

$\boxed{\textbf{2.4}}$ $\overrightarrow{G_1G_2} = \overrightarrow{G_1A} + \overrightarrow{AM} + \overrightarrow{MG_2}$, $\overrightarrow{G_1G_2} = \overrightarrow{G_1B} + \overrightarrow{BN} + \overrightarrow{NG_2}$, $\overrightarrow{G_1G_2} = \overrightarrow{G_1C} + \overrightarrow{CP} + \overrightarrow{PG_2}$. Add the three relations and use the previous exercise.

$\boxed{\textbf{2.5}}$ $\vec{b} \times \vec{c} = -\vec{i} - 2\vec{j} - \vec{k} = (-1, -2, -1)$, $\vec{a} \times (\vec{b} \times \vec{c}) = (1 + 2\alpha, 3 - \alpha, -7)$. If the vector is parallel to the plane $y0z$, then it is perpendicular on the axis $0x$, that is the dot product is zero. This gives $1 + 2\alpha = 0$, so $\alpha = -\frac{1}{2}$.

$\boxed{\textbf{2.6}}$ a) $\cos\alpha = \dfrac{\vec{a}\cdot\vec{j}}{\|\vec{a}\|} = \dfrac{\sqrt{3}}{2}$, so $\alpha = \dfrac{\pi}{6}$. b) $\vec{AB} = -\vec{i} + \vec{k}$, $\vec{AC} - \vec{j} + \vec{k}$, $\alpha = \dfrac{\pi}{3}$.

$\boxed{\textbf{2.7}}$ The area of the parallelogram is $\|\vec{a}\times\vec{b}\| = 6\vec{j}+3\vec{k} = 3\sqrt{5}$. On the other hand, $area = h\|\vec{a}\|$, so we get $h = \frac{3\sqrt{5}}{\sqrt{14}}$.

$\boxed{\textbf{2.8}}$ If $\vec{w} = a\vec{i} + b\vec{j} + c\vec{k}$, from $\vec{w} \perp \vec{k}$, we get $c = 0$. We have $\vec{w}\cdot\vec{i} = \|\vec{w}\|\cos\pi/4 = \sqrt{2}$. On the other hand, $\vec{w}\cdot\vec{i} = a$, so $a = \sqrt{2}$. Finally, since $\|\vec{w}\| = \sqrt{a^2+b^2+c^2} = \sqrt{2}$, it follows that $b = \sqrt{2}$ or $b = -\sqrt{2}$.

$\boxed{\textbf{2.9}}$ The volume of the parallelepiped is given by the mixed product $volume = |(\vec{a},\vec{b},\vec{c})| = 7$. The area of the basis is $area = \|\vec{a}\times\vec{b}\| = \sqrt{14}$. The height is $h = \dfrac{7}{\sqrt{14}}$.

$\boxed{\textbf{2.10}}$ Denoting by α the angle formed by the two vectors, we have $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\alpha$, $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\alpha$ and the identity follows immediately.

$\boxed{\textbf{2.11}}$ We use the properties of the triple product:

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b},\vec{c},\vec{d}) = (\vec{d},\vec{a}\times\vec{b},\vec{c}) = \vec{d}\cdot((\vec{a}\times\vec{b})\times\vec{c}) =$$
$$= -\vec{d}((\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}) = \vec{d}\cdot(\vec{c}\cdot\vec{a})\vec{b} - \vec{d}\cdot(\vec{c}\cdot\vec{b})\vec{a}.$$

$\boxed{\textbf{2.12}}$ We get $(\vec{a}\cdot\vec{b}-\vec{c}\cdot\vec{c})\vec{a}+(\vec{b}\cdot\vec{c}-\vec{a}\cdot\vec{a})\vec{b}+(\vec{c}\cdot\vec{a}-\vec{b}\cdot\vec{b})\vec{c} = 0$. Since the three vectors are non-coplanar, this means $\vec{a}\cdot\vec{b} = \vec{c}\cdot\vec{c}$, $\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{a}$, $\vec{c}\cdot\vec{a} = \vec{b}\cdot\vec{b}$. It follows that $\|\vec{a}\|\|\vec{b}\|\cos\alpha = \|\vec{c}\|^2$, $\|\vec{b}\|\|\vec{c}\|\cos\beta = \|\vec{a}\|^2$ and $\|\vec{c}\|\|\vec{a}\|\cos\gamma = \|\vec{b}\|^2$. Multiplying the last three relationships we get now the desired equality.

CHAPTER 3

Lines and planes in space

3.1 Planes in space

We shall use the language of the vectors to introduce the basic concepts of solid analytic geometry ([8], [10], [11]). We assume that a fixed Cartesian coordinate system in space defined by the origin O and the triad $\{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\}$ has been chosen. Every point M has a position vector $\overrightarrow{r}$; the components of $\overrightarrow{r}$ are the coordinates of M, that is, we have $M(x, y, z)$ and $\overrightarrow{r} = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$.

1) Plane determined by a point and a normal vector
Let $M_0(x_0, y_0, z_0)$ be a point in space and let $\overrightarrow{n} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$, $\overrightarrow{n} \neq 0$. Let P be the plane that passes through M_0 and is perpendicular to $\overrightarrow{n}$.

Let $M(x, y, z)$ be an arbitrary point of P. Then $\overrightarrow{r} - \overrightarrow{r_0}$ is perpendicular to $\overrightarrow{n}$, that is $\overrightarrow{n}(\overrightarrow{r} - \overrightarrow{r_0}) = 0$.

Since $\overrightarrow{r} - \overrightarrow{r_0} = (x - x_0)\overrightarrow{i} + (y - y_0)\overrightarrow{j} + (z - z_0)\overrightarrow{k}$, we obtain

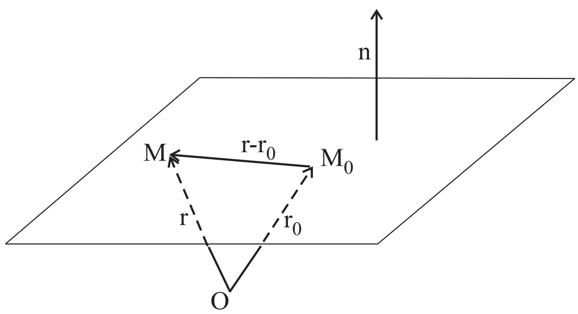
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This is the equation of the plane P. If we denote $d = -ax_0 - by_0 - cz_0$, then it reads:

$$ax + by + cz + d = 0.$$

This is the general form of the equation of a plane. The vector $\overrightarrow{n} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$ is called *normal* to the plane.

In particular, the plane xOy passes through the origin and $\overrightarrow{k}$ is normal to it. Hence we can take $x_0 = y_0 = z_0,\ a = b = 0,\ c = 1$.
Therefore the equation of the plane xOy is simply $z = 0$.



2) Plane determined by three non-collinear points. Let $M_i(x_i, y_i, z_i),\ i = 1, 2, 3$ be three non-collinear points and let P be the plane determined by them. Let $M(x, y, z)$ be an arbitrary point of P. Then M, M_1, M_2, M_3 are coplanar and hence

$$\begin{vmatrix} 1 & x & y & z \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

This is the equation of the plane P.

3) Plane determined by a point and two non-collinear vectors.

Let P be the plane that passes through a given point $M_0(x_0, y_0, z_0)$ and is parallel to two non-collinear given vectors $\overrightarrow{v_i} = x_i\overrightarrow{i} + y_i\overrightarrow{j} + z_i\overrightarrow{k}$, $i = 1, 2$.

Let $M(x, y, z)$ be an arbitrary point of P. Then the vectors $\overrightarrow{r} - \overrightarrow{r_0}, \overrightarrow{v_1}, \overrightarrow{v_2}$ are coplanar, that is $(\overrightarrow{r} - \overrightarrow{r_0}, \overrightarrow{v_1}, \overrightarrow{v_2}) = 0$. Thus the equation of the plane P can be written in the form:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

4) An important result is the following:

The equation of a plane passing through the line of intersection of the planes

(1) $a_1x + b_1y + c_1z + d_1 = 0$

(2) $a_2x + b_2y + c_2z + d_2 = 0$

is of the form

(3) $\quad a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0,\ \lambda \in \mathbb{R}.$

Indeed, (3) is the equation of a plane P. The coordinates of any point of the line verify (1) and (2) - and hence also (3). Thus the line is contained in the plane P.

3.2 Straight lines in space

Consider a direction in space, determined by the vector

$$\overrightarrow{v} = l\,\overrightarrow{i} + m\,\overrightarrow{j} + n\,\overrightarrow{k} \neq \overrightarrow{0}.$$

The numbers (l, m, n) are called the *direction ratios* of this direction. Clearly any other numbers proportional to them are also direction ratios for the same direction.

Now suppose that $\overrightarrow{v}$ is a unit vector, that is, $||\overrightarrow{v}|| = 1$. Then $l^2 + m^2 + n^2 = 1$. On the other hand, $l = \overrightarrow{v} \cdot \overrightarrow{i} = \cos\alpha$, $m = \overrightarrow{v} \cdot \overrightarrow{j} = \cos\beta$, $n = \overrightarrow{v} \cdot \overrightarrow{k} = \cos\gamma$ where α, β, γ are the angles between $\overrightarrow{v}$ and the axes. Hence the direction ratios are now $(\cos\alpha, \cos\beta, \cos\gamma)$. They are called *direction-cosines.*

Since $l^2+m^2+n^2 = 1$, we have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$.

1) Line determined by a point and a vector.
Consider the line d determined by the point $M_0(x_0, y_0, z_0)$ and the vector $\overrightarrow{v} = l\,\overrightarrow{i} + m\,\overrightarrow{j} + n\,\overrightarrow{k} \neq \overrightarrow{0}$. Let $M(x, y, z)$ be an arbitrary point of d.

The vectors $\overrightarrow{r} - \overrightarrow{r_0}$ and $\overrightarrow{v}$ are collinear, hence $\overrightarrow{r} - \overrightarrow{r_0} = t\overrightarrow{v}$, with $t \in \mathbb{R}$. Thus we obtain the *parametric equations* of the line d:

$$x = x_0 + lt,\ y = y_0 + mt,\ z = z_0 + nt,\ t \in \mathbb{R}.$$

By eliminating the parameter t between these equations, we deduce the canonical equations of d:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

Since $\overrightarrow{v} \neq \overrightarrow{0}$, at least one denominator is nonnull. If a denominator equals 0, the corresponding numerator must also equal 0.

Example 3.2.1 For the x-axis we can take $M_0 = 0$ and $\overrightarrow{v} = \overrightarrow{i}$. Hence $x_0 = y_0 = z_0 = 0,\ l = 1,\ m = n = 0.$

The canonical equations are $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{0}$. They are equivalent to $\begin{cases} y = 0 \\ z = 0 \end{cases}$.

2) Equations of the line joining the points $M_0(x_0, y_0, z_0)$ **and** $M_1(x_1, y_1, z_1)$

Let $\overrightarrow{r_0}$ and $\overrightarrow{r_1}$ be the position vectors of these points. Then the line is determined by the point M_0 and the vector $\overrightarrow{r_1} - \overrightarrow{r_0}$. Consequently, we can take $(x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0)$ as direction-ratios.

The canonical equations of the line will be

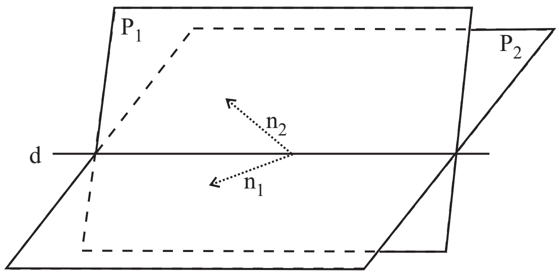
$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}.$$

3) Line determined by the intersection of two planes

Let d be the intersection of the planes P_1 and P_2. Then the equations of d are

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}$$

The normal vectors to P_1, respectively P_2, are $\overrightarrow{n_1} = a_1\overrightarrow{i} + b_1\overrightarrow{j} + c_1\overrightarrow{k}$ and $\overrightarrow{n_2} = a_2\overrightarrow{i} + b_2\overrightarrow{j} + c_2\overrightarrow{k}$.

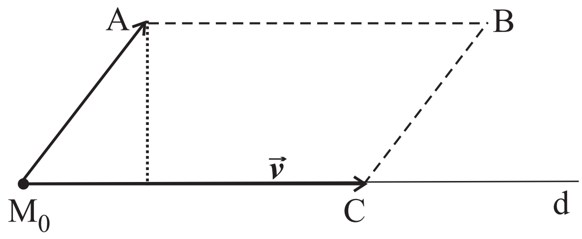


They are both perpendicular to d, so d is parallel to $\overrightarrow{n} = \overrightarrow{n_1} \times \overrightarrow{n_2}$. This enables us to take as direction-ratios of d the components of $\overrightarrow{n}$, that is

$$\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right).$$

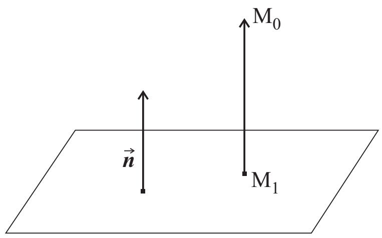
3.3 Distance from a point to a line. Distance from a point to a plane

1) Consider a line d determined by a point M_0 and a vector $\overrightarrow{v}$. The distance from the point A to the line d equals the length of the height of the parallelogram M_0ABC.



Hence $dist(A, d) = \dfrac{||\overrightarrow{v} \times \overrightarrow{M_0A}||}{||\overrightarrow{v}||}$

2) Consider the plane $P : ax + by + cz + d = 0$ and the point $M_0(x_0, y_0, z_0)$. Let M_1 be the projection of M_0 on the plane P.
The vector $\overrightarrow{n} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$ is normal to P.



Let (x_1, y_1, z_1) be the coordinates of M_1. Then $ax_1 + by_1 + cz_1 + d = 0$. We have also $\overrightarrow{M_1M_0} = (x_0 - x_1)\overrightarrow{i} + (y_0 - y_1)\overrightarrow{j} + (z_0 - z_1)\overrightarrow{k}$. Therefore

$\overrightarrow{n} \cdot \overrightarrow{M_1M_0} = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1) =$
$= ax_0 + by_0 + cz_0 + d - (ax_1 + by_1 + cz_1 + d) =$
$= ax_0 + by_0 + cz_0 + d.$
On the other hand,

$|\overrightarrow{n} \cdot \overrightarrow{M_1M_0}| = ||\overrightarrow{n}|| \cdot ||\overrightarrow{M_1M_0}|| = \sqrt{a^2 + b^2 + c^2}\,\mathrm{dist}(M_0, P).$

It follows that

$$\text{dist}(M_0, P) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Exercises

3.1 Write the equation of the plane (P) such that:
a) $M(-1, 2-3) \in (P)$ and $0z \perp (P)$
b) $M(-1, 2-3) \in (P)$ and $0z \parallel (P)$, $0x \parallel (P)$.

3.2 Write the equation of the plane (Q) knowing that it is symmetrical to the plane $(P) : x - 3y + 2z - 1 = 0$ with respect to the point $M(0, -1, 1)$.

3.3 Write the equations of the straight line d that passes through the point $M(3, -1, 0)$ and is parallel to the line $l : \begin{cases} x - 2y + 7 = 0 \\ x + y + z - 6 = 0 \end{cases}$.

3.4 Let $A(2, 2, 2)$, $B(0, 1, 1)$, $C(1, 1, 0)$ and $D(1, 0, 1)$. Find the equations and the length of the height of the tetrahedron $ABCD$ with the basis BCD.

3.5 Let $A(3, -1, 3)$, $B(5, 1, -1)$, $C(0, 4, -3)$. Find the parametric and canonical equations of the lines D_1 and D_2 if:
a) $D_1 = AB$ and $D_2 = BC$
b)D_1 is parallel to AC and passes through B and D_2 is perpendicular to D_1 and passes through C.

3.6 Considering A, B and C from the exercise 3.5, calculate the distances between these three points and find the angles formed by AB, AC and BC.

3.7 Considering A and B from the exercise 3.5, find the equation of a plane with respect to which A and B are symmetrical.

3.8 Find the equation of a plane which passes through the point M and is parallel to the plane (P) if:
a)$M(2,-1,3)$ and $(P): x-3y+5z+2=0$
b)$M(0,-2,4)$ and $(P): 7x+4y-3z-1=0$
c)$M(1,0,-1)$ and $(P): 2y-5x-11z=0$.

3.9 Find the equation of a plane (P) if:
a) $M(2,3,-5) \in (P)$ and $OM \perp (P)$;
b) $A(2,1,-6)$ and $B(6,-1,-2)$ are symmetrical about the plane (P);
c) $M_1(3,2,1) \in (P)$, $M_2(6,6,8) \in (P)$ and (P) cuts equal segments on Ox and Oz.

3.10 Write the equations of three planes that contain $M(3,2,-1)$ and each contains a different coordinate axis.

3.11 Find the equation of a plane which passes through A and is perpendicular to the planes (P_1) and (P_2) if:
a)$A(-1,1,0)$, $(P_1): x-2y+z-5=0$ and $(P_2): y-5z+2=0$
b)$A(1,0,1)$, $(P_1): 3x+y-1=0$ and $(P_2): x+y-z-1=0$

3.12 Write the equations of three planes that contain $A(2,-1,-1)$ and $B(3,1,2)$ and each, is parallel to a different coordinate axis.

3.13 Find the equation of a plane which contains the point A and is perpendicular to AB, if:
a) $A(1,2,-1)$, $B(2,3,5)$
b) $A(1,3,2)$, $B(-3,-1,0)$
c) $A(2,0,1)$, $B(1,1,-1)$.

3.14 A plane cuts, on the coordinate axes, segments equal to $3, 10$ and 5. Find the equation of the plane and the angles formed by the plane and the axes.

3.15 Find the equation of a plane determined by the lines

$$D_1: \begin{cases} x+y-3z=0 \\ 2x+3y-z-1=0 \end{cases} \text{ and } D_2: \begin{cases} x+5y+4z-3=0 \\ x+2y+2z-1=0 \end{cases}.$$

3.16 Write the equation of a plane which contains $M(-1,1,1)$ and is perpendicular to the line D, if:

a) $D: \dfrac{x-2}{3} = \dfrac{y}{2} = \dfrac{z+1}{-1}$

b) $D: \dfrac{x}{4} = \dfrac{y-2}{-4} = \dfrac{z-3}{5}$

c) D: $\begin{cases} x+y=0 \\ x+y-2z+1=0 \end{cases}.$

3.17 Let D_1, D_2 be two lines parallel to the vectors $d_1 = (-1,0,1)$ and $d_2 = (1,1,0)$. Find:

a) the angle between D_1 and D_2

b) the parametric equations of the line D_3 perpendicular to D_1 and D_2, which passes through $M(3,2,1)$.

3.18 Calculate the distance between the point $A(3,-1,1)$ and the line D_1 if:

a) D_1: $\begin{cases} 2x-y+2z-3=0 \\ x-y-3z+2=0 \end{cases}$

b)D_1: $\dfrac{x-1}{4} = \dfrac{y}{-5} = \dfrac{z+2}{3}.$

3.19 Write the equation of a plane which passes through the point $M(1,-1,1)$ and is perpendicular to the line D if:

a) D: $\dfrac{x-3}{2} = \dfrac{y}{3} = \dfrac{z+1}{-1}$

b) D: $\begin{cases} x-z+3=0 \\ 2x-y=0 \end{cases}$

3.20 A plane contains the point $A(1,0,1)$ and the line D. Find the equation of the plane if:

a) D: $\begin{cases} x=2-3t \\ y=4+t \\ z=1-2t \end{cases}$

b) D: $\dfrac{x}{-2} = \dfrac{y-1}{4} = z-5$

c) D: $\begin{cases} x+z+1=0 \\ x-2y+z-3=0 \end{cases}$

3.21 We consider the planes P_1, P_2 and P_3 such that $A(-1,-2,2) \in P_1$ and the vector normal to P_1 is $(1,-2,2)$, the plane P_2 is perpendicular to the line D: $\dfrac{x}{2} = \dfrac{y+7}{-1} = \dfrac{z-1}{-2}$ and contains the point $B(1,1,1)$ and P_3: $2x+2y+z=2$.

1) Find the equations of P_1 and P_2.
2) Show that each of two planes are perpendicular.
3) Find the intersection of the planes.
4) Calculate the distance from $A(2,4,7)$ to P_1.

3.22 Find the equation of a plane which contains the symmetric points of $A(2,3,-1)$, $B(1,2,4)$ and $C(0,1,-1)$ with respect to the plane $P:$ $x-y+2z+2=0$.

3.23 Find the projection of $M(2,1,1)$, on the plane $P: x+y+3z+5=0$ and calculate the distance from M to P.

3.24 Find the equations of two planes P_1 and P_2 if both pass through the line D: $\begin{cases} 2x+y-3z+2=0 \\ 5x+5y-4z+3=0 \end{cases}$, $P_1 \perp P_2$ and P_1 contains $M(4,-3,1)$.

3.25 Find the position of the line D relative to the plane P if:

a) D: $\begin{cases} x=t \\ y=1+2t \\ z=-6t \end{cases}$ and $P: 4x+y+z=4$

b) D: $\begin{cases} x=13+8t \\ y=1+2t \\ z=2+3t \end{cases}$ and $P: x+y+2z=2$.

3.26 Find the distance between two lines D_1 and D_2 and the equation of the common perpendicular if it exists, for:

a) D_1: $\dfrac{x-1}{-5} = y-2 = z$ D_2: $\begin{cases} x+2z=4 \\ y=0 \end{cases}$

b) D_1: $\dfrac{x}{3} = \dfrac{y-1}{2} = z-5$ and D_2: $\begin{cases} x=1+3t \\ y=2t \\ z=1+t \end{cases}$

c) D_1: $\dfrac{x-1}{3} = \dfrac{y+2}{2} = z-4$ and D_2: $\begin{cases} x = 1+t \\ y = 2t-2 \\ z = 4+5t \end{cases}$.

Solutions

$\boxed{\mathbf{3.1}}$ a) $0z$ has the direction vector $\vec{k}$ and is normal to the requested plane (P). The equation is $0 \cdot (x+1) + 0 \cdot (y-2) + 1 \cdot (z+3) = 0$, that is $z+3=0$. b) The plane is determined by a point and two vectors, $\begin{vmatrix} x+1 & y-2 & z+3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0$, that is $y-2=0$. (In fact, the plane is perpendicular to $0y$).

$\boxed{\mathbf{3.2}}$ We choose three points that belong to the plane (P), for instance $A(1,0,0)$, $B(0,1,2)$ and $C(-1,0,1)$. We determine their symmetrical points A_1, B_1, C_2 with respect to M, from the fact that M is the middle of the segments $[AA_1]$, $[BB_1]$, $[CC_1]$, getting $A_1(-1,-2,2)$, $B_1(0,-3,0)$, $C_1(1,-2,1)$. The plane (Q) is determined by these three points: $x-3y+2z-9=0$.

$\boxed{\mathbf{3.3}}$ We find first the direction vector of of l, for instance $\vec{l} = \vec{n}_1 \times \vec{n}_2$, where $\vec{n}_1 = (1,-2,0)$ and $\vec{n}_2 = (1,1,1)$ are the normals to the planes that determine l. So $\vec{l} = (-2,-1,3)$ and the equations of the line d are $\dfrac{x-3}{-2} = \dfrac{y+1}{-1} = \dfrac{z}{3}$ or, in another form, $d : \begin{cases} x-2y-5=0 \\ 3y+z+3=0 \end{cases}$.

$\boxed{\mathbf{3.4}}$ The plane BCD has the equation $x+y+z-2=0$, so the normal is $\vec{n}=(1,1,1)$. The equations of the height from A are $x=y$ and $x=z$ and the intersection point between the height and the plane BCD is $H(\dfrac{2}{3},\dfrac{2}{3},\dfrac{2}{3})$. The length of the height is $AH=\frac{4}{3}\sqrt{3}$.

