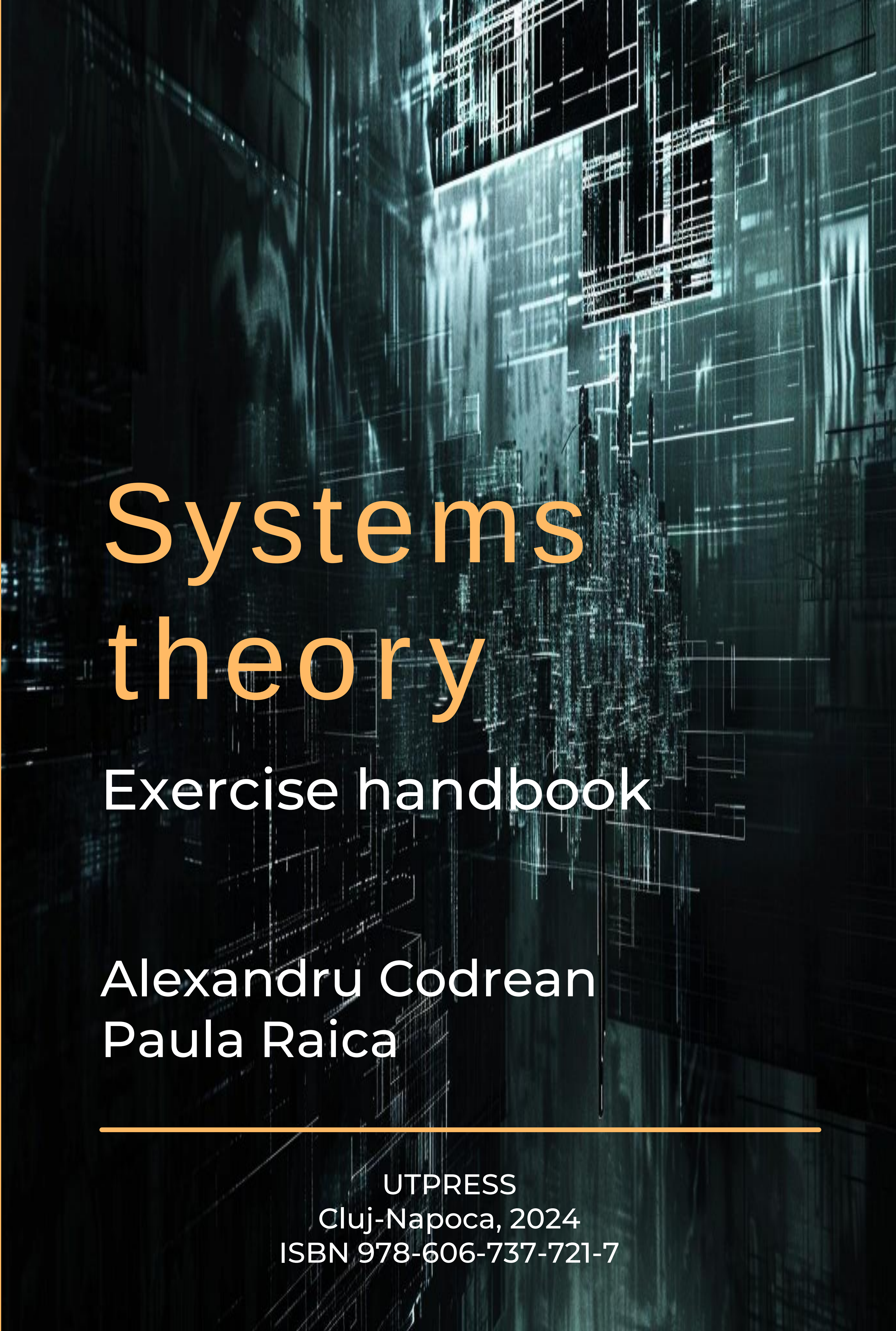

Systems theory

Exercise handbook

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Contents

1 Modeling with differential and difference equations

Topics: mathematical modeling, systems linearization, numerical simulation.