$\boxed{\mathbf{3.5}}$ a) $D_1=AB=\frac{x-3}{2}=\frac{y+1}{2}=\frac{z-3}{-4}$
b) $D_1 \parallel AC$ means the direction of D_1 is $\bar{D_1}=\bar{AC}=(-3,5,-6)$, then $D_1:\frac{x-5}{-3}=\frac{y-1}{5}=\frac{z+1}{-6}$.
Let $CM\perp D_1$ and $M(a,b,c)\in D_1$, then $D_2=CM$. We know D_2 is perpendicular to D_1, so $(-3,5,-6)\cdot(a,b-4,c+3)=0$. Also $M\in D_1 \Leftrightarrow \frac{a-5}{-3}=\frac{b-1}{5}=\frac{c+1}{-6}$ and after finding a, b, and c from this system, we obtain the line $D_2=CM$.

$\boxed{\mathbf{3.6}}$ $d(A,B)=\sqrt{(5-3)^2+(1+1)^2+(-1-3)^2}=2\sqrt{6}$, etc.

Let $\alpha=\sphericalangle(AB,AC)$, then we have $\cos\alpha=\frac{\vec{AB}\cdot\vec{AC}}{\|\vec{AB}\|\cdot\|\vec{AC}\|}$, if $\vec{AB}=(2,2,-4)$ and $\vec{AC}=(-3,5,-6)$.

$\boxed{\mathbf{3.7}}$ Consider $M(x,y,z)\in P$. Then $\|AM\|=\|MB\|$, with $AM\perp P$ which implies $P:x+y-2z-2=0$.

$\boxed{\mathbf{3.8}}$ a) Let P_1 be the plane parallel to P, then the vector normal to P_1 is the vector normal to P, $\vec{n}=(1,-3,5)$. The equation of the plane is P_1: $x-2-3(y+1)+5(z-3)=0$.

$\boxed{\mathbf{3.9}}$ c) Consider $A(a,0,0)\in Ox$ and $B(0,0,a)\in Oz$, $\|OA\|=$

$\|OB\|$. We write the plane P in two ways:

$$(AM_1M_2): \begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 3 & 2 & 1 & 1 \\ 6 & 6 & 8 & 1 \end{vmatrix} = 0, \qquad (BM_1M_2): \begin{vmatrix} x & y & z & 1 \\ 0 & 0 & a & 1 \\ 3 & 2 & 1 & 1 \\ 6 & 6 & 8 & 1 \end{vmatrix} = 0$$

and obtain $P: 2x - 5y + 2z + 2 = 0$

3.10 $(MOx): \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 3 & 2 & -1 \end{vmatrix} = 0$ and obtain $(MOx): y + 2z = 0$, etc.

3.11 a) The normals $\vec{n}_1$ and $\vec{n}_2$ of the planes P_1 and P_2 are parallel to the plane P, so we get $P: \begin{vmatrix} x+1 & y-1 & z \\ 1 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 0$, $9x + 5y + z + 4 = 0$.

3.12 $P_1: \begin{vmatrix} x-3 & y-1 & z-2 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 0$, etc.

3.13 a) The normal of the plane is the vector $\vec{AB} = (1,1,6)$, so the equation of the plane, which contains A, is $x - 1 + y - 2 + 6(z+1) = 0$.

3.14 The equation of the plane is P: $10x + 3y + 6z - 30 = 0$. Consider the normal to the plane $\vec{n}$, and for the coordinates axes we have the unit vectors $\vec{i}$, $\vec{j}$, $\vec{j}$. By denoting $\alpha = \sphericalangle(\vec{i}, \vec{n})$, we have $\sin\alpha = \dfrac{\vec{i}\cdot\vec{n}}{\left\|\vec{i}\right\|\cdot\|\vec{n}\|} = \dfrac{10}{\sqrt{145}}$, etc.

$\boxed{\mathbf{3.15}}$ $P : x + 2y + 2z - 1 = 0$

$\boxed{\mathbf{3.16}}$ c) The direction of the line D is

$$d = (\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}) = (-2, 2, 0)$$

and this line is normal to the plane. The equation of the plane is $P : x - y + 2 = 0$

$\boxed{\mathbf{3.17}}$ a) $\alpha = 2\pi/3$
b)

$$\bar{d_1} \times \bar{d_2} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -i + j - k.$$

Then the equation of the line is D: $\begin{cases} x = 3 - t \\ y = 2 + t \\ z = 1 - t \end{cases}$

$\boxed{\mathbf{3.18}}$ a) $\vec{d_1} = (-5, -8, 1)$, so $\left\|\vec{d_1}\right\| = 3\sqrt{10}$, then the distance is calculated from $d(A, D_1) = \frac{\left\|\vec{MA}\times\vec{d_1}\right\|}{\left\|\vec{d_1}\right\|}$.

$\boxed{\mathbf{3.19}}$ b) $P : x + 2y + z = 0$

$\boxed{\mathbf{3.20}}$ c) The pencil of planes which pass through D is $x-2y+z-3+\lambda(x+z+1) = 0$ and we need the plane which contains A, so, $\lambda = 1/3$ then the plane is $P : 4x - 6y + 4z - 8 = 0$.

$\boxed{\textbf{3.21}}$ 1) $P_1 : x - 2y + 2z - 7 = 0$, $P_2 : 2x - y - 2z + 1 = 0$.
2) We verify that the scalar product between the normals of two planes is zero.
3) $P_1 \cap P_2 \cap P_3 = M(1, -1, 2)$.
4) $d(A, P_1) = 1/3$.

$\boxed{\textbf{3.22}}$ Consider A', B', C' the symmetrical points of A, B, and C with respect to the plane P. We take $C_0 = CC' \cap P$, $C_0 \in P$ and obtain $C_0(1/6, 5/6, -2/3)$ after we solve the system obtained from the equations of the line $CC' : \frac{x}{1} = \frac{y-1}{-1} = \frac{z+1}{2}$ and the plane P. Then we find the coordinates of C' by knowing $\|CC_0\| = \left\|C_0C'\right\|$. By using the same procedure we find A' and B'.

$\boxed{\textbf{3.23}}$ Consider $M' \in P$ the projection of M on P, $MM' : x - 2 = y - 1 = \frac{z-1}{3}$, then $M'(1, 0, -2)$ and $d(M', P) = 11/\sqrt{11}$.

$\boxed{\textbf{3.24}}$ The pencil of planes passing through D is
$P_\mu : 2x + y - 3z + 2 + \mu(5x + 5y - 4z + 3) = 0$. $M \in P_1$ and $P_1 \subset P_\mu \Longrightarrow M \in P_\mu$ which gives us $\mu = -1$, so $P_1 : 3x + 4y - z + 1 = 0$.
Let $P_2 : ax + by + cz + d = 0$, from $P_1 \perp P_2$ we have $3a + 4b - c = 0$ and considering also $P_2 \subset P_\mu$ we obtain the relations $a = 2 + 5\mu, b = 1 + 5\mu, c = -3 - 4\mu$. Then, for $\mu = -1/3$, we obtain $P_2 : x - 2y - 5z + 3 = 0$.

$\boxed{\textbf{3.25}}$ a) Consider the system $\begin{cases} 2x = y - 1 \\ -6x = z \\ 4x + y + z = 4 \end{cases}$

which have the determinant zero and the rank of the corresponding matrix 2 and observe the system is inconsistent, so $D \parallel P$. This could be, also, observed if we check that the normal to the plane is perpendicular to the line D.
b) The system is consistent and determined and we obtain $t = -1$, then we find x, y, z the coordinates of the point $M(x, y, z)$, where $M = D \cap P$.

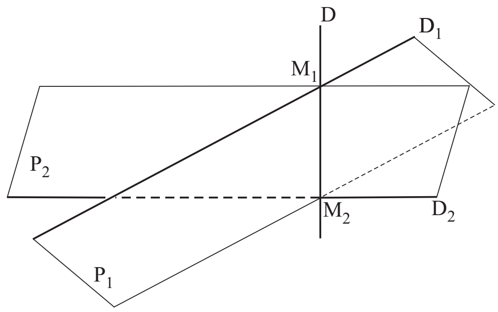
$\boxed{\textbf{3.26}}$ a) The direction of the common perpendicular of D_1 and D_2 is

$$\vec{d} = (-5, 1, 1) \times (2, 0, -1) = \begin{vmatrix} i & j & k \\ -5 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = -i - 3j - 2k = (-1, -3, -2)$$

The equation of the plane which contains D_1 and D is

$$P_1 : \begin{vmatrix} x-1 & y-2 & z \\ -5 & 1 & 1 \\ -1 & -3 & -2 \end{vmatrix} = 0 \Leftrightarrow x - 11y + 16z + 21 = 0.$$

Similarly, the equation of the plane which contains D_2 and D is $P_2 : -3x + 5y - 6z + 12 = 0$. The common perpendicular is $D : \begin{cases} P_1 \\ P_2 \end{cases}$.



The distance is $d(D_1, D_2) = \|M_1M_2\|$, where $\{M_1\} = D_1 \cap D$ and $\{M_2\} = D_2 \cap D$.

b) $D_1 \parallel D_2$, let $A_1 \in D_1$ and $A_2 \in D_2$, then

$$d(D_1, D_2) = d(A_1, D_2) = \frac{\left\|\vec{d_2} \times \overrightarrow{A_1A_2}\right\|}{\left\|\vec{d_2}\right\|}.$$

c) $D_1 \cap D_2 = M(1, -2, 4)$, so the distance is zero.

CHAPTER 4

Linear spaces

4.1 The definition of a linear space

Let K be the field of real numbers or the field of complex numbers.

Definition 4.1 A set V is called a *linear space* (or a *vector space*) over the field K if it satisfies the following conditions:

I) There exists an internal binary operation on V, called addition and denoted by $+$, such that $(V, +)$ is a commutative group.

II) There exists an external binary operation called scalar multiplication, in which each element $k \in K$ can be combined with each element $v \in V$ to give an element $kv \in V$, and such that, for all $k, l \in K$ and $x, y \in V$,

1) $k(x + y) = kx + ky$
2) $(k + l)x = kx + lx$
3) $(kl)x = k(lx)$
4) $1x = x$.

We must be careful to distinguish between the two types of elements: those belonging to V called *vectors*, and those belonging to K called *scalars*.

Example 4.1.1 1) The set $\mathcal{V}_3$ of the vectors in space with the usual definitions of addition and multiplication by a real number, forms a linear space over the field $\mathbb{R}$.

2) Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ $(x_i, y_i \in K)$ be two elements of K^n (the set of n-tuples of elements of K). The addition $x + y$ and scalar multiplication λx $(\lambda \in K)$ may be defined by

$$x + y = (x_1 + y_1, \ldots, x_n + y_n)$$

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n)$$

With these operations it is easily verified that K^n is a linear space over the field K.

3) An obvious generalization of the previous example is the set $\mathcal{M}_{n,m}(K)$ with the usual definitions of addition of matrices and multiplication of a matrix by an element of K.

4) Let S be any set and $F = \{f | f : S \longrightarrow K\}$.
With the usual definitions of addition of functions and multiplication of a function by a number, F is a linear space over K.

We see that the structure of linear space appears in various and quite natural situations ([4],[5], [6], [7]).

The first theorem gives a number of elementary deductions from the definition of a linear space. We must be careful to distinguish between 0, the zero of K, and 0, the zero vector of V.

Theorem 4.2 *In any linear space V over K we have*

(i) $0v = 0$;

(ii) $k0 = 0$;

(iii) $(-1)v = -v$,

for all $v \in V$ *and* $k \in K$. *(*$-v$ *is the negative of* v *in the group* $(V, +)$*).*

Proof.

(i) Since $0v = (0+0)v = 0v + 0v$, we infer that $0v = 0$.

(ii) $k0 = k(0+0) = k0 + k0$, hence $k0 = 0$.

(iii) $v + (-1)v = 1v + (-1)v = [1 + (-1)]v = 0v = 0$, therefore $(-1)v = -v$. □

Theorem 4.3 *(a) If* $k \in K, v \in V$ *and* $kv = 0$*, then either* $k = 0$ *or* $v = 0$.

(b) If $lv = kv$ *and* $v \neq 0$*, then* $l = k$.

(c) If $kv = kw$ *and* $k \neq 0$*, then* $v = w$.

Proof.

(a) Suppose that $k \neq 0$. Then there exists $k^{-1} \in K$. We have $k^{-1}(kv) = k^{-1} \cdot 0$, hence $(k^{-1}k)v = 0$. It follows that $1v = 0$ and finally $v = 0$, q.e.d.

(b) $lv = kv$ implies $(l-k)v = 0$. Since $v \neq 0$ we may apply (a) and deduce $l - k = 0$, that is, $l = k$.

(c) is left to the reader. □

4.2 Linear subspaces

Let V be a linear space over K. A non-empty subset W of V is called a *linear subspace* (or a *vector subspace*) of V if

$kx + ly \in W$ for all $k, l \in K$ and $x, y \in W$.

Let us remark that this condition is equivalent to the following two conditions:

(1) $x + y \in W$ for all $x, y \in W$

(2) $kx \in W$ for all $k \in K$ and $x \in W$.

Any linear subspace W contains the vector 0; indeed, for any $v \in W$ we have $0v \in W$ and hence $0 \in W$.

Example 4.2.1 (1) $\{0\}$ and V are linear subspaces of V. These two subspaces are called *improper subspaces* of V; all other subspaces are *proper subspaces*.

(2) $\{a\overrightarrow{i} \mid a \in \mathbb{R}\}$ and $\{a\overrightarrow{i} + b\overrightarrow{j} \mid a, b \in \mathbb{R}\}$ are linear subspaces of $\mathcal{V}_3$.

(3) $\{(0, x_2, \ldots, x_n) \mid x_2, \ldots, x_n \in K\}$ is a linear subspace of K^n.

Let $S \subset V, S \neq \emptyset$. A vector $v \in V$ of the form $v = k_1v_1 + \cdots + k_nv_n$, where $n \in \mathbb{N}^*$, $k_i \in K$ and $v_i \in S$ is called a *linear combination* of elements of S. It is easy to verify that the set of all linear combinations of elements of S is a linear subspace of V, called the subspace *generated* by S.

Theorem 4.4 *Let U and W be linear subspaces of the space V.*

a) $U \cap W$ is a linear subspace of V.

b) The set $U + W = \{u + w \mid u \in U, w \in W\}$ is a linear subspace of V, called the sum of U and W.

The (easy) proof is left to the reader.

4.3 Linear dependence, bases, dimension

A subset X of a linear space V is called a *linearly dependent set* if it contains a finite subset $\{x_1, \ldots, x_r\}(r \geq 1)$ for which there exist scalars $k_1, \ldots, k_r \in K$, not all zero, such that $k_1x_1 + \cdots + k_rx_r = 0$. Such a linear relation, where not all the k_i are zero, will be called *non-trivial.*

A subset of a linear space is linearly independent if it is not linearly dependent. An alternative definition, equivalent to this is:*A* set X is linearly independent if every linear relation $k_1x_1 + \cdots + k_rx_r = 0$ $(k_i \in K)$ between the vectors x_i of X has zero coefficients. In other words, every linear relation between the vectors of X is trivial.

Example 4.3.1 1) Every subset $X \subset V$ which contains 0 is linearly dependent.

2) If $v \in V, v \neq 0$, then $\{v\}$ is linearly independent.

3) Let $V = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$. Let $f_i \in V,\ f_i(t) = t^i,\ i = 0, 1, \ldots, n.$ Then $\{f_0, f_1, \ldots, f_n\}$ is linearly independent.

4) $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in V_3$ are linearly dependent if and only if they are coplanar.

Definition 4.5 Any linearly independent subset of a vector space V, which has the property that it generates V, is called a *basis* of V.

It can be shown that every vector space $V \neq \{0\}$ possesses a basis. Also, if V has a finite basis with r elements, then every basis of V has r elements. We say that the *dimension* of V is r and write $dimV = r$.

If V has no finite bases, it is called infinite-dimensional. In this case we can find arbitrarily large linearly independent finite subsets of V. On the other hand, we write $dim\{0\} = 0$.

Example 4.3.2 1) $\{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\}$ is a basis of $\mathcal{V}_3$.

2) The vectors $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ form a basis of K^n, called the *canonical basis* of K^n. Thus, $dimK^n = n$.

3) Let $K_n[X]$ be the linear space of all polynomials of degree $\leq n$, with coefficients in K. A basis of this space is $\{1, X, X^2, \ldots, X^n\}$.

4) Let $K[X]$ be the space of all polynomials with coefficients in K. A basis of it is $\{1, X, X^2, \ldots, X^n, \ldots\}$. Hence $K[X]$ is infinite-dimensional.

Let V be finite-dimensional. It can be shown that if U and W are linear subspaces of V, then

$$dim(U + W) + dim(U \cap W) = dimU + dimW.$$

Theorem 4.6 *Let $T = \{v_1, \ldots, v_m\} \subset V$ be a linearly independent set which is not a basis. Then there exists $v \in V$ such that $\{v_1, \ldots, v_m, v\}$ is linearly independent.*

Theorem 4.7 *a) Every linearly independent subset of V_n with n elements is a basis of V_n.*

b) Every linearly independent subset of V_n is a part of a basis.

4.4 Coordinates. Change of bases

Let $B = \{b_1, \ldots, b_n\}$ be a basis of the n-dimensional linear space V_n over K.

Theorem 4.8 *Each $v \in V_n$ can be written uniquely in the form*

$$v = x_1 b_1 + \cdots + x_n b_n$$

with $x_1, \ldots, x_n \in K$. (The scalars $x_1, \ldots, x_n$ are called the coordinates of the vector v relative to the basis B.)

Proof. Let $v \in V_n$. Since B generates V_n, there exist scalars $x_1, \ldots, x_n$ such that $v = x_1 b_1 + \ldots x_n b_n$. We have to prove that they are uniquely determined.

Suppose that $x'_1, \ldots, x'_n \in K$ and $v = x'_1 b_1 + \cdots + x'_n b_n$. Then it follows $(x_1 - x'_1)b_1 + \cdots + (x_n - x'_n)b_n = 0$. Since $b_1, \ldots, b_n$ are linearly independent, it follows that $x'_1 = x, \ldots, x'_n = x_n$ and the theorem is proved. $\square$

Consider now the above basis B and let $B' = \{b'_1, \ldots, b'_n\} \subset V_n$. Then we have $b'_j = \sum\limits_{i=1}^{n} c_{ij} b_i,\ j = 1, \ldots, n$, with $c_{ij} \in K$.

Theorem 4.9 *B' is a basis of V_n if and only if det$(c_{ij}) \neq 0$.*

Proof. Since B' has n elements, the following two statements are equivalent:

(1) B' is a basis

(2) B' is linearly independent

Clearly (2) is equivalent to

(3) $k_1 b'_1 + \cdots + k_n b'_n = 0 \Longrightarrow k_1 = \cdots = k_n = 0$.

We have $\sum\limits_{j=1}^{n} k_j b'_j = \sum\limits_{j=1}^{n} k_j \sum\limits_{i=1}^{n} c_{ij} b_i = \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{n} c_{ij} k_j b_i = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} c_{ij} k_j b_i = \sum\limits_{i=1}^{n} \Big(\sum\limits_{j=1}^{n} c_{ij} k_j \Big) b_i$.

Thus the first equality in (3) is equivalent to $\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} c_{ij}k_j\right)b_i = 0$, which is equivalent (due to the linear independence of B) to $\sum\limits_{j=1}^{n} c_{ij}k_j = 0,\ i = 1,\ldots,n$. Hence (3) is equivalent to

(4) The linear homogeneous system $\sum\limits_{j=1}^{n} c_{ij}k_j = 0,\ i = 1,\ldots,n$, has only the trivial solution.Finally, (4) is equivalent to

(5) $det(c_{ij} \neq 0$

We conclude that (1) and (5) are equivalent and the theorem is proved.

Let us remark that the columns of the matrix $C = (c_{ij}),\ i,j = 1,\ldots,n$ are formed with the coordinates of b'_j relative to the basis B. Suppose that C is nonsingular; this means that B' is also a basis of V_n. C is called the *transition matrix* from B to B'.

Let $x \in V_n$. We have $x = \sum\limits_{i=1}^{n} x_i b_i$ and $x = \sum\limits_{j=1}^{n} x'_j b'_j$, with $x_i, x'_j \in K$. Then $x = \sum\limits_{j=1}^{n} x'_j \sum\limits_{i=1}^{n} c_{ij}b_i = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{n} c_{ij}x'_j b_i = \sum\limits_{i=1}^{n}(\sum\limits_{j=1}^{n} c_{ij}x'_j)b_i$.

Hence $\sum\limits_{i=1}^{n} x_i b_i = \sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} c_{ij}x'_j\right)b_i$. It follows that

(6) $x_i = \sum\limits_{j=1}^{n} c_{ij}x'_j,\ i = 1,\ldots,n.$

We have here the relationship between the coordinates of x relative to the basis B and the coordinates of the same x relative to the basis B'. Let us denote

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ X' = \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}$$

Then (6) is equivalent to $X = CX'$. □

Finally, let us mention-without proof - the following important result.

Let $B = \{b_1, \ldots, b_n\}$ be a basis of V_n and let $v_1, \ldots, v_p \in V$. Write $v_j = \sum_{i=1}^{n} a_{ij} b_i, \ j = 1, \ldots, p$, with $a_{ij} \in K$. Consider the matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1p} \\ \ldots & & \\ a_{n1} & \ldots & a_{np} \end{pmatrix}$$

Theorem 4.10 *The dimension of the linear subspace of V_n generated by $\{v_1, \ldots, v_p\}$ equals r_A.*

Exercises

4.1 Let $V = \{x \in \mathbb{R} \mid x > 0\}$ be endowed with the internal operation $x \oplus y = xy$. Prove that $(V, \oplus)$ is a linear space over $\mathbb{R}$ with the external operation $\alpha * x = x^{\alpha}$, for each $x \in V$, $\alpha \in \mathbb{R}$.

4.2 Prove that all square matrices of order n with real elements, form a vector space over the field of real numbers, if the operations involved are addition of matrices and multiplication of a matrix by a scalar. Find a basis and dimension of this space.

4.3 Prove that all polynomials of degree $\leq n$ with real coefficients form a vector space if the operations involved are ordinary addition of polynomials and multiplication of a polynomial by a scalar. Find a basis and dimension of this space.

4.4 Determine which of the following sets are linear subspaces of the corresponding linear spaces.
a) $W_1 = \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\}$, in $\mathbb{R}^n$ over $\mathbb{R}$
b) $W_2 = \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 1\}$, in $\mathbb{R}^n$ over $\mathbb{R}$
c) $W_3 = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{Z}, i = 1, \ldots, n\}$, in $\mathbb{R}^n$ over $\mathbb{R}$
d) $W_4 = \{(x, y, z) \mid 2x - 3y + z = 0\}$, in $\mathbb{R}^3$ over $\mathbb{R}$
e) $W_5 = \{(x, y, z) \mid 2x - 3y + z + 6 = 0\}$, in $\mathbb{R}^3$ over $\mathbb{R}$
f) $W_6 = \{(x, y, z) \mid \frac{x}{3} = \frac{y}{-2} = \frac{z}{8}\}$, in $\mathbb{R}^3$ over $\mathbb{R}$
g) $W_7 = \{(x, y, z) \mid \frac{x-1}{3} = \frac{y}{-2} = \frac{z}{8}\}$, in $\mathbb{R}^3$ over $\mathbb{R}$
h) $W_8 = \{f : I \to \mathbb{R} \mid f \text{ differentiable on } I\}$, in $C(I)$ over $\mathbb{R}$, the space of continuous functions on the interval $I \in \mathbb{R}$
i) $W_9 = \{P \mid P \text{ is a polynomial of odd degree}\}$, in $\mathbb{R}_n[X]$ over $\mathbb{R}$, the space of polynomials of degree at most n with real coefficients.

4.5 Prove that the following sets of vectors are subspaces in $\mathbb{R}^n$ over $\mathbb{R}$ and find a basis and dimension of each:
a) All n-dimensional vectors with the first and last coordinates equal.
b) All n-dimensional vectors of the form $(\alpha, \beta, \alpha, \beta, ...)$, where α and β are any numbers.

4.6 Find out if the following matrices are linearly independent in the space $\mathcal{M}_2(\mathbb{R})$, for $a \in \mathbb{R}$:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}.$$

4.7 Determine a basis in the linear subspace generated by the set of functions $\{1, \sin^2 x, \cos^2 x, \cos 2x\}$.

4.8 Determine the dimension and a basis for the linear subspace V generated:

a) in $\mathbb{R}^4$ by the vectors: $v_1 = (0, 2, -1, 3)$, $v_2 = (1, 1, 2, -1)$, $v_3 = (2, 5, -2, 3)$ and $v_4 = (-1, 0, 2, 2)$,

b) in $\mathbb{R}^4$ by the vectors: $v_1 = (2, 1, 3, 0)$, $v_2 = (-3, 1, 1, 2)$, $v_3 = (-1, 2, 4, 2)$ and $v_4 = (-1, 0, 2, -2)$,

c) in $\mathbb{R}^3$ by the vectors: $v_1 = (-1, 3, 2)$, $v_2 = (1, 4, 1)$, $v_3 = (0, 1, 2)$.

4.9 Find the dimensions and bases of the linear subspaces spanned (generated) by the following sets of vectors:
a) $a_1 = (1, 0, 0, -1)$, $a_2 = (1, 1, 1, 1)$, $a_3 = (2, 1, 1, 0)$, $a_4 = (1, 2, 3, 4)$ and $a_5 = (0, 1, 2, 3)$.
b) $a_1 = (1, 1, 1, 1, 0)$, $a_2 = (1, 1, -1, -1, -1)$, $a_3 = (2, 2, 0, 0, -1)$, $a_4 = (1, 1, 5, 5, 2)$ and $a_5 = (1, -1, -1, 0, 0)$

4.10 Find the dimensions of the union and intersection of the linear subspaces $S_1 = \text{span}\{a_1, a_2, ..., a_k\}$ and $S_2 = \text{span}\{b_1, b_2, ..., b_m\}$, if:
a) $a_1 = (1, 2, 0, 1)$, $a_2 = (1, 1, 1, 0)$ and $b_1 = (1, 0, 1, 0)$, $b_2 = (1, 3, 0, 1)$
b) $a_1 = (1, 1, 1, 1)$, $a_2 = (1, -1, 1, -1)$, $a_3 = (1, 3, 1, 3)$ and $b_1 = (1, 2, 0, 2)$, $b_2 = (1, 2, 1, 2)$, $b_3 = (3, 1, 3, 1)$.

4.11 Find bases of the unions and intersections of the linear subspaces $S_1 = \text{span}\{a_1, a_2, ..., a_k\}$ and $S_2 = \text{span}\{b_1, b_2, ..., b_m\}$:
a) $a_1 = (1, 2, 1)$, $a_2 = (1, 1, -1)$, $a_3 = (1, 3, 3)$ and $b_1 = (2, 3, -1)$, $b_2 = (1, 2, 2)$, $b_3 = (1, 1, -3)$.
b) $a_1 = (1, 2, 1, -2)$, $a_2 = (2, 3, 1, 0)$, $a_3 = (1, 2, 2, -3)$ and $b_1 = (1, 1, 1, 1)$, $b_2 = (1, 0, 1, -1)$, $b_3 = (1, 3, 0, -4)$.

4.12 Consider in $\mathbb{R}^3$ the linear subspaces P and Q given by $P : 5x - 2y + z = 0$, $Q : x + y - 3z = 0$. Determine bases in P, Q, $P \cap Q$ and in $\text{sp}(P \cup Q)$.

4.13 Find the coordinates of the vector $v = (-3, 1, 2)$ in the basis $B' = \{(1, -1, 0), (1, 0, -1), (0, 1, 1)\}$.

4.14 Show that the vectors $e_1 = (1, 1, 1)$, $e_2 = (1, 1, 2)$, $e_3 = (1, 2, 3)$ form a basis in $\mathbb{R}^3$ and find the coordinates of the vector $a = (6, 2, -7)$ in this basis.

4.15 Show that the vectors $e_1 = (1, 2, -1, -2)$, $e_2 = (2, 3, 0, -1)$, $e_3 = (1, 2, 1, 4)$ and $e_4 = (1, 3, -1, 0)$ form a basis in $\mathbb{R}^4$ and find the coordinates of the vector $b = (7, 14, -1, 2)$ in this basis.

4.16 Prove that each of the two sets of vectors is a basis in $\mathbb{R}^3$ and find the relationship between the coordinates of one and the same vector in the two bases:
$a_1 = (1, 2, 1)$, $a_2 = (2, 3, 3)$, $a_3 = (3, 7, 1)$ and $b_1 = (3, 1, 4)$, $b_2 = (5, 2, 1)$, $b_3 = (1, 1, -6)$.

4.17 Let $P_1 = (X-b)(X-c)$, $P_2 = (X-a)(X-c)$, $P_3 = (X-a)(X-b)$ be polynomials from $\mathbb{R}_2[X]$, $a, b, c \in \mathbb{R}$.

a) Determine the condition under which P_1, P_2, P_3 are linearly independent.

b) Considering the condition of (a) satisfied, write the polynomial $P = 1 + X + X^2$ as a linear combination of P_1, P_2 and P_3.

4.18 In the space of polynomials of degree at most two over $\mathbb{R}$, consider the canonical basis $B = \{1, X, X^2\}$ and another basis $B' = \{1, X - a, (X-a)^2\}$, where $a \in \mathbb{R}$.

a) Determine the transition matrix from B to B',

b) Determine the coordinates of the polynomial $f = \alpha + \beta X + \gamma X^2$ in the new basis B'.

4.19 Find the coordinates of the polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ in the following bases:
a) $1, x, x^2, \ldots, x^n$.
b) $1, x - \alpha, (x-\alpha)^2, \ldots, (x-\alpha)^n$.

4.20 Prove that each of the two sets of vectors is a basis in the space of polynomials of degree ≤ 3 with real coefficients and find the transition

matrix between the two bases:
$e_1 = 1$, $e_2 = x$, $e_3 = x^2$ and $e_4 = x^3$ and $e_1^{'} = 1 - x$, $e_2^{'} = 1 + x^2$, $e_3^{'} = x^2 - x$ and $e_4^{'} = x^3 + x^2$

4.21 Find a basis in the real space of the solutions of the following systems:

a) $\begin{cases} x + y - z + 2t = 0 \\ x - 2y + t = 0 \end{cases}$ b) $\begin{cases} x + y - z + t = 0 \\ x - y + 2z - t = 0 \\ 2x + y - z - t = 0 \end{cases}$

c) $\begin{cases} x + 2y + 4z - 3t = 0 \\ 3x + 5y + 6z - 4t = 0 \\ 3x + 8y + 24z - 19t = 0 \\ 4x + 5y - 2z + 3t = 0 \end{cases}$ d) $\begin{cases} x - 2y + z - t = 0 \\ 2x - y + 3z - 3t = 0 \\ x + y + z + t = 0 \\ 2x - y + 2z = 0 \end{cases}$

4.22 In $\mathbb{R}^3$ consider the subspaces

$$D = \{(x, y, z) \mid \frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma},\ \alpha, \beta, \gamma \in \mathbb{R}^*\}$$

and

$$P = \{(x, y, z) \mid ax + by + cz = 0,\ a, b, c \in \mathbb{R}\}.$$

Find the condition wherefore $\mathbb{R}^3 = D \oplus P$.

Solutions

$\boxed{4.1}$ $(V, \oplus)$ is a commutative group. We check also the other axioms, for $x, y \in V$ and $\alpha, \beta \in \mathbb{R}$.

$$\alpha * (x \oplus y) = (xy)^{\alpha} = x^{\alpha} y^{\alpha} = (\alpha * x) \oplus (\alpha * y),$$
$$(\alpha + \beta) * x = x^{\alpha+\beta} = x^{\alpha} x^{\beta} = \alpha * x \oplus \beta * x,$$
$$\alpha * (\beta * x) = (\beta * x)^{\alpha} = (x^{\beta})^{\alpha} = x^{\alpha\beta} = (\alpha\beta) * x,$$
$$1 * x = x^1 = x.$$

$\boxed{\mathbf{4.2}}$ The basis is formed, for example, by the matrices E_{ij} $(i, j = 1, 2, ..., n)$ whose elements in the ith row and the jth column is equal to unity and all other elements are zero. The dimension is n^2.

$\boxed{\mathbf{4.3}}$ The basis is formed, for example, by the polynomials $1, x, x^2, ..., x^n$. The dimension is $n + 1$.

$\boxed{\mathbf{4.4}}$ a) Yes, b) No, c) No, d) Yes, e) No, f) Yes, g) No, h) Yes, i) No.

$\boxed{\mathbf{4.5}}$ a) The basis is formed, for example, by the vectors $(1, 0, 0, ..., 0, 1)$, $(0, 1, 0, ..., 0, 0)$, $(0, 0, 1, ..., 0, 0)$, ..., $(0, 0, 0, ..., 1, 0)$ and the dimension is $n - 1$.
b) The basis is formed, for example, by the two vectors $(1, 0, 1, 0, ...)$, $(0, 1, 0, 1, ...)$ and the dimension is 2.

$\boxed{\mathbf{4.6}}$ Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} + \beta \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

that is $\begin{cases} \alpha + 2\beta = 0 \\ a\beta + \gamma = 0 \\ -\alpha + 2\gamma = 0 \\ \alpha + \beta - \gamma = 0 \end{cases}$. We can notice that if α, β, γ satisfy the first and third equation they also satisfy the last one, so we have the linear homogeneous system $\begin{cases} \alpha + 2\beta = 0 \\ a\beta + \gamma = 0 \\ -\alpha + 2\gamma = 0 \end{cases}$. If the determinant of the system $\begin{vmatrix} 1 & 2 & 0 \\ 0 & a & 1 \\ -1 & 0 & 2 \end{vmatrix} = 2a - 2$ is not zero, then the only solution is the trivial one $\alpha = \beta = \gamma = 0$. For $a \neq 1$ the three matrices are linearly independent, and for $a = 1$ they are linearly dependent, for instance $B = 2A + C$.

4.7 Since $\sin^2 x = \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot \cos 2x$, $\cos^2 x = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \cos 2x$ it means that only two of the elements can be linearly independent. From $\alpha \cdot 1 + \beta \cdot \cos 2x = 0$ follows $\alpha = \beta = 0$ so a basis for the subspace is $\{1, \cos 2x\}$.

4.8 a) The 4^{th} order determinant having the four vectors as columns has the value 0, so $\dim(V) < 4$. We can find 3^{rd} order minors that are different from 0, so $\dim(V) = 3$. A basis can be, for instance $\{v_1, v_2, v_3\}$, or $\{v_1, v_2, v_4\}$; b) The rank of the matrix is 3, $\dim(V) = 3$, a basis is for instance $\{v_2, v_3, v_4\}$; c) $\dim(V) = 3$, so the subspace coincides with the whole space $\mathbb{R}^3$.

4.9 a) The basis is formed, for example, by the vectors a_1, a_3 and a_4, so the dimension is 3.
b) The basis is formed, for example, by the vectors a_1, a_2 and a_5 and the dimension is 3.

4.10 a) The dimensions of the union is 3 and of the intersection is 1. b) The dimensions of the union is 3 and of the intersection is 2.

4.11 a) The basis of the union (sum) is formed, for example,

by the vectors a_1, a_2 and b_1 and the basis of the intersection consist of the single vector $x = 2a_1 + a_2 = b_1 + b_2 = (3, 5, 1)$.
b) The basis of the union (sum) is formed, for example, by the vectors a_1, a_2, a_3 and b_2 and the basis of the intersection consist of $b_1 = -2a_1 + a_2 + a_3$ and $b_3 = 5a_1 - a_2 - 2a_3$.

$\boxed{\textbf{4.12}}$ $P = \{(x, y, z) \in \mathbb{R}^3 \mid 5x - 2y + z = 0\} = \{(x, y, -5x + 2y) \mid x, y \in \mathbb{R}\} = \{x(1, 0, -5) + y(0, 1, 2) \mid x, y \in \mathbb{R}\}$, so $\{(1, 0, -5), (0, 1, 2)\}$ is a basis for P. Similarly, $Q = \mathrm{sp}\{(1, -1, 0), (0, 3, 1)\}$ To find $P \cap Q$ we solve the system $\begin{cases} 5x - 2y + z = 0 \\ x + y - 3z = 0 \end{cases}$ and get
$z = \dfrac{7}{5}x$, $y = \dfrac{16}{5}x$, so
$P \cap Q = \mathrm{sp}\{(1, \frac{7}{5}, \frac{16}{5})\} = \mathrm{sp}\{(5, 7, 16)\}$.
$\mathrm{sp}(P \cup Q) = \mathrm{sp}\{(1, 0, -5), (0, 1, 2), (1, -1, 0), (0, 3, 1)\} =$

$$= \mathrm{sp}\{(1, 0, -5), (0, 1, 2), (1, -1, 0)\} = \mathbb{R}^3.$$

$\boxed{\textbf{4.13}}$ The transition matrix from the canonical basis to the basis B' is $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Denoting by a, b, c the coordinates in the new basis we have $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and we get $a = -1$, $b = -2$, $c = 0$. Indeed, $v = -(1, -1, 0) - 2(1, 0, -1)$.

$\boxed{\textbf{4.14}}$ $(15, -5, -4)$.

$\boxed{\textbf{4.15}}$ $(0, 2, 1, 2)$.

$\boxed{\textbf{4.16}}$ We consider the same vector in the first basis $(\alpha_1, \alpha_2, \alpha_3)$ and in the second basis $(\beta_1, \beta_2, \beta_3)$. Then $\alpha_1 = -27\beta_1 - 71\beta_2 - 41\beta_3$, $\alpha_2 = 9\beta_1 + 20\beta_2 + 9\beta_3$ and $\alpha_3 = 4\beta_1 + 12\beta_2 + 8\beta_3$.

$\boxed{\textbf{4.17}}$ Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha P_1 + \beta P_2 + \gamma P_3 = 0$. This means

$$\alpha(X-b)(X-c)+\beta(X-a)(X-c)+\gamma(X-a)(X-b) = 0,\ \forall\, x \in \mathbb{R}.$$

Assigning to X the values a, b or c it follows that $\alpha(a-b)(a-c) = 0$, $\beta(b-a)(b-c) = 0$ and $\gamma(c-a)(c-b) = 0$. If a, b, c are distinct two by two we get $\alpha = \beta = \gamma = 0$, so P_1, P_2, P_3 are linearly independent. If, for instance, $a = b$, for $\alpha = 1$, $\beta = -1$, $\gamma = 0$, we have $P_1 - P_2 = 0$, so they are not linearly independent. The same for $a = c$ or $b = c$. In conclusion the condition of linear independence is $(a-b)(a-c)(b-c) \neq 0$. b) We must determine l, m, n such that $1+X+X^2 = l(X-b)(X-c) + m(X-a)(X-c) + n(X-a)(X-b)$. Assigning to X the values a, b or c we get $l = \dfrac{1+a+a^2}{(a-b)(a-c)}$, $m = \dfrac{1+b+b^2}{(b-a)(b-c)}$, $n = \dfrac{1+c+c^2}{(c-a)(c-b)}$.

$\boxed{\textbf{4.18}}$ a) The transition matrix is $C = \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$, b) $f = \alpha + \beta a + \gamma a^2 + (\beta + 2\gamma a)(X-a) + \gamma(X-a)^2$ or $f = f(a) + \dfrac{f'(a)}{1!}(X-a) + \dfrac{f''(a)}{2!}(X-a)^2$.

$\boxed{\textbf{4.19}}$ a) $a_0, a_1, a_2, \ldots, a_n$. b) $f(\alpha)$, $f^{'}(\alpha)$, $f^{''}(\alpha)/2!$, …, $f^{(n)}(\alpha)/n!$.

$\boxed{\textbf{4.20}}$ $\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$\boxed{\textbf{4.21}}$ a) $(1, 0, -1, -1), (0, 1, 5, 2)$. b) $(2, -7, -4, 1)$. c) $(8, -6, 1, 0)$, $(-7, 5, 0, 1)$. d) $(-10/3, -2/3, 3, 1)$.

$\boxed{\textbf{4.22}}$ $\alpha a + \beta b + \gamma c \neq 0$.

CHAPTER 5

Inner product spaces

5.1 Inner products

Definition 5.1 An inner product on a real or complex linear space V is any scalar-valued function, defined on V^2 (the set of ordered pairs (x, y) of elements of V) and denoted by $(x|y)$, which satisfies the following three axioms: for all $x, x_1, x_2, y \in V$ and $k_1, k_2 \in K$,

(1) $(x|y) = \overline{(y|x)}$

(2) $(k_1x_1 + k_2x_2|y) = k_1(x_1|y) + k_2(x_2|y)$

(3) $(x|x) \geq 0$, and $(x|x) = 0$ if and only if $x = 0$.

In (1) the bar denotes the complex conjugate, and so may be omitted if the vector space is real. Because of (1), $(x|x)$ is real (even if V is a complex vector space) and so the inequality of (3) is meaningful. Corresponding to (2) is the relation

(2') $(x|k_1y_1 + k_2y_2) = \overline{k_1}(x|y_1) + \overline{k_2}(x|y_2)$,

which can be deduced, using (1), from (2) and is equivalent to it. Both (2) and (2') extend, in an obvious manner, to the case where more than two terms occur in either the first or second position in the inner product. We have also $(x|0) = (0|y) = 0$ for all $x, y \in V$.

Example 5.1.1 (1) For $\overrightarrow{u}, \overrightarrow{v} \in \mathcal{V}_3$ define $(\overrightarrow{u}|\overrightarrow{v}) = \overrightarrow{u} \cdot \overrightarrow{v}$. In this way we have an inner product on $\mathcal{V}_3$.

(2) Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The formula $(x|y) = x_1y_1 + \cdots + x_ny_n$ defines an inner product on $\mathbb{R}^n$, called *the canonical inner product* on $\mathbb{R}^n$.

(3) Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n, y = (y_1, \ldots, y_n) \in \mathbb{C}^n$.Then $(x|y) = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$ defines the *canonical inner product* on $\mathbb{C}^n$.

(4) Let $C[a, b] = \{f : [a, b] \longrightarrow \mathbb{R} \mid f\text{continuous on } [a, b]\}$. For $f, g \in C[a, b]$, define

$$(f|g) = \int_a^b f(x)g(x)dx$$

Then we have an inner product on $C[a, b]$.

An *inner product space* is any linear space on which an inner product is defined. A finite-dimensional real inner product space is known as a *Euclidean space*; a finite-dimensional complex inner product space is known as a *unitary space*.

Theorem 5.2 *(Schwarz' inequality). Let V be an inner product space and $u, v \in V$. Then*

$$|(u|v)|^2 \leq (u|u)(v|v).$$

Proof. If $v = 0$, the inequality reduces to $0 \leq 0$. So, let $v \neq 0$; then $(v|v) > 0$. We have

(i) $(u - kv|u - kv) \geq 0, \quad \forall k \in K.$

It follows immediately that

(ii) $(u - kv|u) - \overline{k}(u - kv|v) \geq 0, \ \forall k \in K$

For $k_0 = \dfrac{(u|v)}{(v|v)}$ the second inner product equals zero and hence (ii) implies

$$(u|u) - k_0(v|u) \geq 0, \text{ that is,}$$
$$(u|u) - \frac{(u|v)}{(v|v)}(v|u) \geq 0.$$

Since $(u|v)(v|u) = (u|v)\overline{(u|v)} = |(u|v)|^2$ we deduce the desired inequality $|(u|v)|^2 \leq (u|u)(v|v)$. □

5.2 Norm and distance

Definition 5.3 Let V be a linear space over K. A norm on V is any real-valued function defined on V (its value at x being denoted by $||x||$) which satisfies the following axioms:

(1) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$

(2) $||kx|| = |k|||x||$

(3) $||x_1 + x_2|| \leq ||x_1|| + ||x_2||$ for all $x, x_1, x_2 \in V$ and all $k \in K$.

Any linear space on which a norm is defined is known as a *normed vector space.*

Let V be a normed vector space. Define $d : V \times V \to \mathbb{R}$,

$$d(x, y) = ||x - y|| \quad \forall x, y \in V.$$

It is easy to verify that

(4) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$

(5) $d(x, y) = d(y, x)$

(6) $d(x,y) \leq d(x,z) + d(z,y)$

for all $x, y, z \in V$.

Thus d is a *metric* on V, and V is a *metric space.* The value $d(x,y)$ is called the *distance* between x and y. We refer to $||x||$ as the *length* of the vector x, and call x a *unit vector* if $||x|| = 1$.

The following result is a very important one.

Theorem 5.4 *Every inner product space is a normed space with norm defined by*

$$||x|| = \sqrt{(x|x)}.$$

Proof. Since $(x|x) \geq 0$ for all $x \in V$, $||x|| \geq 0$. Moreover, $||x|| = 0 \Longleftrightarrow (x|x) = 0 \Longleftrightarrow x = 0$ and so axiom (1) from the definition of a norm is satisfied.

Now $||kx|| = \sqrt{(kx|kx)} = \sqrt{k\overline{k}(x|x)} = \sqrt{|k|^2||x||^2} = |k|||x||$, which proves (2).

Finally,

$$\begin{aligned}
||x+y||^2 &= (x+y)|x+y) = (x|x) + (x|y) + (y|x) + (y|y) = \\
&= (x|x) + (x|y) + (\overline{(x|y)} + (y|y) = \\
&= ||x||^2 + 2Re(x|y) + ||y||^2 \\
&\text{(where } Re \text{ signifies the real part)} \\
&\leq ||x||^2 + 2|(x|y)| + ||y||^2 \leq \\
&\leq ||x||^2 + 2\sqrt{(x|x)}\sqrt{(y|y)} + ||y||^2 \\
&\text{by the Schwarz'inequality} \\
&= ||x||^2 + 2||x||||y|| + ||y||^2 = (||x|| + ||y||)^2
\end{aligned}$$

This implies $||x+y \leq ||x|| + ||y||$ and so axiom (3) is also satisfied. $\square$

5.3 Orthonormal bases

Let V be an inner product space. Two vectors $x, y \in V$ are *orthogonal* if $(x|y) = 0$; this definition extends the well-known situation that has appeared in the study of $\mathcal{V}_3$.

A set of vectors $\{x_1, \ldots, x_r\} \subset V$ is called *orthonormal* if

$$(x_i|x_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise.} \end{cases}$$

Thus each x_i is of unit length, and each pair of vectors is orthogonal. Finding an orthonomal set in an inner product space is analogous to choosing a set of mutually perpendicular unit vectors in elementary vector analysis.

Theorem 5.5 *An orthonormal set in an inner product space V is linearly independent.*

Proof. Suppose that $\{x_1, \ldots, x_r\}$ is the given orthonormal set and

$$k_1x_1 + \cdots + k_rx_r = 0.$$

Then for each i, $0 = (0|x_i) = (k_1x_1 + \cdots + k_rx_r|x_i) = k_1(x_1|x_i) + \cdots + k_i(x_i|x_i) + \cdots + k_r(x_r|x_i) = k_i$ since $(x_j|x_i) = 0$ unless $j = i$. Thus each coefficient k_i is zero, and so the vectors are linearly independent. $\square$

Let now V_n be an n-dimensional inner product space and $B \subset V_n$ an orthonormal set with n elements. As a consequence of the above theorem we deduce that B is a basis of V_n, called an *orthonormal basis.*

Theorem 5.6 *Let $B = \{b_1, \dots, b_n\}$ be an orthonormal basis of V_n. The coordinates of a vector $v \in V_n$ relative to B are the numbers*

$$(v|b_1), \dots, (v|b_n).$$

Proof. Let $k_1, \dots, k_n \in K$ be the coordinates of v, that is $v = \sum_{i=1}^{n} k_i b_i$. Then $(v|b_j) = (\sum_{i=1}^{n} k_i b_i | b_j) = \sum_{i=1}^{n} k_i (b_i|b_j) = k_j,\ j = 1, \dots, n$. Thus the theorem is proved and we have a very simple procedure for calculating the coordinates of any vector relative to an orthonormal basis. $\square$

Finally, let $B = \{v_1, \dots, v_n\}$ be any basis of V_n. The following procedure enables us to construct an orthonormal basis in V.

Let $x_1 = \dfrac{v_1}{||v_1||}$. Then $\{x_1\}$ is an orthonormal set with one element. Take $x_2 = \dfrac{v_2 - cx_1}{||v_2 - cx_1||}$; note that $v_2 - cx_1 = v_2 - \dfrac{c}{||v_1||} v_1 \neq 0$ for all $c \in K$.
Clearly $||x_2|| = 1$; we shall determine $c \in K$ such that $(x_2|x_1) = 0$. In fact, we find immediately $c = \dfrac{(v_2|x_1)}{(x_1|x_1)}$. So $\{x_1, x_2\}$ is an orthonormal set and $c = (v_2|x_1)$ (since $(x_1|x_1) = 1$).

Now take $x_3 = \dfrac{v_3 - c_1 x_1 - c_2 x_2}{||v_3 - c_1 x_1 - c_2 x_2||}$. As above, we deduce that the set $\{x_1, x_2, x_3\}$ is orthonormal if $c_1 = (v_3|x_1)$ and $c_2 = (v_3|x_2)$.

Proceeding in this way, after n steps we arrive at an orthonormal set $\{x_1, \dots, x_n\}$ with n elements, that is to say, an orthonormal basis of V_n.

The above procedure for constructing an orthonormal basis of V from an arbitrary basis is known as the *Gram-Schmidt orthogonalisation process.*

5.4 Orthogonal complement

Definition 5.7 Let W be a linear subspace of the inner product space V. The *orthogonal complement* of W is defined by

$$W^{\perp} = \{v \in V \mid (v|w) = 0, \ \forall \ w \in W\}.$$

The following properties could be easily verified:
a) $W^{\perp}$ is a subspace of V;
b) $V^{\perp} = \{0\}$ and $\{0\}^{\perp} = V$;
c) $U_1 \subseteq U_2 \Rightarrow U_2^{\perp} \subseteq U_1^{\perp}$;
d) $U_1 = (U_1^{\perp})^{\perp}$.

Theorem 5.8 *If U is a subspace of V, then*

$$V = U \oplus U^{\perp}$$

Proof. Suppose that U is a subspace of V. We will show that

$$V = U + U^{\perp}$$

Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of U and $v \in V$. We have

$$v = (v|e_1)e_1 + \cdots + (v|e_m)e_m + (v - (v|e_1)e_1 - \cdots - (v|e_m)e_m)$$

Denote the first vector by u and the second by w. Clearly $u \in U$. For each $j \in \{1, 2, \dots, m\}$ one has

$$\begin{aligned}(w|e_j) &= (v|e_j) - (v, e_j)\\ &= 0\end{aligned}$$

Thus w is orthogonal to every vector in the basis of U, that is $w \in U^{\perp}$, consequently

$$V = U + U^{\perp}.$$

We will show now that $U \cap U^{\perp} = \{0\}$. Suppose that $v \in U \cap U^{\perp}$. Then v is orthogonal to every vector in U, hence $\langle v, v \rangle = 0$, that is $v = 0$. The relations $V = U + U^{\perp}$ and $U \cap U^{\perp} = \{0\}$ imply the conclusion of the theorem. $\square$

5.5 Linear manifolds

Let V be a vector space over the field $\mathbb{F}$.

Definition 5.9 A set $L = v_0 + V_L = \{v_0 + v | v \in V_L\}$, where $v_0 \in V$ is a vector and $V_L \subset V$ is a subspace of V is called *a linear manifold (or linear variety)*. The subspace V_L is called the director subspace of the linear variety.

Remark 5.10 The following properties are easy to prove.

- A linear manifold is a translated subspace, that is $L = f(V_L)$ where $f : V \to V$, $f(v) = v_0 + v$.
- if $v_0 \in V_L$ then $L = V_L$.
- $v_0 \in L$ because $v_0 = v_0 + 0 \in v_0 + V_L$.
- for $v_1, v_2 \in L$ we have $v_1 - v_2 \in V_L$.
- for every $v_1 \in L$ we have $L = v_1 + V_L$.
- $L_1 = L_2$, where $L_1 = v_0 + V_{L_1}$ and $L_2 = v_0' + V_{L_2}$ iff $V_{L_1} = V_{L_2}$ and $v_0 - v_0' \in V_{L_1}$.

Definition 5.11 We would like to emphasize that:
a) The dimension of a linear manifold is the dimension of its director subspace.
b) Two linear manifolds L_1 and L_2 are called orthogonal if $V_{L_1} \perp V_{L_2}$.
c) Two linear manifolds L_1 and L_2 are called parallel if $V_{L_1} \subset V_{L_2}$ or $V_{L_2} \subset V_{L_1}$.

Let $L = v_0 + V_L$ be a linear manifold in a finitely dimensional vector space V. For $\dim L = k \leq n = \dim V$ one can choose in the director subspace V_L a basis of finite dimension $\{v_1, \dots, v_k\}$. We have

$$L = \{v = v_0 + \alpha_1 v_1 + \cdots + \alpha_k v_k | \alpha_i \in \mathbb{F}, i = \overline{1,k}\}$$

We can consider an arbitrary basis (fixed) in V, let's say $E = \{e_1, \dots, e_n\}$ and if we use the column vectors for the coordinates in this basis, i.e. $v_{[E]} = (x_1, \dots, x_n)^\top$, $v_{0_{[E]}} = (x_1^0, \dots, x_n^0)^\top$, $v_{j_{[E]}} = (x_{1j}, \dots, x_{nj})^\top$, $j = \overline{1,k}$, one has the *parametric equations of the linear manifold*

$$\begin{cases} x_1 = & x_1^0 + \alpha_1 x_{11} + \cdots + \alpha_k x_{1k} \\ \vdots & \\ x_n = & x_n^0 + \alpha_1 x_{n1} + \cdots + \alpha_k x_{nk} \end{cases}$$

The rank of the matrix $(x_{ij})_{\substack{i=\overline{1,n} \\ j=\overline{1,k}}}$ is k because the vectors $v_1, \dots, v_k$ are linearly independent.

It is worthwhile to mention that:

- a linear manifold of dimension one is called line.
- a linear manifold of dimension two is called plane.
- a linear manifold of dimension k is called k plane.
- a linear manifold of dimension $n-1$ in an n dimensional vector space is called hyperplane.

5.6 The Gram determinant. Distances.

In this section we will explain how we can measure the distance between some linear structures.

Let $(V, (\cdot|\cdot))$ be an inner product space and consider the vectors $v_i \in V$, $i = \overline{1, k}$.

The determinant

$$G(v_1, \ldots, v_k) = \begin{vmatrix} (v_1|v_1) & (v_1|v_2) & \ldots & (v_1|v_k) \\ (v_2|v_1) & (v_2|v_2) & \ldots & (v_2|v_k) \\ \ldots\ldots & \ldots & \ldots & \\ (v_k|v_1) & (v_k|v_2) & \ldots & (v_k|v_k) \end{vmatrix}$$

is called the *Gram determinant* of the vectors $v_1 \ldots v_k$.

Proposition 5.12 *In an inner product space, the vectors $v_1, \ldots, v_k$ are linearly independent iff $G(v_1, \ldots, v_k) \neq 0$.*

Proof. Let us consider the homogenous system

$$G \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This system can be written as

$$\begin{cases} (v_1|v) = 0 \\ \vdots \\ (v_k|v) = 0 \end{cases} \quad \text{where } \; v = x_1v_1 + \ldots x_kv_k.$$

The following statements are equivalent.

The vectors $v_1, \ldots, v_k$ are linearly dependent. $\iff$ There exist $x_1, \ldots, x_k \in \mathbb{F}$, not all zero such that $v = 0$. $\iff$ The homogenous system has a nontrivial solution. $\iff$ $\det G = 0$. $\square$

Proposition 5.13 *If $\{e_1, \ldots, e_n\}$ are linearly independent vectors and $\{f_1, \ldots, f_n\}$ are vectors obtained by Gram Schmidt orthogonalization process, one has:*

$$G(e_1, \ldots, e_n) = G(f_1, \ldots, f_n) = \|f_1\|^2 \cdot \ldots \cdot \|f_n\|^2$$

Proof. In $G(f_1, \ldots, f_n)$ replace f_n by $e_n - a_1f_1 - \cdots - a_{n-1}f_{n-1}$ and we obtain

$$G(f_1, \ldots, f_n) = G(f_1, \ldots, f_{n-1}, e_n).$$

By an inductive process the relation in the theorem follows. Obviously $G(f_1, \ldots, f_n) = \|f_1\|^2 \cdot \ldots \cdot \|f_n\|^2$ because in the determinant we have only on the diagonal $(f_1|f_1), \ldots, (f_n|f_n)$. $\square$

Remark 5.14 Observe that:

- $\|f_k\| = \sqrt{\dfrac{G(e_1, \ldots e_k)}{G(e_1, \ldots, e_{k-1})}}$

- $f_k = e_k - a_1 f_1 - \ldots a_{k-1} f_{k-1} = e_k - v_k$ one obtains $e_k = f_k + v_k$, $v_k \in \text{span}\{e_1, \ldots, e_{k-1}\}$ and $f_k \in \text{span}\{e_1, \ldots, e_{k-1}\}^\perp$, so f_k is the orthogonal complement of e_k with respect to the space generated by $\{e_1 \ldots, e_{k-1}\}$.

The distance between a vector and a subspace

Let U be a subspace of the inner product space V. The distance between a vector v and the subspace U is

$$d(v, U) = \inf_{w \in U} d(v, w) = \inf_{w \in U} \|v - w\|.$$

Remark 5.15 The linear structure implies a very simple but useful fact:

$$d(v, U) = d(v + w, w + U)$$

for every $v, w \in V$ and $U \subseteq V$, that is the linear structure implies that the distance is invariant by translations.

We are interested in the special case when U is a subspace.

Proposition 5.16 *The distance between a vector $v \in V$ and a subspace U is given by*

$$d(v, U) = \|v^\perp\| = \sqrt{\frac{G(e_1, \ldots, e_k, v)}{G(e_1, \ldots, e_k)}},$$

where $v = v_1 + v^\perp$, $v_1 \in U$, $v^\perp \in U^\perp$ and $e_1, \ldots, e_k$ is a basis in U.

Proof. First we prove that $\|v^\perp\| = \|v - v_1\| \leq \|v - u\|, \quad \forall u \in U$. We have

$$\begin{gathered}
\|v^\perp\| \leq \|v - u\| \Leftrightarrow \\
(v^\perp | v^\perp) \leq (v^\perp + v_1 - u | v^\perp + v_1 - u) \Leftrightarrow \\
(v^\perp | v^\perp) \leq (v^\perp | v^\perp) + (v_1 - u | v_1 - u).
\end{gathered}$$

The second part of the equality, i.e. $\|v^{\perp}\| = \sqrt{\frac{G(e_1,\ldots,e_k,v)}{G(e_1,\ldots,e_k)}}$, follows from the previous remark. $\square$

Definition 5.17 If $e_1, \ldots, e_k$ are vectors in V the volume of the k- parallelepiped constructed on the vectors $e_1, \ldots, e_k$ is defined by $\mathcal{V}_k(e_1, \ldots, e_k) = \sqrt{G(e_1, \ldots, e_k)}$.

We have the following inductive relation

$$\mathcal{V}_{k+1}(e_1, \ldots, e_k, e_{k+1}) = \mathcal{V}_k(e_1, \ldots, e_k) d(e_{k+1}, \operatorname{span}\{e_1, \ldots, e_k\}).$$

The distance between a vector and a linear manifold

Let $L = v_0 + V_L$ be a linear manifold, and let v be a vector in a finitely dimensional inner product space V. The distance induced by the norm is invariant by translations, that is, for all $v_1, v_2 \in V$ one has

$$d(v_1, v_2) = d(v_1+v_0, v_1+v_0) \Leftrightarrow \|v_1-v_2\| = \|v_1+v_0-(v_2+v_0)\|$$

That means that we have

$$\begin{aligned} d(v, L) &= \inf_{w \in L} d(v, w) = \inf_{v_L \in V_L} d(v, v_0 + v_L) \\ &= \inf_{v_L \in V_L} d(v - v_0, v_L) \\ &= d(v - v_0, V_L). \end{aligned}$$

Finally,

$$d(v, L) = d(v - v_0, V_L) = \sqrt{\frac{G(e_1, \ldots, e_k, v - v_0)}{G(e_1, \ldots, e_k)}},$$

where $e_1, \ldots, e_k$ is a basis in V_L.

Let us consider now the hyperplane H of equation

$$(v - v_0|n) = 0 \ .$$

The director subspace is $V_H = (v|n) = 0$ and the distance

$$d(v, H) = d(v - v_0, V_H).$$

One can decompose $v - v_0 = \alpha n + v_H$, where v_H is the orthogonal projection of $v - v_0$ on V_H and αn is the normal component of $v - v_0$ with respect to V_H. It means that

$$d(v, H) = \|\alpha n\|$$

Then, by taking into account the previous observations about the tangential and normal part, we compute:

$$\begin{aligned}(v - v_0|n) &= (\alpha n + v_H|n) \\ &= \alpha(n|n) + (v_H|n) \\ &= \alpha\|n\|^2 + 0\end{aligned}$$

So, we obtained

$$\frac{|(v - v_0|n)|}{\|n\|} = |\alpha|\|n\| = \|\alpha n\|$$

that is

$$d(v, H) = \frac{|(v - v_0|n)|}{\|n\|}$$

In the case that we have an orthonormal basis at hand, the equation of the hyperplane H is

$$a_1 x_1 + \cdots + a_k x_k + b = 0 \ ,$$

so the relation is now

$$d(v, H) = \frac{|a_1 v_1 + \cdots + a_k v_k + b|}{\sqrt{a_1^2 + \cdots + a_k^2}} .$$

The distance between two linear manifolds

For A and B sets in a metric space, the distance between them is defined as

$$d(A, B) = \inf\{d(a, b) | a \in A \ , \ b \in B\}.$$

For two linear manifolds $L_1 = v_1 + V_1$ and $L_2 = v_2 + V_2$ it easily follows:

$$\begin{aligned} d(L_1, L_2) &= d(v_1 + V_1, v_2 + V_2) = d(v_1 - v_2, V_1 - V_2) \\ &= d(v_1 - v_2, V_1 + V_2). \end{aligned}$$

This gives us the next proposition.

Proposition 5.18 *The distance between the linear manifolds $L_1 = v_1 + V_1$ and $L_2 = v_2 + V_2$ is equal to the distance between the vector $v_1 - v_2$ and the sum space $V_1 + V_2$.*

If we choose a basis in $V_1 + V_2$, let's say $e_1, \ldots, e_k$, then this formula follows:

$$d(L_1, L_2) = \sqrt{\frac{G(e_1, \ldots, e_k, v_1 - v_2)}{G(e_1, \ldots, e_k)}}.$$

Some analytic geometry

In this section we are going to apply distance problems in Euclidean spaces. Consider the vector space $\mathbb{R}^n$ with the canonical inner product, that is: for $\overline{x} = (x_1, \ldots, x_n)$, $\overline{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ the inner product is given by

$$(\overline{x}|\overline{y}) = \sum_{i=1}^{n} x_k y_k.$$

Consider D_1 , D_2 two lines (one dimensional linear manifolds), M a point (zero dimensional linear manifold, we assimilate with the vector $\overline{x}_M = \overline{0M}$), P a two dimensional linear manifold (a plane), and H an $n-1$ dimensional linear manifold (hyperplane). The equations of these linear manifolds are:

$$D_1 \ : \ \overline{x} = \overline{x}_1 + s\overline{d}_1,$$

$$D_2 \ : \ \overline{x} = \overline{x}_2 + t\overline{d}_2,$$

$$M \ : \ \overline{x} = \overline{x}_M,$$

$$P \ : \ \overline{x} = \overline{x}_P + \alpha\overline{v}_1 + \beta\overline{v}_2,$$

respectively

$$H \ : \ (\overline{x}|\overline{n}) + b = 0,$$

where $s, t, \alpha, \beta, b \in \mathbb{R}$. Recall that two linear manifolds are parallel if the director space of one of them is included in the director space of the other.

Now we can write down several formulas for distances between linear manifolds.

$$d(M, D_1) = \sqrt{\frac{G(\overline{x}_M - \overline{x}_1, \overline{d}_1)}{G(\overline{d}_1)}};$$

$$d(M, P) = \sqrt{\frac{G(\overline{x}_M - \overline{x}_P, \overline{v}_1, \overline{v}_2)}{G(\overline{v}_1, \overline{v}_2)}};$$

$$d(D_1, D_2) = \sqrt{\frac{G(\overline{x}_1 - \overline{x}_2, \overline{d}_1, \overline{d}_2)}{G(\overline{d}_1, \overline{d}_2)}} \quad \text{if} \quad D_1 \nparallel D_2$$

$$d(D_1, D_2) = \sqrt{\frac{G(\overline{x}_1 - \overline{x}_2, \overline{d}_1)}{G(\overline{d}_1,)}} \quad \text{if} \quad D_1 \parallel D_2$$

$$d(M, H) = \frac{|\langle \overline{x}_M, \overline{n} \rangle + b|}{\|\overline{n}\|}$$

$$d(D_1, P) = \sqrt{\frac{G(\overline{x}_1 - \overline{x}_P, \overline{d}_1, \overline{v}_1, \overline{v}_2)}{G(\overline{d}_1, \overline{v}_1, \overline{v}_2)}} \quad \text{if} \quad D_1 \nparallel P$$

Exercises

5.1 Let S be the set of solutions of the following systems and find bases in S and in the orthogonal complement $S^{\perp}$:

a) $\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + 2x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$.

b) $\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + 3x_3 - x_4 = 0 \\ x_2 + 7x_3 - 3x_4 = 0 \end{cases}$.

5.2 Let S be the set of solutions of the system

$$\begin{cases} x + y + t = 0 \\ 2x + y + z - 3v = 0 \\ x - y + 2z - 3t - 6v = 0 \end{cases}.$$

Find an orthonormal basis in S.

5.3 Verify that the following sets of vectors $\{v_1, v_2\}$ are orthogonal and complete them to form orthogonal bases of $\mathbb{R}^4$:
a) $v_1 = (1, 0, -2, 1)$ and $v_2 = (1, 1, 1, 1)$.
b) $v_1 = (1, 0, 2, -1)$ and $v_2 = (1, 2, 0, 1)$.
c) $v_1 = (1, -2, 2, -3)$ and $v_2 = (2, -3, 2, 4)$.
d) $v_1 = (1, 1, 1, 2)$ and $v_2 = (1, 2, 3, -3)$.

5.4 If V and W are linear subspaces of the inner product space U then:
a) $(V + W)^\perp = V^\perp \cap W^\perp$
b) $(V \cap W)^\perp = V^\perp + W^\perp$.

5.5 Let $\mathbb{R}^4$ be the inner product space with the canonical inner product. Apply the Gram-Schmidt orthogonalization to construct orthogonal bases for the subspaces spanned by the following sets of vectors:
a) $(1, 2, 2, -1)$, $(1, 1, -5, 3)$, $(3, 2, 8, -7)$.
b) $(1, 1, -1, -2)$, $(5, 8, -2, -3)$, $(3, 9, 3, 8)$.

5.6 Find an orthonormal basis for the subspace spanned by the vectors $v_1 = (1, -1, 1, -1)$, $v_2 = (5, 1, 1, 1)$, $v_3 = (-3, -3, 1, -3)$.

5.7 Show that the vectors $(1, 0, 1)$, $(1, 1, 0)$ and $(0, 1, 1)$ form a basis of $\mathbb{R}^3$ and find an orthonormal basis of this space, by using the Gram-Schmidt process.

5.8 For $f, g \in C[1, e]$ denote

$$(f|g) = \int_1^e f(x)g(x)(\ln x)\,\mathrm{d}x.$$

a) Prove that this defines an inner product in $C[1, e]$.
b) Find the norm of $f(x) = x$.
c) Find the polynomials of degree 1 which are orthogonal on the constant functions.

5.9 Let $p, q \in \mathbb{R}_2[X]$, $p = a_1X^2 + b_1X + c_1$, $q = a_2X^2 + b_2X + c_2$. Define

$$(p|q) = a_1a_2 + b_1b_2 + c_1c_2.$$

a) Prove that this defines an inner product in $\mathbb{R}_2[X]$.

b) Let $p_1 = 3X^2 + 2X + 1$, $p_2 = -X^2 + 2X + 1$, $p_3 = 3X^2 + 2X + 5$, $p_4 = 3X^2 + 5X + 2$. Find $p \in \mathbb{R}_2[X]$ which is equidistant with respect to p_1, p_2, p_3 and p_4. Find also the common distance.

5.10 Prove Pythagoras' Theorem: If V is an inner product space and $x, y \in V$ are orthogonal, then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

5.11 Let a subspace $L = span\{(2,1,0,1), (0,2,1,-1), (2,-1,-1,2)\} \subset \mathbb{R}^4$ and a linear manifold $K = \{(x,y,z,t) \in \mathbb{R}^4 | x - y + z + t = 1, x + t = 4, x + y - 2z = 0\}$. If $K = u_0 + V_K$, find: a) bases and dimensions for the subspaces L, V_K and $L + V_K$; b) $d(L,K)$; c) $(L + V_K)^{\perp}$.

5.12 Let a linear manifold $L = \{(x,y,z,t) \in \mathbb{R}^4 | 3x - y + z + t = 5, -y + t = 1\}$ and $U = span\{(2,1,1,0), (1,0,0,-1), (1,2,1,0), (1,1,1,1)\} \subset \mathbb{R}^4$. If $L = v_0 + V_L$, where V_L is the director subspace of L, find: a) bases and dimensions for the subspaces U, V_L, $U + V_L$ and $U \cap V_L$; b) $d(L,U)$; c) $(V_L)^{\perp}$.

5.13 Let the sests $S_1 = \{(x,y,z,t) \in \mathbb{R}^4 \mid 3x + y - z + t = 2\}$ and $S_2 = span\{(1,0,1,0), (0,-1,0,-1), (1,-2,1,-2), (1,1,-1,0)\}$.
Find bases and dimensions for the subspaces related to the given sets of vectors and for the union and intersection of these subspaces.

5.14 Let $V = span\{(1,0,0,-1), (2,1,1,0), (1,2,1,0), (0,2,1,1)\}$ and $L = \{(x,y,z,t) \in \mathbb{R}^4 \mid x + y - 2z + t = 2, -y + 3t = 0, z - t = 1\}$.
If $L = u_0 + V_L$, find: a) $d(V,L)$; b) $V_L^{\perp}$.

Solutions

$\boxed{\mathbf{5.1}}$ a) The system has the determinant zero, and since $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \neq 0$, we choose x_2, x_3 the primary unknowns and $x_1 = \alpha$ the secondary unknown. We get $x_2 = -\alpha$, $x_3 = 0$ so $S = \{(\alpha, -\alpha, 0) \mid \alpha \in \mathbb{R}\}$ with $\{(1, -1, 0)\}$ a basis. The orthogonal complement is $S^\perp = \{(a, b, c) \mid (a, b, c) \perp (1, -1, 0)\}$. We obtain $a - b = 0$, so $S = \{(a, a, c) \mid a, c \in \mathbb{R}\} = \{a(1, 1, 0) + c(0, 0, 1) \mid a, c \in \mathbb{R}\}$. A basis in $S^\perp$ is $\{(1, 1, 0), (0, 0, 1)\}$.
b) A basis in S is, for example $\{(4, -7, 1, 0), (-2, 3, 0, 1)\}$ and a basis in $S^\perp$ is $\{(1, 0, -4, 2), (0, 1, 7, -3)\}$.

$\boxed{\mathbf{5.2}}$ The solution set of the system is $S = \{(-\alpha + \beta + 3\gamma, \alpha - 2\beta - 3\gamma, \alpha, \beta, \gamma) \mid \alpha, \beta \in \mathbb{R}\}$, with a basis $\{v_1, v_2, v_3\}$, where $v_1 = (-1, 1, 1, 0, 0)$, $v_2 = (1, -2, 0, 1, 0)$, $v_3 = (3, -3, 0, 0, 1)$. To get an orthonormal basis we use the Gram-Schmidt procedure. First, $x_1 = v_1$. Then $x_2 = v_2 - c_1 x_1 = (0, -1, 1, 1, 0)$ $\left(c_1 = \dfrac{(v_2|x_1)}{(x_1|x_1)} = -1\right)$. Finally, $x_3 = v_3 - c_1 x_1 - c_2 x_2 = (1, 0, 1, -1, 1)$. This basis is orthogonal, in order to get an orthonormal one, we divide each vector to its own norm: $x_1' = \dfrac{1}{\sqrt{3}}(-1, 1, 1, 0, 0)$, $x_2' = \dfrac{1}{\sqrt{3}}(0, -1, 1, 1, 0)$ and $x_3' = \dfrac{1}{2}(1, 0, 1, -1, 1)$.

$\boxed{\mathbf{5.3}}$ a) Is clear that $(v_1|v_2) = 0$. We need two more vectors to form a basis. Let $v = (a, b, c, d)$. From $(v_1|v) = 0$ and $(v_2|v) = 0$ we have $\begin{cases} a - 2c + d = 0 \\ a + b + c + d = 0 \end{cases}$ so $a = 2c - d$, $b = -3c$. Choosing $c = 0$, $d = 1$ we get $v_3 = (-1, 0, 0, 1)$. Now v_4 has to be orthogonal also on v_3, that gives $c = d$.

Choosing $c = d = 1$ we have $v_4 = (1, -3, 1, 1)$. Obviously, the solution is not unique.
b) For example, they may be completed by adjoining the vectors $v_3 = (1, -1, 0, 1)$ and $v_4 = (-1, 0, 1, 1)$.
c) $v_3 = (2, 2, 1, 0)$ and $v_4 = (5, -2, -6, -1)$.
d) $v_3 = (1, -2, 1, 0)$ and $v_4 = (25, 4, -17, -6)$.

$\boxed{\mathbf{5.4}}$ a) Let $x \in (V + W)^{\perp}$. Then, for any $v \in V$ and $w \in W$, $(x|v + w) = 0$. Taking $w = 0$ follows that $(x|v) = 0$, for any $v \in V$, that is $x \in V^{\perp}$. Taking $v = 0$ follows $x \in W^{\perp}$. So $(V + W)^{\perp} \subset V^{\perp} \cap W^{\perp}$. Conversely, let $x \in V^{\perp} \cap W^{\perp}$. Let $y = v + w \in V + W$. Then $(x|y) = (x|v) + (x|w) = 0$ so $x \in (V + W)^{\perp}$. b) In the relation (a) we replace V by $V^{\perp}$ and W by $W^{\perp}$. We get $(V^{\perp} + W^{\perp})^{\perp} = (V^{\perp})^{\perp} \cap (W^{\perp})^{\perp}$ that is $(V^{\perp} + W^{\perp})^{\perp} = V \cap W$ and further $V^{\perp} + W^{\perp} = (V \cap W)^{\perp}$.

$\boxed{\mathbf{5.5}}$ a) $(1, 2, 2, -1), (2, 3, -3, 2), (2, -1, -1, -2)$.
b) $(1, 1, -1, -2)$, $(2, 5, 1, 3)$.

$\boxed{\mathbf{5.6}}$ A basis of the generated subspace is v_1, v_2. Applying the orthogonalisation, we obtain the orthogonal basis $u_1 = (1, -1, 1, -1)$ and $u_2 = (4, 2, 0, 2)$. An orthonormal basis is w_1, w_2, where $w_1 = u_1/\|u_1\| = 1/2(1, -1, 1, -1)$ and $w_2 = u_2/\|u_2\| = 1/\sqrt{6}(2, 1, 0, 1)$.

$\boxed{\mathbf{5.7}}$ An orthonormal basis is formed by the three vectors $1/\sqrt{2}(1, 0, 1)$, $1/\sqrt{6}(1, 2, -1)$ and $1/\sqrt{3}(-1, 1, 1)$.

$\boxed{\mathbf{5.8}}$ b) $\|f\| = \dfrac{1}{3}\sqrt{2e^3 + 1}$. c) $p(x) = a\left(x - \dfrac{e^2 + 1}{4}\right)$, $a \in \mathbb{R}$.

$\boxed{\mathbf{5.9}}$ b) $p = X^2 + 3X + 3$. The common distance is 3.

$\boxed{\mathbf{5.11}}$ a) A basis of L is $\{(2,1,0,1),(0,2,1,-1)\}$, then $dimL = 2$. $K = u_0 + V_K$, a basis of V_K is $\{(1,1,1,-1)\}$, $dimV_K = 1$ and $u_0 = (4,10,7,0) \in K$. Also, $dim(L + V_K) = 3$.

$\boxed{\mathbf{5.12}}$ a) A basis of U is $\{(2,1,1,0),(1,0,0,-1),(1,2,1,0)\}$ and a basis for the director subspace of L is, for example, $\{(1,0,-3,0),(0,1,0,1)\}$, with $v_0 = (0,0,4,1)$. Also, $dim(U+ V_L) = 4$ and $dim(U \cap V_L) = 1$. b) $d(L,U) = 0$.

$\boxed{\mathbf{5.13}}$ A basis for the director subspace of S_1 is, for example, $\{(1,0,3,0),(0,1,1,0),(0,0,1,1)\}$ and a basis of S_2 is $\{(1,0,1,0),(0,-1,0,-1),(1,1,-1,0)\}$. The dimension of the union is 4 and the dimension of the intersection is 2.

$\boxed{\mathbf{5.14}}$ a) If $L = u_0 + V_L$, a basis of V_L is, for example, $\{(-2,3,1,1)\}$ and $u_0 = (4,0,0,0)$. A basis of V is given by $\{(1,0,0,-1),(2,1,1,0),(1,2,1,0)\}$. Then, $d(V,L) = 0$.

CHAPTER 6

Linear transformations

6.1 Linear transformations

Mappings between two vector spaces are, in many respects, more interesting than vector spaces themselves. This applies especially to linear transformations ([3], [4], [5], [9], [10]).

Let V, W be two vector spaces over the same field K. Then a mapping $\mathcal{T} : V \longrightarrow W$ is called a *linear transformation* from V to W if it satisfies the following conditions:

(1) $\mathcal{T}(x + y) = \mathcal{T}(x) + \mathcal{T}(y), \ \forall x, y \in V$

(2) $\mathcal{T}(kx) = k\mathcal{T}(x), \ \forall k \in K, x \in V.$

An immediate consequence of (2) is that the zero vector of V is mapped by every linear transformation into the zero vector of W, that is $\mathcal{T}(0) = 0$.

Sometimes we shall write $\mathcal{T}x$ instead of $\mathcal{T}(x)$.

Example 6.1.1 1) $\mathcal{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \ \mathcal{T}(x_1, x_2) = (2x_1, -x_2, x_1 + x_2)$ defines a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

2) $\mathcal{D} : K[X] \longrightarrow K[X]$, $\mathcal{D}(p) = p' \ \forall p \in K[X]$ is a linear transformation, the well-known derivation of polynomials.

It is not difficult to verify that $\mathcal{T} : V \longrightarrow W$ is linear if and only if for any given integer $r \geq 2$ we have

$$\mathcal{T}(k_1x_1 + \cdots + k_rx_r) = k_1\mathcal{T}(x_1) + \cdots + k_r\mathcal{T}(x_r) \ \forall x_i \in V, \forall k_i \in K$$

An important consequence of this property is expressed in the following theorem:

Theorem 6.1 *A linear transformation $\mathcal{T} : V_n \longrightarrow W$ is uniquely determined by the images $\mathcal{T}(b_1), \ldots, \mathcal{T}(b_n)$ of a basis $\{b_1, \ldots, b_n\}$ of V_n.*

Proof. Each vector $x \in V_n$ can be expressed uniquely in the form $x = k_1b_1 + \cdots + k_nb_n, k_i \in K$. Then $\mathcal{T}x = \mathcal{T}(k_1b_1 + \cdots + k_nb_n) = k_1\mathcal{T}b_1 + \cdots + k\mathcal{T}b_n$, hence $\mathcal{T}x$ is uniquely determined. $\square$

For a linear transformation $\mathcal{T} : V \longrightarrow W$ denote

$$Ker(\mathcal{T}) = \{x \in V \mid \mathcal{T}x = 0\}, \quad Im(\mathcal{T}) = \{\mathcal{T}x \mid x \in V\}.$$

$Ker(\mathcal{T})$ is called the *kernel* of $\mathcal{T}$ and $Im(\mathcal{T})$ the *range* of $\mathcal{T}$.

Theorem 6.2 *$Ker(\mathcal{T})$ is a linear subspace of V. $Im(\mathcal{T})$ is a linear subspace of W.*

The (easy) proof is left to the reader.

Suppose that $dimV = n$; it can be shown that

$$dimKer(\mathcal{T}) + dimIm(\mathcal{T}) = n.$$

Definition 6.3 A linear transformation $\mathcal{T} : V \longrightarrow W$ is called an *isomorphism* if it is both one-to-one and onto W. V and W are called *isomorphic.*

The concept of isomorphism is of importance since any two isomorphic vector spaces have identical structure in the sense that any algebraic statement that is true for one space will necessarily be true for the other.

The next theorem is fundamental:

Theorem 6.4 *Two finite dimensional vector spaces over the same field are isomorphic if and only if they have the same dimension.*

We omit the proof but we mention the following obvious

Corollary 6.5 *Any vector space V over K and of dimension n is isomorphic to K^n.*

The reader may question why, in view of this result, we do not restrict our attention to the vector spaces K^n since these exhibit all the algebraic properties of abstract finite-dimensional vector spaces. The answer is that to do so would lead to unnecessary complications, in exactly the same way as in elementary vector analysis it is simpler to work with vectors as such, rather than to reduce every vector to a set of components.

Finally, denote by $\mathcal{L}(V, W)$ the set of all linear transformations from V to W. In $\mathcal{L}(V, W)$ we define addition and scalar multiplication by

$$\begin{array}{rcl} (\mathcal{T}+\mathcal{S})(x) = \mathcal{T}x+\mathcal{S}x \\ (k\mathcal{T})(x) \ = \ k\mathcal{T}x \end{array} \quad \forall x \in V,\ k \in K.$$

It is very easy to verify that, with these operations, $\mathcal{L}(V, W)$ forms a vector space over K.

Moreover, let V, W, U be three linear spaces over the same field K and let $\mathcal{T} \in \mathcal{L}(V, W)$, $\mathcal{S} \in \mathcal{L}(W, U)$. Consider the *composition* $\mathcal{S} \circ \mathcal{T} : V \longrightarrow U$ (called also the *product* and denoted simply by $\mathcal{S}\mathcal{T}$). Then $\mathcal{S}\mathcal{T} \in \mathcal{L}(V, U)$; we leave the proof to the reader.

The elements of $\mathcal{L}(V, V)$ are called the *endomorphisms* of the linear space V. Instead of $\mathcal{L}(V, V)$ we shall write simply $\mathcal{L}(V)$.

Denote by I the identity transformation of V, that is $Ix = x, \ \forall x \in V$. For $\mathcal{T} \in \mathcal{L}(V)$ let $\mathcal{T}^0 = I, \ \mathcal{T}^1 = \mathcal{T}, \mathcal{T}^2 = \mathcal{T}\mathcal{T}, \ldots$.

6.2 The matrix of a linear transformation

Let U, V be vector spaces over the same field K and let $\{u_1, \ldots, u_m\}$, $\{v_1, \ldots, v_n\}$ be bases of U, V respectively. If $\mathcal{T} \in \mathcal{L}(U, V)$ then $\mathcal{T} v_j \in U$ and so we may write

$$\mathcal{T} v_j = \sum_{i=1}^{m} t_{ij} u_i, \ t_{ij} \in K \tag{6.1}$$

The scalars $(t_{1j}, \ldots, t_{mj})$ are, for each j, the coordinates of $\mathcal{T} v_j$ relative to the given basis of U, and so are uniquely determined by $\mathcal{T}$.

Conversely, if we are given any set $\{t_{ij} | i = 1, \ldots, m; j = 1, \ldots, n\}$ of scalars, and bases $\{u_1, \ldots, u_m\}, \{v_1, \ldots, v_n\}$ of U

and V, then equation (6.1) determines a unique linear transformation $\mathcal{T} \in \mathcal{L}(V, U)$ (see Th.6.1, Section 5.1).

Write T for the matrix $(t_{ij})_{i=1,\dots,m; j=1,\dots,n}$. Then $T \in \mathcal{M}_{m,n}(K)$ will be called *the matrix of* $\mathcal{T}$, or *the matrix representing* $\mathcal{T}$, relative to the given bases of U and V. The columns of T are formed with the coordinates of $\mathcal{T}v_1, \dots, \mathcal{T}v_n$ relative to the basis $\{u_1, \dots, u_m\}$.

Since the definition of the scalars t_{ij} by (5.1) depends upon the arbitrarily chosen bases of U and V, many different matrices represent the same linear transformation.

Let $(x_1, \dots, x_n)$ be the coordinates of $x \in V$ relative to the basis $\{v_1, \dots, v_n\}$. Let $(y_1, \dots, y_m)$ be the coordinates of $\mathcal{T}x \in U$ relative to the basis $\{u_1, \dots, u_m\}$. Denote

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

Theorem 6.6 *The coordinates of x and the coordinates of $\mathcal{T}x$ are connected by the equation*

$$Y = TX. \tag{6.2}$$

Proof. We have $x = \sum\limits_{j=1}^{n} x_j v_j$ and $\mathcal{T}x = \sum\limits_{i=1}^{m} y_i u_i$. On the other hand, $\mathcal{T}x = \mathcal{T}(\sum\limits_{i=1}^{n} x_j v_j) = \sum\limits_{j=1}^{n} x_j \mathcal{T} v_j = \sum\limits_{j=1}^{n} x_j \sum\limits_{i=1}^{m} t_{ij} u_i = \sum\limits_{i=1}^{m} (\sum\limits_{i=1}^{n} t_{ij} x_j) u_i$.

Since the representation of the vector $\mathcal{T}x$ as a linear combination of the elements of the basis $\{u_1, \dots, u_m\}$ is unique, we may equate the coefficients of $u_i,\ i = 1, \ldots, m$ and so obtain

$$y_i = \sum_{j=1}^{n} t_{ij} x_j \quad , i = 1, \ldots, m.$$

This system is equivalent to (6.2). □

When $U = V$, to obtain a matrix representation of $\mathcal{T} \in \mathcal{L}(V, V)$ it is only necessary to choose one basis $\{v_1, \ldots, v_n\}$ of V. In this case the theorem must be modified by writing v_i for u_i throughout the statement and proof.

We now interpret, in the language of matrices, the operations on linear transformations defined in Section 5.1.

Theorem 6.7 *Let U, V, W be three vector spaces over the same field K, of dimensions m, n, p respectively, and let $\{u_1, \ldots, u_m\}$, $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_p\}$ be bases of U, V, W. Then, relative to these bases:*

1) *The zero linear transformation $0 \in \mathcal{L}(V, U)$ is represented by the zero matrix $0 \in \mathcal{M}_{m,n}(K)$.*

2) *The identity transformation $I \in \mathcal{L}(V, V)$ is represented by the unit matrix $I \in \mathcal{M}_{n,n}(K)$.*

3) *If $\mathcal{T} \in \mathcal{L}(V, U)$ is represented by the matrix $T \in \mathcal{M}_{m,n}(K)$, then for all $k \in K$ the transformation $k\mathcal{T}$ is represented by the matrix kT.*

4) *If $\mathcal{T}, \mathcal{S} \in \mathcal{L}(V, U)$ are represented by the matrices $T, S \in \mathcal{M}_{m,n}(K)$ respectively, then $\mathcal{T} + \mathcal{S}$ is represented by the matrix $T + S$.*

5) *If $\mathcal{T} \in \mathcal{L}(V, U)$ and $\mathcal{S} \in \mathcal{L}(U, W)$ are represented by $T \in \mathcal{M}_{m,n}(K)$ and $S \in \mathcal{M}_{p,m}$ respectively, then $\mathcal{S}\mathcal{T}$ is represented by ST.*

6) *If $\mathcal{T} \in \mathcal{L}(V, V)$ is non-singular and is represented by the matrix $T \in \mathcal{M}_{m,n}(K)$, then the inverse transformation $\mathcal{T}^{-1}$ is represented by the inverse matrix T^{-1}.*

Proof. All the statements follow immediately from the definitions, and we omit the details. We need also the following result:

Let $\mathcal{T} \in \mathcal{L}(V)$ be represented by the matrix T relative to the basis $B = \{b_1, \ldots, b_n\}$ of V, and by a matrix T' relative to the basis $B' = \{b'_1, \ldots, b'_n\}$ of V. Let C be the transition matrix from B to B'. Then $T' = C^{-1}TC$. $\square$

6.3 Invariant subspaces. Eigenvalues and eigenvectors

We now begin a more detailed study of linear transformations. Throughout the remainder of this chapter we shall be concerned only with linear transformations of a vector space V into itself, that is, with endomorphisms of V.

Definition 6.8 Let $\mathcal{T} \in \mathcal{L}(V)$ and W be a subspace of V with the property that $\mathcal{T}(W) \subset W$. Then $\mathcal{T}$ is called an *invariant subspace* of V under the endomorphism $\mathcal{T}$, or - more briefly - W is said to be $\mathcal{T}$-invariant.

Example 6.3.1 1) The improper subspaces V and $\{0\}$ are invariant under every endomorphism of V. Every subspace of V is invariant under both the identity and zero transformations.

2) $K_n[X]$ is an invariant subspace of $K[X]$ under the endomorphism $\mathcal{D}$ described in Example 5.1.1.

3) $\mathcal{T}\vec{i} = \vec{j}, \mathcal{T}\vec{j} = -\vec{i}$ define an endomorphism of the space $V = \{a\vec{i} + b\vec{j} \mid a, b \in \mathbb{R}\}$. It can be shown that V has no proper invariant subspaces under $\mathcal{T}$. (Exercise!)

Definition 6.9 Let $\mathcal{T} \in \mathcal{L}(V)$. A scalar $\lambda \in K$ is called an *eigenvalue* (or proper value) of $\mathcal{T}$ if there exists a non-zero vector $x \in V$ such that $\mathcal{T}x = \lambda x$. The vector x is called an *eigenvector* (or proper vector) of $\mathcal{T}$.

Let λ be an eigenvalue of $\mathcal{T}$. Denote $E(\lambda) = \{x \in V \mid \mathcal{T}x = \lambda x\}$. Clearly $E(\lambda)$ consists of all the eigenvectors of $\mathcal{T}$ corresponding to λ, together with the vector zero.

It is easy to verify that $E(\lambda)$ is a linear subspace of V and, moreover, it is $\mathcal{T}$-invariant. (Exercise!) It will be called the proper subspace of $\mathcal{T}$ corresponding to the eigenvalue λ.

Let now V_n be an n-dimensional linear space over K and let $B = \{b_1, \ldots, b_n\}$ be a basis of V_n. Let $\lambda \in K$ be an eigenvalue and let $x = x_1 b_1 + \cdots + x_n b_n$ be an eigenvector of $\mathcal{T}$ corresponding to λ. Hence we have $\mathcal{T}x = \lambda x$ and $x \neq 0$. Denote

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then $(\mathcal{T} - \lambda I)(x) = 0$, which is equivalent to $(T - \lambda I)X = 0$, where T is the matrix of $\mathcal{T}$ relative to the basis B (see Section 5.2).

The equation $(T - \lambda I)X = 0$ may be written in the form

$$\begin{pmatrix} t_{11} - \lambda & t_{12} & \ldots & t_{1n} \\ t_{21} & t_{22} - \lambda & \ldots & t_{2n} \\ \ldots & & & \\ t_{n1} & t_{n2} & \ldots & t_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (6.3)$$

This is a linear homogeneous system. Since $x \neq 0$, it has non-trivial solutions, that is $\det(T - \lambda I) = 0$. Let us denote

$P(\lambda) = \det(T - \lambda I)$; remark that $P(\lambda)$ is a polynomial of degree n.

Theorem 6.10 *$P(\lambda)$ does not depend on the choice of the basis B.*

Proof. Let B' be another basis of V_n and let T' be the matrix of $\mathcal{T}$ relative to B'. Let C be the transition matrix from B to B'. Then $T' = C^{-1}TC$; see Section 5.2. We have to prove that $\det(T' - \lambda I) = \det(T - \lambda I)$ for all $\lambda \in K$. Indeed,

$$\begin{aligned}\det(T' - \lambda I) &= \det(C^{-1}TC - C^{-1}(\lambda I)C) = \det(C^{-1}(T - \lambda I)C) = \\ &= \det C^{-1} \cdot \det(T - \lambda I) \cdot \det C = \\ &= (\det C)^{-1} \det(T - \lambda I) \det C = \det(T - \lambda I),\end{aligned}$$

so the theorem is proved. $\square$

Since the polynomial $P(\lambda)$ is independent of the choice of the basis B, it will be called the *characteristic polynomial* of $\mathcal{T}$. If a matrix T represents $\mathcal{T}$ with respect to some basis, $P(\lambda)$ will be also called the caracteristic polynomial of T, and we have simply $P(\lambda) = \det(T - \lambda I)$.

Returning to the eigenvalues of $\mathcal{T}$, we see that they are exactly the roots in K of the characteristic polynomial of $\mathcal{T}$. There exist n roots, real or complex. If $K = \mathbb{C}$, all of them are eigenvalues; if $K = \mathbb{R}$, only the real roots (if there exist real roots!) are eigenvalues of $\mathcal{T}$.

Now suppose that λ is an eigenvalue of $\mathcal{T}$. Then (6.3) has non-trivial solutions. Every such non-trivial solution gives us an eigenvector x by means of the formula $x = x_1b_1 + \cdots + x_nb_n$.

6.4 The Cayley-Hamilton Theorem

Let $P \in K[X]$ be an arbitrary polynomial, $P(X) = a_m X^m + \cdots + a_1 X + a_0$, $a_i \in K$. For a matrix $A \in \mathcal{M}_{n,n}(K)$ let us denote $P(A) = a_m A^m + \cdots + a_1 A + a_0 I$. The Cayley-Hamilton Theorem asserts that *if* $P(\lambda) = \det(T - \lambda I)$ *is the characteristic polynomial of a matrix* $T \in \mathcal{M}_{n,n}(K)$, *then* $P(T) = 0$.

We shall use this result in order to prove

Theorem 6.11 *Let $A \in \mathcal{M}_{n,n}(K)$. Then for each $p \geq n$, A^p can be expressed as a linear combination of $I, A, A^2, \ldots, A^{n-1}$.*

Proof. Let $P(\lambda) = \det(A - \lambda I)$ be the characteristic polynomial of the matrix A. By virtue of the Cayley-Hamilton Theorem, we have $P(A) = 0$.

Clearly $P(\lambda) = (-1)^n \lambda^n + k_{n-1}\lambda^{n-1} + \cdots + k_1 \lambda + k_0$, with $k_i \in K$. Hence

$$(-1)^n A^n + k_{n-1} A^{n-1} + \cdots + k_1 A + k_0 I = 0.$$

It follows that

$$A^n = c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I, \quad c_i \in K. \tag{6.4}$$

Thus A^n is a linear combination of $I, A, A^2, \ldots, A^{n-1}$.

From (6.4)we deduce

$$A^{n+1} = c_{n-1} A^n + c_{n-2} A^{n-1} + \cdots + c_1 A^2 + c_0 A \tag{6.5}$$

If we substitute A^n taking into account (6.4), we obtain A^{n+1} as a linear combination of $I, A, \ldots, A^{n-1}$. By repeating this argument we finish the proof. □

6.5 The diagonal form

Let V be a linear space over K.

Theorem 6.12 *Let $\mathcal{T} \in \mathcal{L}(V)$ and let $x_1, \ldots, x_n$ be eigenvectors of $\mathcal{T}$ associated with mutually distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the vectors $x_1, \ldots, x_n$ are linearly independent.*

Proof. Suppose that

(1) $\{x_1, \ldots, x_n\}$ is a linearly dependent set
Then there exist $k_1, \ldots, k_n \in K$, not all zero, such that

(2) $k_1 x_1 + \cdots + k_n x_n = 0$.
Renumbering the variables if necessary, we may suppose that

(3) $k_1 \neq 0$.
From (2) we obtain $k_1 \mathcal{T} x_1 + \cdots + k_n \mathcal{T} x_n = 0$. Since $\mathcal{T} x_i = \lambda_i x_i$, it follows that

(4) $k_1 \lambda_1 x_1 + \cdots + k_n \lambda_n x_n = 0$
Now (2) and (4) imply

(5) $k_2(\lambda_2 - \lambda_1)x_2 + \cdots + k_n(\lambda_n - \lambda_1)x_n = 0$.
We claim that $\{x_2, \ldots, x_n\}$ must be linearly dependent. Indeed, if we suppose that they are linearly independent, then $k_2 = \cdots = k_n = 0$ since $\lambda_i - \lambda_1 \neq 0$, $i = 2, \ldots, n$. But (2) implies $k_1 x_1 = 0$. Since x_1 is an eigenvector, it is

non-zero. Hence $k_1 = 0$, which contradicts (3).

Thus (1) implies:

(6) $\{x_2, \ldots, x_n\}$ is a linearly dependent set.

Now we repeat the same arguments and conclude that (6) implies:

(7) $\{x_3, \ldots, x_n\}$ is a linearly dependent set.

In this manner we deduce finally that $\{x_n\}$ is a linearly dependent set. On the other hand,the same set is linearly independent, since $x_n \neq 0$ as an eigenvector. This contradiction shows that (1) is false and the theorem is proved. $\square$

Theorem 6.13 *Let $\mathcal{T}$ be an endomorphism of a linear space V_n of finite dimension $n \geq 1$ over K. Suppose that the characteristic polynomial $P(\lambda)$ of $\mathcal{T}$ has n simple roots $\lambda_1, \ldots, \lambda_n$ in the field K. Then there exists a basis of V_n relative to which the matrix of $\mathcal{T}$ is*

$$\begin{pmatrix} \lambda_1 & 0 & 0 \ldots & 0 \\ 0 & \lambda_2 & 0 \ldots & 0 \\ \ldots & & & \\ 0 & 0 & 0 \ldots & \lambda_n \end{pmatrix}$$

Proof. Since the roots $\lambda_1, \ldots, \lambda_n$ are in K, they are eigenvalues of $\mathcal{T}$. For each i choose an eigenvector x_i of $\mathcal{T}$ corresponding to the eigenvalue λ_i. By hypothesis $\lambda_1, \ldots, \lambda_n$ are mutually distinct. Theorem 6.12 shows that $x_1, \ldots, x_n$ are linearly independent. Since $dim V_n = n$, $\{x_1, \ldots, x_n\}$ is a basis. We have $\mathcal{T} x_i = \lambda_i x_i,\ i = 1, \ldots, n$, hence the matrix of $\mathcal{T}$ with respect to this basis is the diagonal matrix of the theorem. $\square$

Corollary 6.14 *Let $T \in \mathcal{M}_{n,n}(K)$. Suppose that the characteristic polynomial of T has n simple roots in K. Then there exists a matrix $C \in \mathcal{M}_{n,n}(K)$ such that*

$$C^{-1}TC = \begin{pmatrix} \lambda_1 & 0 & 0 \dots & 0 \\ 0 & \lambda_2 & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & \lambda_n \end{pmatrix},$$

$\lambda_1, \dots, \lambda_n$ being the roots.

Proof. Let B be the canonical basis of K^n. Let $\mathcal{T} \in \mathcal{L}(K^n)$ be the endomorphism which has the matrix T relative to the basis B. Theorem 6.13 shows that there exists a basis B' of K^n relative to which the matrix of $\mathcal{T}$ is

$$T' = \begin{pmatrix} \lambda_1 & 0 & 0 \dots & 0 \\ 0 & \lambda_2 & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & \lambda_n \end{pmatrix}$$

Let C be the transition matrix from B to B'. We know that $T' = C^{-1}TC$ and the proof is complete. $\square$

We shall denote

$$diag(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & 0 \dots & 0 \\ 0 & \lambda_2 & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & \lambda_n \end{pmatrix}$$

The algebra of matrices applies especially smoothly to diagonal matrices: to add or multiply any two diagonal matrices, one simply adds or multiplies corresponding diagonal entries.

For instance, let T be as in the above corollary. Then it is easy to compute T^p for any $p \geq 1$. Indeed, let $T' = diag(\lambda_1, \ldots, \lambda_n)$. Then $C^{-1}TC = T'$, that is $T = CT'C^{-1}$. We have

$$T^p = (CT'C^{-1}) \cdot (CT'C^{-1}) \cdot \cdots \cdot (CT'C^{-1}) = C(T')^p C^{-1}$$

But $(T')^p = diag(\lambda_1^p, \ldots, \lambda_n^p)$ and hence

$$T^p = C \cdot diag(\lambda_1^p, \ldots, \lambda_n^p) \cdot C^{-1}.$$

6.6 Reduction to diagonal form

We want to characterize the endomorphisms that can be "diagonalized", that is, for which there exists a basis relative to which the matrix is a diagonal one.

Let V_n be a linear space of finite dimension $n \geq 1$ over the field K. Let $\mathcal{T} \in \mathcal{L}(V_n)$ and let λ_0 be an eigenvalue of $\mathcal{T}$. We know that λ_0 is a root in K of the characteristic polynomial of $\mathcal{T}$. Denote by $m(\lambda_0)$ the multiplicity of λ_0 as a root of this polynomial.

Consider also the proper subspace corresponding to λ_0:

$$E(\lambda_0) = \{x \in V_n \mid \mathcal{T}x = \lambda_0 x\}.$$

Let $B = \{b_1, \ldots, b_n\}$ be an arbitrary basis of V_n and let T be the matrix of $\mathcal{T}$ relative to this basis.

Theorem 6.15 $dimE(\lambda_0) = n - rank(T - \lambda_0 I) \leq m(\lambda_0)$

Proof. Let $x \in V_n,\ x = x_1 b_1 + \cdots + x_n b_n$. As usual, denote

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then the following statements are equivalent:

(1) $x \in E(\lambda_0)$

(2) $(\mathcal{T} - \lambda_0 I)(x) = 0$

(3) $(T - \lambda_0 I) \cdot X = 0$

We conclude that $E(\lambda_0)$ can be identified with the set of the solutions of the linear homogeneous system

$$\begin{pmatrix} t_{11} - \lambda_0 & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} - \lambda_0 & \dots & t_{2n} \\ \dots & & & \\ t_{n1} & t_{n2} & \dots & t_{nn} - \lambda_0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

But this set is a linear subspace of K^n of dimension $n - rank(T - \lambda_0 I)$. Thus $dim E(\lambda_0) = n - rank(T - \lambda_0 I)$, and the first statement of the theorem is proved.

Now denote $q = dim E(\lambda_0)$ and let $\{v_1, \dots, v_q)$ be a basis of $E(\lambda_0)$. Let us complete it in order to obtain a basis $\{v_1, \dots, v_q, v_{q+1}, \dots, v_n\}$ of V_n.

We have $\mathcal{T} v_j = \lambda_0 v_j,\ j = 1, \dots, q$ and $\mathcal{T} v_j = t_{1j} v_1 + \cdots + t_{nj} v_n,\ j = q+1, \dots, n,\ t_{ij} \in K$.

Hence the matrix T' of $\mathcal{T}$ relative to the basis $\{v_1, \ldots, v_n\}$ is

$$T' = \begin{pmatrix} \lambda_0 & 0 & \ldots & 0 & t_{1,q+1} & \ldots & t_{1n} \\ 0 & \lambda_0 & \ldots & 0 & t_{2,q+1} & \ldots & t_{2n} \\ \ldots & \ldots & & & & & \\ 0 & 0 & \ldots & \lambda_0 & t_{q,q+1} & \ldots & t_{qn} \\ 0 & 0 & \ldots & 0 & t_{q+1,q+1} & \ldots & t_{q+1,n} \\ \ldots & \ldots & & & & & \\ 0 & 0 & \ldots & 0 & t_{n,q+1} & \ldots & t_{n,n} \end{pmatrix}$$

The characteristic polynomial of $\mathcal{T}$ is $P(\lambda) = \det(T' - \lambda I)$. If we take account of the form of T' we conclude that $P(\lambda)$ is of the form $P(\lambda) = (\lambda_0 - \lambda)^q \cdot Q(\lambda)$, where $Q(\lambda)$ is a polynomial. Now it is clear that the multiplicity of λ_0 as a root of $P(\lambda)$ is at least q, that is $m(\lambda_0) \geq q$.

Thus $n - \operatorname{rank}(T - \lambda_0 I) \leq m(\lambda_0)$ and the theorem is proved. $\square$

Definition 6.16 Let $\mathcal{T}$ be an endomorphism of a vector space V_n of finite dimension n over K. The endomorphism $\mathcal{T}$ is said to be *diagonalizable* if there exists a basis of V_n consisting of eigenvectors of $\mathcal{T}$, in other words a basis relative to which the matrix of $\mathcal{T}$ is diagonal.

Theorem 6.13, Section 5.5 gives a sufficient condition for this to be the case: namely that the roots of the characteristic polynomial of $\mathcal{T}$ all lie in K and are all distinct. But it is easily seen that this condition is not necessary: a trivial example is the identity endomorphism whose matrix with respect to any basis of V_n is diagonal, but whose characteristic polynomial, namely $(1-\lambda)^n$ has no simple roots (assuming that $n > 1$).

6.7 The Jordan canonical form

Let $\mathcal{T} \in \mathcal{L}(V_n)$; suppose that all the roots of the characteristic polynomial are in $\mathbb{K}$. Let λ be such a root, i.e., an eigenvalue of $\mathcal{T}$. Let m be the algebraic multiplicity of λ, and $q = dim E(\lambda)$. Then $m \geq q \geq 1$.

It is possible to find q eigenvectors in $E(\lambda)$ and $m-q$ principal vectors, all of them linearly independent; an eigenvector v and the principal vectors $u_1, \ldots, u_r (r \geq 0)$ corresponding to it satisfy:

$$\mathcal{T}v = \lambda v;\ \mathcal{T}u_1 = \lambda u_1 + v;\ \mathcal{T}u_2 = \lambda u_2 + u_1; \ldots ; \mathcal{T}u_r = \lambda u_r + u_{r-1}.$$

All these eigenvectors and principal vectors, associated to all the eigenvalues of $\mathcal{T}$, form a basis of V_n, called a *Jordan basis* with respect to $\mathcal{T}$. The matrix of $\mathcal{T}$ relative to a Jordan basis is called a *Jordan matrix* of $\mathcal{T}$. Such a matrix has the form $\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \cdots & \\ & & & J_p \end{pmatrix}$, where $J_1, \ldots, J_p$ are called *Jordan cells.* Each cell represents the contribution of an eigenvector v and the corresponding principal vectors

$$u_1, \ldots, u_r : \begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \lambda & 1 & & \\ & & & \vdots & 1 & \\ & & & & \vdots & \\ & & & & & \lambda \end{pmatrix} \in M_{r+1}(\mathbb{K}).$$

We see that: the Jordan matrix is a diagonal matrix $\Longleftrightarrow$ there are no principal vectors $\Longleftrightarrow m(\lambda) = dimE(\lambda)$ for each eigenvalue λ.

Let T be the matrix of $\mathcal{T}$ with respect to a given basis B, and J the Jordan matrix with respect to a Jordan basis B'. Let C be the transition matrix from B to B'. Then $J = C^{-1}TC$, hence $T = CJC^{-1}$. It follows that $T^n = CJ^nC^{-1}$.

The exponential of the matrix T is defined by

$$e^T = I + \frac{1}{1!}T + \frac{1}{2!}T^2 + \cdots + \frac{1}{n!}T^n + \ldots.$$

Example 6.7.1 1. Let $\mathcal{T} \in \mathcal{L}(\mathbb{R}^3)$ have the matrix

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

with respect to the canonical basis.
We find $\lambda_1 = 2,\ m(\lambda_1) = 1$,

$$E(\lambda_1) = \left\{\alpha \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R}\right\},\ \text{hence } q(\lambda_1) = 1.$$

$\lambda_2 = 1,\ m(\lambda_2) = 2,$

$$E(\lambda_2) = \left\{\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R}\right\}, \text{ hence } q(\lambda_2) = 1.$$

So $v_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; the principal vector u_1 associated with v_2 satisfies $Tu_1 = u_1 + v_2$. Let $u_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; then

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We get $x = y - 1,\ z = y + 1,\ y \in \mathbb{R}$. Choosing $y = 1$, we obtain

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

The Jordan basis is $B = \{v_1, v_2, u_1\}$.
Since

$$\begin{aligned} \mathcal{T}v_1 &= 2v_1 + 0v_2 + 0u_1 \\ \mathcal{T}v_2 &= 0v_1 + v_2 + 0u_1 \\ \mathcal{T}u_1 &= 0v_1 + v_2 + u_1, \end{aligned}$$

the Jordan matrix will be

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transition matrix from the canonical basis to the Jordan basis is

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 4 & 1 & 2 \end{pmatrix}.$$

We have $J = C^{-1}TC,\ T = CJC^{-1}, T^n = CJ^nC^{-1},\ J^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$

2. $T = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -3 & -1 \end{pmatrix}$

In this case $\lambda_1 = -1,\ m(\lambda_1) = q(\lambda_1) = 1,\ \lambda_2 = 0,\ m(\lambda_2) = 2,\ q(\lambda_2) = 1.$

$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix},\ u_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$

The Jordan matrix is

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. $T = \begin{pmatrix} 1 & 1 & 0 \\ -4 & -2 & 1 \\ 4 & 1 & -2 \end{pmatrix}.$

We find $\lambda_1 = -1,\ m(\lambda_1) = 3,\ q(\lambda_1) = 1.$

$v_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. The principal vectors u_1 and u_2 associated with v_1 satisfy

$$Tu_1 = -u_1 + v_1$$
$$Tu_2 = -u_2 + u_1.$$

We obtain $u_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
The Jordan basis is $\{v_1, u_1, u_2\}$ and the Jordan matrix is

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

6.8 Matrix functions

Let $\mathcal{T} \in \mathcal{L}(V_n)$. Consider T the matrix of $\mathcal{T}$ with respect to the basis B, and J the Jordan matrix corresponding to the Jordan basis B'. Let C the transition matrix from the basis B to the basis B'.

Remark: The transition matrix from the canonical basis to a Jordan considered basis will contain the vectors of the Jordan basis on the columns in the order given in the basis.
Then we have $J = C^{-1}TC$, which results $T = CJC^{-1}$.

Theorem 6.17 *In the above hypotheses, we have*

$$T^n = CJ^nC^{-1}.$$

Proof. $T^n = (CJC^{-1})(CJC^{-1}) \cdots (CJC^{-1}) = CJ^nC^{-1}$. $\square$

Consider now, $f : D \subset \mathbb{R} \to \mathbb{R}$ an analytical real function which has the Taylor expansion (Mac-Laurin for $x_0 = 0$):

$$f(x) = \sum_{k \geq 0} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

We assume as known, the theoretical aspects related to power series (studied at Calculus). In fact, a Taylor series is an infinite polynomial. After the definition of a polynomial related to a matrix, we will define further, a *matrix function* like:

$$f(A) = \sum_{k\geq 0} \frac{f^{(k)}(x_0)}{k!}(A - x_0 I)^k$$

where f is an analytical function, $A \in M_n(\mathbb{R})$ and I is the unit matrix of order n.

Example 6.8.1 Mac-Laurin series expansions of some elementary functions, will become for a matrix $A \in M_n(\mathbb{R})$, as follows:

$$e^A = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{k!}A^k + \cdots = \sum_{k\geq 0} \frac{1}{k!}A^k$$

$$\cos A = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \cdots + \frac{(-1)^k}{(2k)!}A^{2k} + \cdots = \sum_{k\geq 0} \frac{(-1)^k}{(2k)!}A^{2k}$$

$$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} + \cdots + \frac{(-1)^k}{(2k+1)!}A^{2k+1} + \cdots = \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)!}A^{2k+1}$$

Matrix functions for a diagonalizable matrix

Let A be tha matrix of a linear map $\mathcal{T} \in \mathcal{L}(V_n)$. If A is diagonalizabe, then exists a basis formed by eigenvectors in V_n, for which is obtained the diagonal matrix of $\mathcal{T}$, denoted $D = diag(\lambda_1, \ldots, \lambda_n)$.

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \ldots & 0 \\ 0 & \lambda_2 & 0 \ldots & 0 \\ \ldots & & & \\ 0 & 0 & 0 \ldots & \lambda_n \end{pmatrix}$$

Obvious, we have $A = CDC^{-1}$.

Theorem 6.18 *Let T being a matrix of the linear map $\mathcal{T} \in \mathcal{L}(V_n)$ and J a Jordan matrix corresponding to it. Then:*

$$f(T) = Cf(J)C^{-1}.$$

Proof. Consider the general Taylor series expansion of the form

$$f(x) = \sum_{k\geq 0} a_k x^k.$$

Then

$$f(T) = \sum_{k\geq 0} a_k T^k = \sum_{k\geq 0} a_k (CJ^kC^{-1}) = C\left(\sum_{k\geq 0} a_k J^k\right) C^{-1} =$$

$$= Cf(J)C^{-1}.$$ □

From Theorem 6.18, for a diagonalizable matrix A, with the diagonal matrix D, we have

$$f(A) = Cf(D)C^{-1}$$

where

$$f(D) = \sum_{k\geq 0} a_k D^k = \sum_{k\geq 0} a_k \begin{pmatrix} \lambda_1^k & 0 & 0 \dots & 0 \\ 0 & \lambda_2^k & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & \lambda_n^k \end{pmatrix}$$

Then, for the diagonal matrix D, we consider the next result:

Theorem 6.19

$$f(D) = \begin{pmatrix} f(\lambda_1) & 0 & 0 \dots & 0 \\ 0 & f(\lambda_2) & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & f(\lambda_n) \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues corresponding to the endomorphism $\mathcal{T}$.

Matrix function for Jordan cells

Let $\mathcal{T} \in \mathcal{L}(V_n)$ having a given Jordan matrix. Consider a third order Jordan cell (block) of this Jordan matrix, of the form:

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

This matrix could be written in the equivalent form:

$$J = \lambda I + N$$

where

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

N is a nilpotent matrix.

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad N^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \forall k \geq 3.$$

If for an analytical function f we have the Taylor expansion in the vicinity of λ:

$$f(x) = \sum_{k\geq 0} \frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k$$

then

$$f(J) = \sum_{k\geq 0} \frac{f^{(k)}(\lambda)}{k!}(J-\lambda I)^k = \sum_{k\geq 0} \frac{f^{(k)}(\lambda)}{k!} N^k =$$

$$= \frac{f(\lambda)}{0!} I + \frac{f^{(')}(\lambda)}{1!} N + \frac{f^{('')}(\lambda)}{2!} N^2$$

Finally,

$$f(J) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} \\ 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & f(\lambda) \end{pmatrix}$$

Generalization

Consider a Jordan cell of order $r+1$ for the eigenvalue λ with $m(\lambda) = r+1$ and v the corresponding eigenvector with $u_1, u_2, \ldots, u_r$ principal vectors (associated to it), such that:

$$J_\lambda = \begin{pmatrix} \lambda & 1\ 0 \ldots & 0 \\ 0 & \lambda\ 1 \ldots & 0 \\ \ldots & & \\ 0 & 0\ 0 \ldots & 1 \\ 0 & 0\ 0 \ldots & \lambda \end{pmatrix} \in M_{r+1}(\mathbb{R})$$

then

$$f(J_\lambda)=\begin{pmatrix} \frac{f(\lambda)}{0!} & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \dots & \frac{f^{(r-1)}(\lambda)}{(r-1)!} & \frac{f^{(r)}(\lambda)}{r!} \\ 0 & \frac{f(\lambda)}{0!} & \frac{f'(\lambda)}{1!} & \dots & \frac{f^{(r-2)}(\lambda)}{(r-2)!} & \frac{f^{(r-1)}(\lambda)}{(r-1)!} \\ 0 & 0 & \frac{f(\lambda)}{0!} & \dots & \frac{f^{(r-3)}(\lambda)}{(r-3)!} & \frac{f^{(r-2)}(\lambda)}{(r-2)!} \\ \dots \\ 0 & 0 & 0 & \dots & \frac{f(\lambda)}{0!} & \frac{f'(\lambda)}{1!} \\ 0 & 0 & 0 & \dots & 0 & \frac{f(\lambda)}{0!} \end{pmatrix} \in M_{r+1}(\mathbb{R}).$$

Theorem 6.20 *If a Jordan matrix of $\mathcal{T}\in\mathcal{L}(V_n)$ has the form*

$$J=\begin{pmatrix} J_1 & 0 & 0 \dots & 0 \\ 0 & J_2 & 0 \dots & 0 \\ \dots \\ 0 & 0 & 0 \dots & J_p \end{pmatrix}$$

where $J_1,\dots,J_p$ are the corresponding Jordan cells of the eigenvalues $\lambda_1,\dots,\lambda_p$, $p\leq n$, then

$$f(J)=\begin{pmatrix} f(J_1) & 0 & 0 \dots & 0 \\ 0 & f(J_2) & 0 \dots & 0 \\ \dots \\ 0 & 0 & 0 \dots & f(J_p) \end{pmatrix}$$

Proof. If f is an analytical function in a sufficiently large neighborhood of a point x_0, we have:

$$f(J)=\sum_{k\geq 0}\frac{f^{(k)}(x_0)}{k!}(J-x_0I)^k=$$

$$= \sum_{k\geq 0} \frac{f^{(k)}(x_0)}{k!} \begin{pmatrix} J_1 - x_0 I & 0 & 0 \dots & 0 \\ 0 & J_2 - x_0 I & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & J_p - x_0 I \end{pmatrix}^k =$$

$$= \sum_{k\geq 0} \frac{f^{(k)}(x_0)}{k!} \begin{pmatrix} (J_1 - x_0 I)^k & 0 & 0 \dots & 0 \\ 0 & (J_2 - x_0 I)^k & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & (J_p - x_0 I)^k \end{pmatrix} =$$

$$= \begin{pmatrix} f(J_1) & 0 & 0 \dots & 0 \\ 0 & f(J_2) & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & f(J_p) \end{pmatrix}.$$

$\square$

The next result is important in obtaining a matrix function. This is useful in many applications.

Theorem 6.21 *If A is a given matrix of $\mathcal{T} \in \mathcal{L}(V_n)$,with the Jordan matrix of the form*

$$J = \begin{pmatrix} J_1 & 0 & 0 \dots & 0 \\ 0 & J_2 & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & J_p \end{pmatrix}$$

then

$$f(A) = C \begin{pmatrix} f(J_1) & 0 & 0 \dots & 0 \\ 0 & f(J_2) & 0 \dots & 0 \\ \dots & & & \\ 0 & 0 & 0 \dots & f(J_p) \end{pmatrix} C^{-1}.$$

Exercises

6.1 Let $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by: $\mathcal{T}(0,1,2) = (1,0)$, $\mathcal{T}(-1,1,1) = (-1,1)$, $\mathcal{T}(3,0,-1) = (2,1)$. Determine:

a) the matrix of $\mathcal{T}$ relative to the canonical basis in $\mathbb{R}^3$ and $\mathbb{R}^2$

b) bases in the subspaces Ker$\mathcal{T}$ and Im$\mathcal{T}$

6.2 Let $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by: $\mathcal{T}(-1,2) = (-7,6,3)$, $\mathcal{T}(1,3) = (2,9,7)$. Determine:

a) the image of an arbitrary vector of $\mathbb{R}^2$ through $\mathcal{T}$

b) Ker$\mathcal{T}$ and Im$\mathcal{T}$

6.3 Let $\mathcal{T} \in \mathcal{L}(\mathbb{R}^3)$ be defined by $\mathcal{T}x = (x_1 + x_2 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3)$. Determine bases in Ker$\mathcal{T}$ and Im$\mathcal{T}$.

6.4 Consider the basis $B' = \{(1,1,0),(1,0,1),(0,1,1)\}$ in $\mathbb{R}^3$ and the linear transformation $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathcal{T}(x_1, x_2, x_3) = (x_1+x_2-x_3, x_3, 2x_2+3x_3)$. Determine the matrix of $\mathcal{T}$ with respect to the basis B'.

6.5 Determine the eigenvalues and the eigenvectors for the matrix of order n:

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$$

6.6 Determine the eigenvalues and the eigenvectors for the matrix of order n:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

6.7 Denoting by P_2 the space of polynomial functions of degree at most two, let $\mathcal{T} : P_2 \to P_2$ be the linear transformation given by $\mathcal{T}(1+X) = 1 - X^2$, $\mathcal{T}(1+X^2) = -4X$ and $\mathcal{T}(2X^2) = 4X^2$. Find the eigenvalues and the eigenvectors of $\mathcal{T}$.

6.8 Let $V = C(0,1)$, let $T : V \to V$ be an endomorphism defined by $T(f)(x) = xf(x)$. Determine the eigenvalues and eigenvectors of T.

6.9 Let $V = C^\infty(a,b)$, where $0 \notin (a,b)$, let $T : V \to V$ be an endomorphism defined by $T(f)(x) = \dfrac{1}{x}f'(x)$. Determine the eigenvalues and eigenvectors of T.

6.10 Find a Jordan basis and the corresponding Jordan form for:

a) $A = \begin{pmatrix} 6 & 6 & -15 \\ 1 & 5 & -5 \\ 1 & 2 & -2 \end{pmatrix}$, b) $B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$,

c) $C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

6.11 Find the Jordan form and the transfer matrix for:

a) $A = \begin{pmatrix} 4 & 1 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 4 \end{pmatrix}$, b) $B = \begin{pmatrix} -2 & -1 & 1 \\ 5 & -1 & 4 \\ 5 & 1 & 2 \end{pmatrix}$.

6.12 Determine A^n, $n \in \mathbf{N}$ for:

a) $A = \begin{pmatrix} -1 & 6 & 2 \\ -2 & 6 & 1 \\ 2 & -4 & 1 \end{pmatrix}$, b) $A = \begin{pmatrix} 0 & 2 & -3 \\ 4 & 7 & -12 \\ 3 & 6 & -10 \end{pmatrix}$,

c) $A = \begin{pmatrix} -2 & -1 & 1 \\ 5 & -1 & 4 \\ 5 & 1 & 2 \end{pmatrix}$, d) $A = \begin{pmatrix} -61 & 36 \\ -105 & 62 \end{pmatrix}$.

6.13 Determine e^A, for:

a) $A = \begin{pmatrix} -2 & -4 \\ 3 & 5 \end{pmatrix}$,

b) $A = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$.

6.14 For the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ determine e^A and $\sin A$.

6.15 A matrix $A \in M_n(\mathbf{C})$ is called *self-adjoint* if $\overline{A}^t = A$. Prove that if A is self-adjoint then all the roots of its characteristic polynomial are real, and the eigenvectors corresponding to distinct values are orthogonal.

6.16 A matrix $T \in M_n(\mathbf{C})$ is called *unitary* if $(\overline{T}^t)T = I$. Prove that if T is unitary then

a) For each eigenvalue λ of T we have $|\lambda| = 1$.

b) The eigenvectors corresponding to distinct values are orthogonal.

Solutions

$\boxed{\mathbf{6.1}}$ a) Denoting by e_1, e_2 and e_3 the vectors of the canonical basis in $\mathbb{R}^3$, we get the system $\begin{cases} \mathcal{T}e_2 + 2\mathcal{T}e_3 = & (1,0) \\ -\mathcal{T}e_1 + \mathcal{T}e_2 + \mathcal{T}e_3 = & (-1,1) \\ 3\mathcal{T}e_1 - \mathcal{T}e_3 = & (2,1) \end{cases}$ with the solutions $\mathcal{T}e_1 = (1,0)$, $\mathcal{T}e_2 = (-1,2)$ and $\mathcal{T}e_3 =$

$(1,-1)$. So the matrix of $\mathcal{T}$ relative to the canonical basis is $T = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix}$.

b) For $(x_1, x_2, x_3) \in \mathbb{R}^3$ we have

$$\mathcal{T}(x_1, x_2, x_3) = x_1(1,0)+x_2(-1,2)+x_3(1,-1) = (x_1-x_2+x_3, 2x_2-x_3).$$

The kernel of $\mathcal{T}$ is $\text{Ker}\mathcal{T} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1 - x_2 + x_3, 2x_2 - x_3) = (0,0)\} = \{(-\alpha, \alpha, 2\alpha) \mid \alpha \in \mathbb{R}\}$, and $\{(-1,1,2)\}$ is a basis of $\text{Ker}\mathcal{T}$.

The image is $\text{Im}\mathcal{T} = \{(x_1 - x_2 + x_3, 2x_2 - x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\} = \text{sp}\{(1,0), (-1,2), (1,-1)\} = \text{sp}\{(1,0), (-1,2)\} = \mathbb{R}^2$.

$\boxed{\textbf{6.2}}$ We have $\mathcal{T}e_1 = (5,0,1)$, $\mathcal{T}e_2 = (-1,3,2)$ and so $\mathcal{T}(x_1, x_2) = (5x_1 - x_2, 3x_2, x_1 + 2x_2)$. $\text{Ker}\mathcal{T} = \{(0,0)\}$,. The image is $\text{Im}\mathcal{T} = \{(5x_1 - x_2, 3x_2, x_1 + 2x_2) \mid x_1, x_2 \in \mathbb{R}\}$, with a basis $\{(5,0,1), (-1,3,2)\}$. Denoting $5x_1 - x_2 = x$, $3x_2 = y$, $x_1 + 2x_2 = z$ and eliminating the variables x_1, x_2 we get $\text{Im}\mathcal{T} = \{(x,y,z) \in \mathbb{R}^3 \mid 3x + 11y - 15z = 0\}$.

$\boxed{\textbf{6.3}}$ A basis in $\text{Ker}\mathcal{T}$ is $\{(1,0,-1), (0,1,-1)\}$ and in $\text{Im}\mathcal{T}$ is $\{(1,1,1)\}$.

$\boxed{\textbf{6.4}}$ The matrix of $\mathcal{T}$ relative to the canonical basis is $T = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix}$. The matrix relative to the new basis B' is $T' = C^{-1}TC$, where C is the transition matrix from the canonical basis to B'. We have

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}; \quad C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \text{ so } T' = \begin{pmatrix} 0 & -1 & -2 \\ 2 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

Another method to find the same matrix is to write the images of the vectors from the basis B' through $\mathcal{T}$:

$$\begin{aligned}
&\mathcal{T}(1,1,0) = (2,0,2) = 2(1,0,1)\\
&\mathcal{T}(1,0,1) = (0,1,3) = -(1,1,0) + (1,0,1) + 2(0,1,1)\\
&\mathcal{T}(0,1,1) = -2(1,1,0) + 2(1,0,1) + 3(0,1,1).
\end{aligned}$$

$\boxed{\mathbf{6.5}}$ The characteristic polynomial is $P(\lambda) = (-\lambda-1)^{n-1}(-\lambda+n-1)$, so the eigenvalues are $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = -1$ and $\lambda_n = n-1$. For $\lambda = -1$, the eigenvectors are the solutions of the '"system"

$$x_1 + x_2 + \cdots + x_3 = 0,$$

that is, $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{pmatrix}$. For $\lambda = n-1$, we get the system

$$\begin{cases} (1-n)x_1 + x_2 + \cdots + x_n = 0 \\ x_1 + (1-n)x_2 + \cdots + x_n = 0 \\ \cdots\cdots\cdots \\ x_1 + x_2 + \cdots + (1-n)x_n = 0 \end{cases}$$

Expressing $x_n = (n-1)x_1 - x_2 - \cdots - x_{n-1}$ from the first equation and plugging it into the other equations, we get $x_2 = x_1$, $x_3 = x_1$, ... $x_n = x_1$ and the only linearly independent eigenvector is $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$\boxed{\mathbf{6.6}}$ It is easier to calculate the characteristic polynomial by transposing the determinant, which is a circular determinant:

$$P(\lambda) = C(-\lambda, 0, \ldots, 0, 0, 1) = C(-\lambda, 1, \ldots, 0, 0, 0) =$$

$= (-1)^n(\lambda^n - 1).$

The eigenvalues are $\lambda_k = \varepsilon_k$, $k = 0, \ldots, n-1$ (the n-th roots of 1). For each eigenvalue ε_k, we determine the eigenvectors as the solutions of the system:

$$\begin{cases} -\varepsilon_k x_1 + x_n = 0 \\ x_1 - \varepsilon_k x_2 = 0 \\ x_2 - \varepsilon_k x_3 = 0 \\ \ldots\ldots\ldots \\ x_{n-1} - \varepsilon_k x_n = 0 \end{cases}$$

that is $v_k = \begin{pmatrix} \varepsilon_k^{n-1} \\ \varepsilon_k^{n-2} \\ \vdots \\ \varepsilon_k \\ 1 \end{pmatrix}$

$\boxed{\textbf{6.7}}$ From the given data we get $\mathcal{T}(1) = -4X - 2X^2$, $\mathcal{T}(X) = 1 + 4X + X^2$, $\mathcal{T}(X^2) = 2X^2$, so the matrix of the transformation, in the canonical basis $1, X, X^2$ is $T = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$. It has a triple eigenvalue $\lambda = 2$, and the subspace of eigenvectors has dimension 2. Two linearly independent eigenvectors are $v_1 = 1 + 2X$ and $v_2 = X^2$.

$\boxed{\textbf{6.8}}$ If $\lambda \in \mathbb{R}$ is an eigenvalue, then $T(f)(x) = \lambda f(x)$, for any $x \in (0,1)$. We get $(x - \lambda)f(x) = 0$, for any $x \in (0,1)$. We study two situations. If $\lambda \notin (0,1)$, then $x - \lambda \neq 0$, so $f(x) = 0$, for every $x \in (0,1)$, which is not convenient for an eigenvector. If $\lambda \in (0,1)$, then $f(x) = 0$, for $x \neq \lambda$, and $f(x) = \alpha$, $\alpha \in \mathbb{R}$, an arbitrary value. But f has to be continuous, which yields $\alpha = 0$, and $f = 0$, not convenient for an eigenvector. In conclusion, T does not possess eigenvalues.

$\boxed{\textbf{6.9}}$ If $\lambda \in \mathbb{R}$ is an eigenvalue, then $T(f)(x) = \lambda f(x)$, for any $x \in (a,b)$. We get $\frac{1}{x} f'(x) = \lambda f(x)$ or $\frac{f'(x)}{f(x)} = \lambda x$. Integrating, follows that $ln|f(x)| = \frac{\lambda x^2}{2} + \ln c$, that is $f(x) = ce^{\frac{\lambda x^2}{2}}$, $c \in \mathbb{R}$. So each $\lambda \in \mathbb{R}$ is an eigenvalue, with an infinity of eigenvectors, $f(x) = ce^{\frac{\lambda x^2}{2}}$, $c \in \mathbb{R}^*$.

$\boxed{\textbf{6.10}}$ a) $J_A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$, with the basis consisting of the eigenvectors $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and the principal vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

b) $J_B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

c) $J_C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

6.11 a) $J_A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. a) $J_B = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$.

6.12 a) The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = 3$, the matrix is diagonalizable. The transition matrix from the canonical basis to the basis consisting of eigenvectors is $C = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$, with the inverse $C^{-1} = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}$.

Finally we get

$$A^n = \begin{pmatrix} 2-3^n & -4+2^{n+1}+2\cdot 3^n & -2+2^{n+1} \\ 1-3^n & -2+2^n+2\cdot 3^n & -1+2^n \\ -1+3^n & 2-2\cdot 3^n & 1 \end{pmatrix}.$$

b) $J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ and

$$A^n = \begin{pmatrix} (-1)^{n-1}(n-1) & 2n(-1)^{n-1} & 3n(-1)^n \\ 4n(-1)^{n-1} & (-1)^{n-1}(8n-1) & 12n(-1)^n \\ 3n(-1)^{n-1} & 6n(-1)^{n-1} & (-1)^n(9n+1) \end{pmatrix}.$$

c) $A^n = \begin{pmatrix} (-2)^n & -n(-2)^{n-1} & n(-2)^{n-1} \\ 3^n-(-2)^n & n(-2)^{n-1}+(-2)^n & 3^n-(-2)^{n-1}(n-2) \\ 3^n-(-2)^n & n(-2)^{n-1} & 3^n-n(-2)^{n-1} \end{pmatrix}.$

d) $A^n = \begin{pmatrix} 21(-1)^n-20\cdot 2^n & 12(2^n-(-1)^n) \\ 35((-1)^n-2^n) & 21\cdot\ 2^n-20(-1)^n \end{pmatrix}.$

$\boxed{\mathbf{6.13}}$ a) The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 1$, with two corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. The diagonal form of A is $J_A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and the transition matrix is $C = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}$. From $A^n = CJ_A^nC^{-1}$ we get $A^n = \begin{pmatrix} -3\cdot 2^n+4 & -4\cdot 2^n+4 \\ 3\cdot 2^n+3 & 4\cdot 2^n-3 \end{pmatrix}$ and finally

$e^A = \begin{pmatrix} -3e^2+4e & -4e^2+4e \\ 3e^2+3e & 4e^2-3e \end{pmatrix}$.

b) $\lambda_1 = 0$, $\lambda_2 = 1$. Using the Cayley-Hamilton Theorem we have that $A^2 - A = 0$, so $A^n = A$, for $n \geq 1$. $e^A = \begin{pmatrix} 4e-3 & 2-2e \\ 6e-6 & 4-3e \end{pmatrix}$.

6.14 $A^n = \frac{1}{2}\begin{pmatrix} 3^n+(-1)^n & 3^n-(-1)^n \\ 3^n-(-1)^n & 3^n+(-1)^n \end{pmatrix}$,

$e^A = \frac{1}{2}\begin{pmatrix} e^3+e^{-1} & e^3-e^{-1} \\ e^3-e^{-1} & e^3+e^{-1} \end{pmatrix}$,

$\sin A = \frac{1}{1!}A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \cdots =$

$= \frac{1}{2}\begin{pmatrix} \sin 3+\sin(-1) & \sin 3-\sin(-1) \\ \sin 3-\sin(-1) & \sin 3+\sin(-1) \end{pmatrix}$.

CHAPTER 7

Quadratic forms

Consider the n-dimensional space $\mathbb{R}^n$ and denote by $x = (x_1, \ldots, x_n)$ the coordinates of a vector $x \in \mathbb{R}^n$ with respect to the canonical basis $E = \{e_1, \ldots, e_n\}$. A **quadratic form** is a map $Q : \mathbb{R}^n \to \mathbb{R}$

$$Q(x) = a_{11}x_1^2 + \ldots a_{nn}x_n^2 + 2a_{12}x_1x_2 + \cdots + 2a_{ij}x_ix_j + \ldots 2a_{n-1,n}x_{n-1}x_n,$$

where the coefficients a_{ij} are all real.

Thus, quadratic forms are homogenous polynomials of second degree in a number of variables.

Using matrix multiplication, we can write Q in a compact form as

$$Q(x) = X^\top AX,$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}.$$

The symmetric matrix A (notice that $a_{ij} = a_{ji}$) is called the matrix of the quadratic form. Being symmetrical (and real), A is the matrix of a self-adjoint operator with respect to the basis E. This operator, denoted by T, is diagonalizable and there exists a basis $B = \{b_1, \dots, b_n\}$ formed by eigenvectors with respect to which $\mathcal{T}$ has a diagonal matrix consisting of eigenvalues (also denoted by T)

$$T = diag\{\lambda_1. \dots, \lambda_n\}.$$

Let C be the transition matrix from E to B and

$$X' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

the coordinates of the initial vector written in B. We have than

$$X = CX'$$

Knowing that $T = C^{-1}AC$, and that $C^{-1} = C^{\top}$ we compute

$$\begin{aligned} Q(x) &= X^\top A X \\ &= (CX')^\top A\,(CX') \\ &= X'^\top C^\top A C X' \\ &= X'^\top T X' \\ &= \lambda_1 x'^2_1 + \cdots + \lambda_n x'^2_n, \end{aligned}$$

and we say that we have reduced Q to its canonical form

$$Q(x) = \lambda_1 x'^2_1 + \cdots + \lambda_n x'^2_n.$$

This is called the geometric method.

The quadratic form is called

- positive definite if $Q(x) > 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$
- negative definite if $Q(x) < 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$.

We characterize the positively definite quadratic form in terms of the diagonal minors of its matrix, as follows:

$$D_1 = a_{11}, D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}, \ldots, D_n = \det A.$$

We have the following criteria:

- Q is positive definite iff $D_i > 0$ for every $i = \overline{1,n}$
- Q is negative definite iff $(-1)^i D_i > 0$ for every $i = \overline{1,n}$.

7.1 Conics and quadrics

7.1.1 Second degree curves

The general form of a second degree curve (a conic) is:
$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$
where $(x, y) \in \mathbb{R}^2$ and not all the coefficients a_{11}, a_{12}, $a_{22} = 0$.

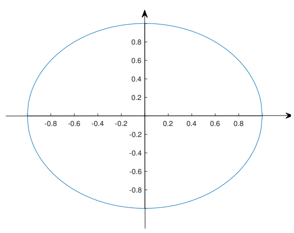
We consider the third and the second order determinants obtained from the coefficients a_{ij}, $i, j = 1, 2, 3$ as follows:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \text{ and } \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}$$

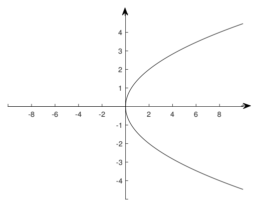
Classification of the conics

A) *Non degenerate conics* $(\Delta \neq 0)$ and their canonical form:

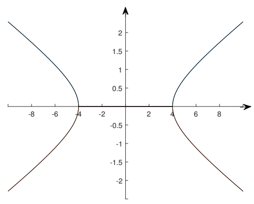
- Ellipse $(\delta > 0)$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



- Parabola $(\delta = 0)$: $y^2 - 2ax = 0$



- Hyperbola ($\delta < 0$): $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$



B) *Degenerate conics* ($\Delta = 0$):

- One point, nothing ($\delta > 0$)
- Two parallel lines, one (double) line ($\delta = 0$)
- Two intersected lines ($\delta < 0$)

The reader is assumed to know from the high school, the graphic representations of the conics, so, we leave this as an exercise.

7.1.2 Second degree surfaces

The general equation of a second degree surface (a quadric surface) is:

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$
$$2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0$$

where $(x, y, z) \in \mathbb{R}^3$.

From a geometric point of view, quadrics, which are also called **quadric surfaces**, are two-dimensional surfaces defined as the locus of zeros of a second degree polynomial in x, y and z. Maybe the most prominent example of a quadric is the sphere (the spherical surface).

The type is determined by the quadratic form that contains all terms of degree two

$$Q = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz.$$

We distinguish, based on the sign of the eigenvalues of the matrix of Q, between: ellipsoids, elliptic or hyperbolic paraboloids, hyperboloids with one or two sheets, cones and cylinders.

The canonical form of a quadric

Further, the geometric method to reduce the general equations of a quadric to canonical form, is presented.

Consider the matrix A associated to Q. Being symmetric, A has real eigenvalues $\lambda_1, \lambda_2, \lambda_3$. If they are distinct, the corresponding eigenvectors are orthogonal (if not we apply the Gram-Schmidt algorithm). Thus, we obtain three orthogonal unit vectors $\{b_1, b_2, b_3\}$, a basis in $\mathbb{R}^3$.

Let R be the transition matrix from $\{i, j, k\}$ to the new basis $\{b_1, b_2, b_3\}$. We recall from previous chapters that R has the three vectors b_1, b_2, b_3 as its columns

$$R = [b_1|b_2|b_3] .$$

Now, we compute $\det R$ and check whether

$$\det R = 1.$$

If $\det R = -1$, we must change one of the vectors by its opposite (for example take $R = [-b_1|b_2|b_3]$) to obtain $\det R = 1$. This assures that the matrix R defines a **rotation** and the new basis is obtained from the original one, by this rotation.
Let (x, y, z) and (x', y', z') be the coordinates of the same point in the original basis and in the new one, we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} .$$

We know that with respect to the new coordinates

$$Q = \lambda_1 {x'}^2 + \lambda_2 {y'}^2 + \lambda_3 {z'}^2,$$

and thus, the equation of the quadric reduces to the simpler form

$$\lambda_1 {x'}^2 + \lambda_2 {y'}^2 + \lambda_3 {z'}^2 + 2{a'}_{14}x' + 2{a'}_{24}y' + 2{a'}_{34}z' + a_{44} = 0.$$

To obtain the canonical form of the quadric we still have to perform another transformation, namely a **translation**. To complete this step we investigate three cases: (A) when we have three nonzero eigenvalues, (B) when one eigenvalue is zero and (C) when two eigenvalues are equal to zero.

(A) For $\lambda_i \neq 0$ we obtain

$$\lambda_1(x' - x_0)^2 + \lambda_2(y' - y_0)^2 + \lambda_3(z' - z_0)^2 + {a'}_{44} = 0$$

Consider the translation defined by

$$\begin{aligned} x'' &= x' - x_0, \\ y'' &= y' - y_0, \\ z'' &= z' - z_0. \end{aligned}$$

In the new coordinates the equation of the quadric reduces to the canonical form

$$\lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2 + {a'}_{44} = 0.$$

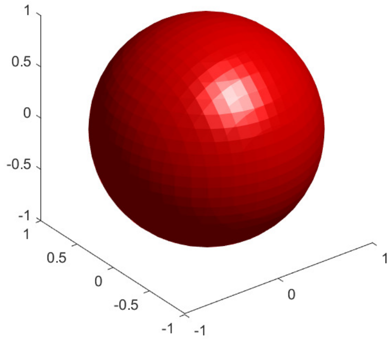
The cases (B) and (C) can be treated similarly.

Quadrics on the reduced equations

1. The Sphere

A sphere with the center at $C(a, b, c)$ and radius R has the equation

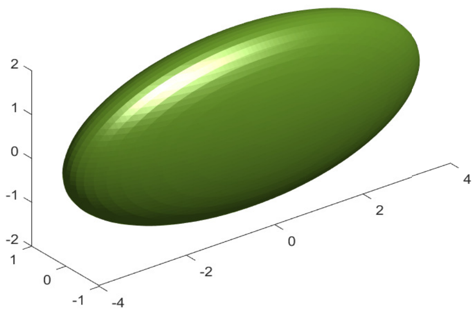
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2.$$



2. The Ellipsoid

The equation of an ellipsoid has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.$$



To investigate the shape of the surface, one can use the *method of parallel sections*. This consists of intersecting the surface with planes that are parallel to the coordinate planes and determining the intersection curves.

The intersection with the plane xOy:
$\begin{cases} z = 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \end{cases}$ is an ellipse.

Intersections with planes parallel to xOy:
$\begin{cases} z = k \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, \end{cases}$ with $k \in \mathbb{R}$. If $|k| < c$ we get an ellipse, if $|k| = c$, a point is obtained and if $|k| > c$ results the empty set.

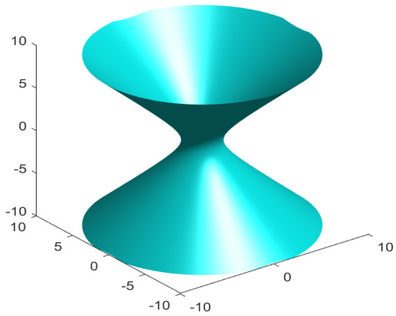
Intersections with planes parallel to xOz:
$\begin{cases} y = k \\ \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{k^2}{b^2}, \end{cases}$ with $k \in \mathbb{R}$. If $|k| < b$ we get an ellipse, if $|k| = b$, we obtain a point and if $|k| > b$, the empty set.

Intersections with planes parallel to yOz:
$\begin{cases} x = k \\ \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}, \end{cases}$ with $k \in \mathbb{R}$. If $|k| < a$ we get an ellipse, if $|k| = a$, a point and if $|k| > a$, the empty set.

3. The Hyperboloid of one sheet

The equation of a hyperboloid of one sheet has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0.$$



-*An ellipse* is obtained by intersecting the hyperboloid of one sheet with planes parallel to xOy:
$\begin{cases} z = k \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \end{cases}$ with $k \in \mathbb{R}$.

- *A hyperbola*: by intersection with planes parallel to xOz:
$\begin{cases} y = k \\ \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{b^2}, \end{cases}$

- *A hyperbola*: by intersections with planes parallel to yOz:
$\begin{cases} x = k \\ \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}, \end{cases}$

This surface has the special property that it can be generated by two distinct families of straight lines (doubly ruled).

Theorem 7.1 *For any point M on the surface, there exist two straight lines lying entirely on the surface, which pass through the point M.*

Proof: The equation of the hyperboloid can be written $\left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right)$. From this, we get that the two families of straight lines are:

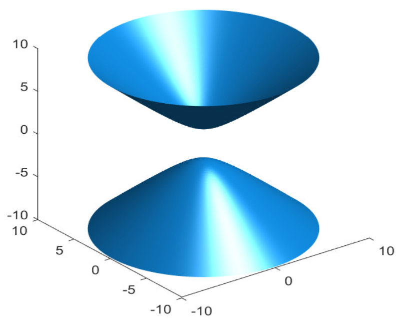
$$G_\lambda : \begin{cases} \dfrac{x}{a} - \dfrac{z}{c} = \lambda\left(1 - \dfrac{y}{b}\right) \\ \dfrac{x}{a} + \dfrac{z}{c} = \dfrac{1}{\lambda}\left(1 + \dfrac{y}{b}\right) \end{cases}, \ \lambda \in \mathbb{R}$$

$$G_\mu : \begin{cases} \dfrac{x}{a} - \dfrac{z}{c} = \mu(1 + \dfrac{y}{b}) \\ \dfrac{x}{a} + \dfrac{z}{c} = \dfrac{1}{\mu}(1 - \dfrac{y}{b}) \end{cases}, \ \mu \in \mathbb{R}.$$

4. The Hyperboloid of two sheets

The equation of a hyperboloid of two sheets has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0.$$

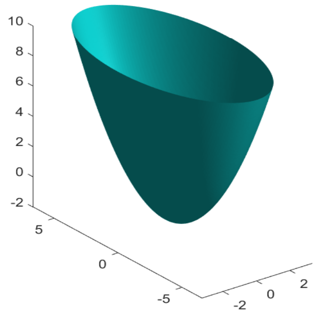


Study the intersection curves with the planes parallel to the coordinate planes.

5. The Elliptic paraboloid

The equation has the form

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$



- *An ellipse*: intersections with planes parallel to xOy: $\begin{cases} z = k \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = k, \end{cases}$ with $k \in \mathbb{R}$. If $k < 0$, we get the empty set, if $k = 0$ a point, if $k > 0$.

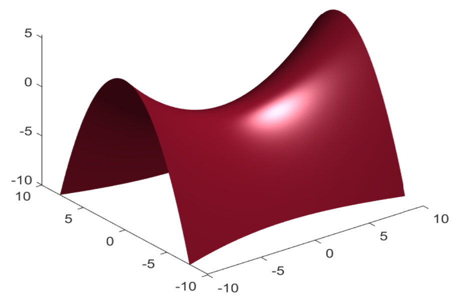
- *A parabola*: intersections with planes parallel to xOz: $\begin{cases} y = k \\ \frac{x^2}{a^2} + \frac{k^2}{b^2} = z \end{cases}$

- *A parabola*: intersections with planes parallel to yOz: $\begin{cases} x = k \\ \frac{y^2}{b^2} + \frac{k^2}{a^2} = z \end{cases}$

6. The Hyperbolic paraboloid

The equation has the form

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$



- *A hyperbola*: intersections with planes parallel to xOy:
$$\begin{cases} z = k \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = k, \end{cases}$$
- *A parabola*: intersections with planes parallel to xOz:
$$\begin{cases} y = k \\ z = \frac{x^2}{a^2} - \frac{k^2}{b^2}, \end{cases}$$
- *A parabola*: intersections with planes parallel to yOz:
$$\begin{cases} x = k \\ z = -\frac{y^2}{b^2} + \frac{k^2}{a^2} = z, \end{cases}$$

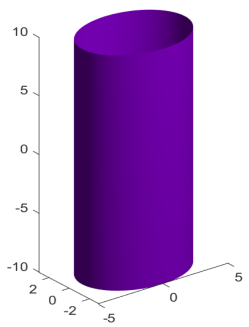
Theorem 7.2 *For any point M on the surface, there exist two straight lines lying entirely on the surface, which pass through the point M.*

Proof: The equation of the paraboloid can be written $z = \left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right)$. From this, we get that the two families of straight lines are:

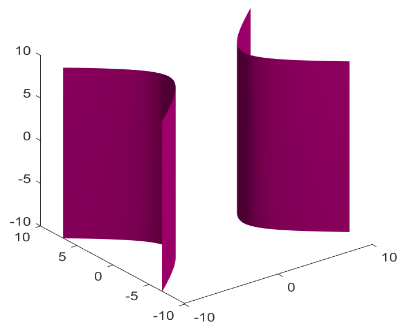
$$G_\lambda : \begin{cases} \frac{x}{a} - \frac{y}{b} = \lambda z \\ \frac{x}{a} + \frac{y}{b} = \frac{1}{\lambda} \end{cases}, \ \lambda \in \mathbb{R} \quad G_\mu : \begin{cases} \frac{x}{a} + \frac{y}{b} = \mu z \\ \frac{x}{a} - \frac{y}{b} = \frac{1}{\mu} \end{cases}, \ \mu \in \mathbb{R}.$$

7. Cylinders (Degenerate quadrics)

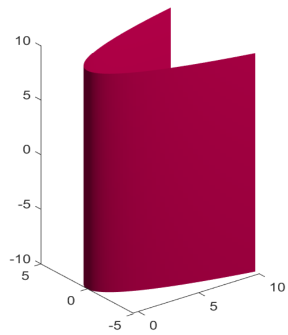
-Elliptic cylinder $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ (the generators are parallel to Oz and intersect a given ellipse).



- Hyperbolic cylinder: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-1=0.$



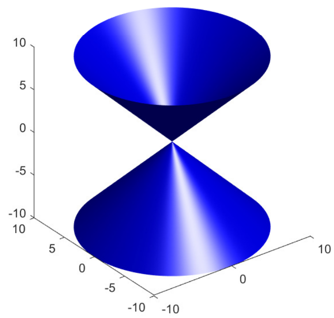
- Parabolic cylinder: $y^2-2px=0.$



8. The Elliptic cone

The equation of an elliptic cone (degenerate quadric) has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$



Exercises

7.1 Write the equation of the circle that passes through the points $A(-1, 2)$, $B(3, 0)$ and has the center on the line $3x - y + 2 = 0$.

7.2 Write the equation of the conic who passes through the points $M_1(2, 0)$, $M_2(3, 0)$, $M_3(0, 1)$, $M_4(0, 4)$, $M_5(5, 4)$.

7.3 Find the canonical form and draw the conic:

a) $4x^2 + 6xy + 4y^2 - 10x + 10y + 1 = 0$

b) $7x^2 - 8xy + y^2 - 6x - 12y - 9 = 0$

c) $xy = 1$

d) $9x^2 - 6xy + y^2 + 20x = 0$

e) $5x^2 + 6xy + 5y^2 - 16x - 16y - 16 = 0$

f) $3x^2 + 10xy + 3y^2 - 2x - 14y - 13 = 0$

7.4 Determine the nature of the conics:

a) $x^2 + 4xy + 4y^2 - 3x - 6y = 0$

b) $x^2 - 4xy + y^2 + 3x - 3y + 2 = 0$

c) $2x^2 + 3xy + y^2 - x - 1 = 0$

d) $x^2 - 6xy + 9y^2 + 4x - 12y + 4 = 0$

e) $x^2 - 4xy + 4y^2 + 2x - 4y - 3 = 0$

7.5 Study the type of the following conics when $\alpha \in \mathbb{R}$:

a) $x^2 - 2xy + \alpha y^2 - 4x - 5y + 3 = 0$

b) $\alpha x^2 + 2xy + y^2 + 2\alpha y + \alpha = 0$.

7.6 Find the values of the parameters α and β for which the conics $\alpha x^2 + 12xy + 9y^2 + 4x + \beta y - 13 = 0$

a) Have a center;

b) Are non degenerate conics but without a center.

7.7 Find the values of the parameters $\alpha, \beta \in \mathbb{R}$ for which the conic $x^2 + 4xy + \alpha y^2 - 3x + 2\beta y = 0$ represents two parallel lines.

7.8 Determine the parameters $\alpha, \beta, \gamma \in \mathbb{R}$ such that, the equation $x^2 - 2\alpha xy + 2\beta y^2 + \beta x - 2\alpha y + \gamma = 0$ represents a double line.

7.9 Find the nature of the conics and show they have the same center $6x^2 - 5xy + y^2 - 22x + 9y - 4 = 0$ and $3x^2 - 2xy - y^2 - 10x - 2y + 12 = 0$.

7.10 Find the projection of the curve

$$\begin{cases} x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} - 1 = 0 \\ x + y + z - 1 = 0 \end{cases}$$ on the plane xOy.

7.11 Find the canonical form of the following quadrics:

a) $38x^2 + 35y^2 + 26z^2 - 28xy - 8xz + 20yz - 54 = 0$

b) $2x^2 + 16y^2 + 2z^2 - 8xy + 8yz - 2x - y + 2z + 3 = 0.$

c) $x^2 + y^2 + 5z^2 - 6xy + 2xz - 2yz - 4x + 8y - 12z + 14 = 0$

d) $2y^2 + 4xy - 8xz - 4yz + 6x - 5 = 0$

e) $x^2 + 3y^2 + 4yz - 6x + 8y + 8 = 0$

7.12 a) Find the canonical form of the quadric $xy = z$.

b) Determine the straight lines that belong to the surface of the quadric and are parallel to the plane $x + y + z = 1$.

7.13 Determine the center and the radius of the circle given by
$$\begin{cases} x^2 + y^2 + z^2 - 2x - 4z - 4 = 0 \\ x - 2y + z + 3 = 0 \end{cases}$$

7.14 Find the geometrical locus generated by the lines
$$\begin{cases} 2x + 3\alpha y + 6z - 6\alpha = 0 \\ 2\alpha x - 3y - 6\alpha z - 6 = 0 \end{cases}, \ \alpha \in \mathbb{R}.$$

7.15 Find the intersection of the line $x - 3 = y - 1 = \dfrac{z-6}{3}$ with the elliptic hyperboloid $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{9} + 1 = 0.$

7.16 Determine the straight lines that belong to the surface of the hyperboloid of one sheet $\dfrac{x^2}{25} + \dfrac{y^2}{16} - \dfrac{z^2}{4} - 1 = 0$ and pass through the point $M(-5, 4, 2)$.

7.17 Find the straight lines of the quadric Q which are parallel to the plane P, if:

a) Q: $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} - 1 = 0$ and P: $6x + 4y + 3z - 17 = 0$

b) Q: $\dfrac{x^2}{16} - \dfrac{y^2}{4} = z$ and P: $3x + 2y - 4z = 0.$

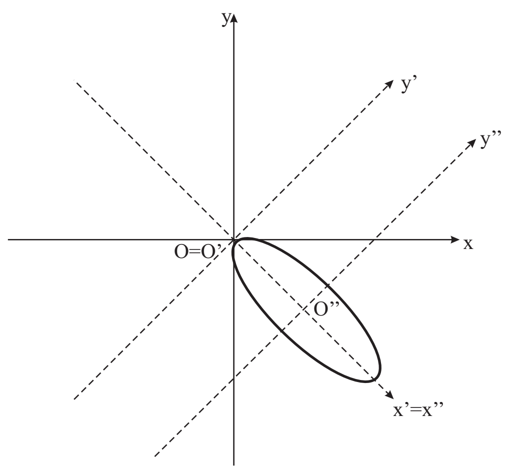
7.18 Find the equation of a plane tangent to the sphere $x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0$ and containing the line $x = 4t + 4$, $y = 3t + 1$, $z = t + 1.$

Solutions

$\boxed{\mathbf{7.1}}$ If the center is $C(a,b)$, then a and b satisfy the system: $\begin{cases} 3a-b+2=0 \\ \sqrt{(a+1)^2+(b-2)^2}=\sqrt{(a-3)^2+b^2} \end{cases}$. We obtain $a=-3$, $b=-7$ and the radius $r=\sqrt{85}$. The equation is $(x+3)^2+(y+7)^2=85$.

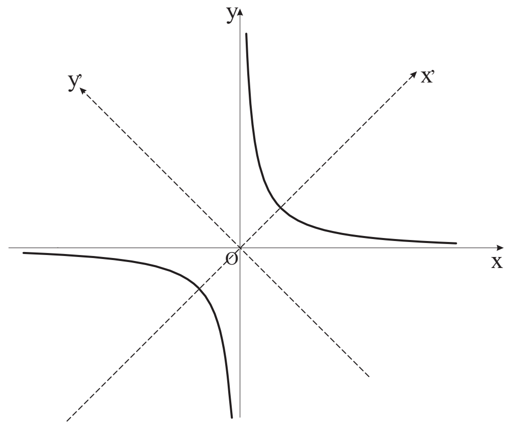
$\boxed{\mathbf{7.2}}$ $2x^2+3y^2-10x-15y+12=0$.

$\boxed{\mathbf{7.3}}$ a) The eigenvalues are 1 and 7 and a possible basis of eigenvectors is $v_1=(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$, $v_2=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$. The new system of coordinates is obtained by a rotation, by $-\frac{\pi}{4}$. The equation in these new coordinates is $x'^2+7y'^2-\frac{20x'}{\sqrt{2}}+1=0$. Then, by a translation $x''=x'-5\sqrt{2}$, $y''=y'$, we get the equation of an ellipse $\frac{x''^2}{49}+\frac{y''^2}{7}=1$.



b) Determining a basis of eigenvectors $v_1=(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})$ and

$v_2 = (-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ we perform a rotation and get the equation $-x'^2 + 9y'^2 - \frac{30x'}{\sqrt{5}} - 9 = 0$. Then, by a translation $x'' = x' + 3\sqrt{5}$, $y'' = y'$, we get the equation of a hyperbola $\frac{x''^2}{36} - \frac{y''^2}{4} = 1$.
c) By a rotation of $\frac{\pi}{4}$ the new system of coordinates is obtained and the new equation is $\frac{1}{2}x'^2 - \frac{1}{2}y'^2 = 1$, an equilateral hyperbola.



d) $\lambda_1 = 0$, $\lambda_2 = 10$, $v_1 = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$, $v_2 = (\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}})$, after a rotation and a translation the equation becomes $10y''^2 + 2\sqrt{10}x'' - 9 = 0$, a parabola.
e) $\frac{x''^2}{4} + \frac{y''^2}{16} - 1 = 0$, an ellipse.
f)$x''^2 - \frac{y''^2}{4} - 1 = 0$, a hyperbola.

$\boxed{\mathbf{7.4}}$ a) $\Delta = \delta = 0$, so the conic is degenerate, of parabolic type, it represents two parallel lines. b) $\Delta \neq 0$, $\delta < 0$ it is a hyperbola. c) $\Delta = 0$, $\delta < 0$ degenerate, of hyperbolic type, that means two intersecting lines. The equation can be written as $(2x + y + 1)(x + y - 1) = 0$, which gives the two lines $2x + y + 1 = 0$ and $x + y - 1 = 0$. d) One (double) line: $(x - 3y + 2)^2 = 0$. e) Two parallel lines: $x - 2y - 1 = 0$ and $x - 2y + 3 = 0$.

$\boxed{\textbf{7.16}}$ One family of straight generators is $G_\lambda : \begin{cases} \frac{x}{5} + \frac{z}{2} = \lambda\left(1 - \frac{y}{4}\right) \\ \frac{x}{5} - \frac{z}{2} = \frac{1}{\lambda}\left(1 + \frac{y}{4}\right) \end{cases}$ From the condition $M \in G_\lambda$ we get $\lambda = -1$, so one of the requested lines is $\begin{cases} 4x - 5y + 10z + 20 = 0 \\ 4x + 5y - 10z + 20 = 0 \end{cases}$. From the other family of generators $G_\mu : \begin{cases} \frac{x}{5} + \frac{z}{2} = \mu\left(1 + \frac{y}{4}\right) \\ \frac{x}{5} - \frac{z}{2} = \frac{1}{\mu}\left(1 - \frac{y}{4}\right) \end{cases}$ we get $\mu = 0$ so the line will be $\begin{cases} 2x + 5y = 0 \\ y = 4 \end{cases}$.

$\boxed{\textbf{7.5}}$ a) Hyperbola for $\alpha \in (-\infty, -77/4) \cup (-77/4, 1)$; intersecting lines for $\alpha = -77/4$; ellipse for $\alpha \in (1, \infty)$; parabola for $\alpha = 1$.
b) Hyperbola for $\alpha \in (-\infty, 0) \cup (0, 1)$; intersecting lines for $\alpha = 0$; ellipse for $\alpha \in (1, \infty)$; parabola for $\alpha = 1$.

$\boxed{\textbf{7.6}}$ a) $\alpha \neq 4$; b) $\alpha = 4$ and $\beta \neq 6$.

$\boxed{\textbf{7.7}}$ $\alpha = 4$, $\beta = -3$ and the lines are $x + 2y = 0$ and $x + 2y - 3 = 0$.

$\boxed{\textbf{7.8}}$ For $\alpha = \beta = \gamma = 0$ the line is $x = 0$; $\alpha = -2, \beta = 2, \gamma = 1 \Rightarrow x + 2y + 1 = 0$; $\alpha = 2, \beta = 2, \gamma = 1 \Rightarrow x - 2y + 1 = 0$.

$\boxed{\mathbf{7.9}}$ The common center is $C(1,-2)$.

$\boxed{\mathbf{7.10}}$ By eliminating z between the equations we obtain the curve $45x^2+18xy+13y^2-18x-18y-27=0$, so the projection is an ellipse $\begin{cases} z=0 \\ 45x^2+18xy+13y^2-18x-18y-27=0 \end{cases}$

$\boxed{\mathbf{7.11}}$ a) We have an ellipsoid of the canonical form $\frac{x'^2}{1}+\frac{y'^2}{2}+\frac{z'^2}{3}-1=0$. b) An elliptic paraboloid $y''2=\frac{2x''^2}{3}+\frac{6z''2}{1}$. c) $-\frac{1}{3}x''^2+\frac{1}{2}y''^2+z''^2+1=0$. d) $x''=\sqrt{6}z''^2-\frac{2\sqrt{6}}{3}y''^2$. e) $-x''^2+y''^2+4z''^2-1=0$.

$\boxed{\mathbf{7.12}}$ a) The eigenvalues are $-\frac{1}{2},\frac{1}{2}$ and 0. A possible basis of eigenvectors consists of $v_1=(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)$, $v_2=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ and $v_3=(0,0,1)$. In the new system of coordinates the equation becomes $z'=\frac{1}{2}(y'^2-x'^2)$, that is a hyperbolic paraboloid.

b) One of the families of generators is $\begin{cases} x=\lambda \\ y=\frac{1}{\lambda}z \end{cases}$, with the direction vector $\vec{l}_\lambda=(0,\frac{1}{\lambda},1)$. From the parallelism follows that the direction vector is perpendicular on the normal to the plane, $\vec{n}=(1,1,1)$, that is $\vec{l}_\lambda\cdot\vec{n}=0$. We get $\lambda=-1$ and so the straight line is $\begin{cases} x=-1 \\ y+z=0 \end{cases}$. From the other family of generators $\begin{cases} x=\mu z \\ y=\frac{1}{\mu} \end{cases}$ we obtain $\begin{cases} x+z=0 \\ y=-1 \end{cases}$.

$\boxed{\mathbf{7.13}}$ $C(0,2,1)$, $r=\sqrt{3}$.

7.14 By eliminating α we obtain an elliptic hyperboloid of one sheet $4x^2 + 9y^2 - 36z^2 - 36 = 0$.

7.15 $x = 4$, $y = 2$, $z = 9$.

7.16 One family of straight generators is $G_\lambda : \begin{cases} \dfrac{x}{5} + \dfrac{z}{2} = \lambda\left(1 - \dfrac{y}{4}\right) \\ \dfrac{x}{5} - \dfrac{z}{2} = \dfrac{1}{\lambda}\left(1 + \dfrac{y}{4}\right) \end{cases}$
From the condition $M \in G_\lambda$ we get $\lambda = -1$, so one of the requested lines is $\begin{cases} 4x - 5y + 10z + 20 = 0 \\ 4x + 5y - 10z + 20 = 0 \end{cases}$. From the other family of generators $G_\mu : \begin{cases} \dfrac{x}{5} + \dfrac{z}{2} = \mu\left(1 + \dfrac{y}{4}\right) \\ \dfrac{x}{5} - \dfrac{z}{2} = \dfrac{1}{\mu}\left(1 - \dfrac{y}{4}\right) \end{cases}$ we get $\mu = 0$ so the line will be $\begin{cases} 2x + 5y = 0 \\ y = 4 \end{cases}$.

7.17 a) $6x - 4y - 3z + 12 = 0$, $6x + 4y + 3z + 12 = 0$ and $6x - 4y - 3z - 12 = 0$, $6x + 4y + 3z - 12 = 0$.
b) $x - 2y - 4z = 0$, $x + 2y - 4 = 0$ and $x + 2y - 2z = 0$, $x - 2y - 8 = 0$.

7.18 $x - y - z - 2 = 0$.

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