1.1 Solved exercises

SE 1.1 Consider the nonlinear model of the Van der Pol oscillator:

$$\frac{d^2x(t)}{dt^2} + (1 - x(t)^2)\frac{dx(t)}{dt} + x(t) = 0 \tag{1.1}$$

or

$$\ddot{x}(t) + (1 - x(t)^2)\dot{x}(t) + x(t) = 0 \tag{1.2}$$

(a) Determine a linear approximation of the Van der Pol equation around the equilibrium point $x_0 = 0$, $\dot{x}_0 = 0$, $\ddot{x}_0 = 0$.

(b) Compare the numerical solution of the linearized system with that of the original nonlinear system for a time interval of $t \in [0, 14]$. Start with the initial conditions $x(0) = 0.001$ and $\dot{x}(0) = 0$.

Solution:

(a) According to the nonlinear Van der Pol equation, for $\dot{x}_0 = 0$ and $\ddot{x}_0 = 0$ we obtain $x_0 = 0$.

Use the first-order truncated Taylor series approximation for a function of n variables: $y = g(x_1, x_2, ..., x_n)$ around a point $\mathbf{x_0} = (x_{10}, x_{20}, ..., x_{n0})$:

$$y = g(x_{10}, x_{20}, ..., x_{n0}) + \frac{\partial g}{\partial x_1}|_{\mathbf{x_0}} \cdot (x_1 - x_{10}) + ... + \frac{\partial g}{\partial x_n}|_{\mathbf{x_0}} \cdot (x_n - x_{n0})$$

For the Taylor series approximation in case of a differential equation, consider $x_1 = x(t)$, $x_2 = \dot{x}(t)$ and $x_3 = \ddot{x}(t)$ and the nonlinear function g is the left hand side of the nonlinear Van der Pol equation. We obtain:

$$g(x, \dot{x}, \ddot{x}) = g(0,0,0) + \frac{\partial g}{\partial x}|_{(0,0,0)}(x(t) - 0) + \frac{\partial g}{\partial \dot{x}}|_{(0,0,0)}(\dot{x}(t) - 0) + \frac{\partial g}{\partial \ddot{x}}|_{(0,0,0)}(\ddot{x}(t) - 0)$$

or

$$g(x, \dot{x}, \ddot{x}) = 0 + (-2x\dot{x} + 1)|_{(0,0,0)}(x - 0) + (1 - x^2)|_{(0,0,0)}(\dot{x} - 0) + 1 \cdot (\ddot{x} - 0)$$

$$g(x, \dot{x}, \ddot{x}) = 0 + \Delta x + \Delta \dot{x} + \Delta \ddot{x} = 0$$

where:

$$\begin{aligned}
\Delta x &= x(t) - x_0 = x(t) - 0 \\
\Delta \dot{x} &= \dot{x}(t) - \dot{x}_0 = \dot{x}(t) - 0 \\
\Delta \ddot{x} &= \ddot{x}(t) - \ddot{x}_0 = \ddot{x}(t) - 0
\end{aligned}$$

The equation:

$$\Delta x + \Delta \dot{x} + \Delta \ddot{x} = 0 \quad \text{or} \quad x(t) + \dot{x}(t) + \ddot{x}(t) = 0 \tag{1.3}$$

is linear and homogeneous.

(b) We will now obtain the numerical solution using MATLAB[1] and compare it with the solution of the nonlinear Van der Pol equation. Consider the nonlinear Van der Pol equation (1.2). We will bring the nonlinear second order differential equation to a state space form by defining the state variables: $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t) = \dot{x}_1(t)$. Therefore, $\dot{x}_2(t) = \ddot{x}(t)$, and we obtain the nonlinear state space model:

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -(1 - x_1(t)^2)x_2(t) - x_1(t) \end{cases} \tag{1.4}$$

With the same procedure, the linear approximation (1.3) will be written as:

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - x_2(t) \end{cases} \tag{1.5}$$

The initial conditions in both cases are: $x_1(0) = 0.001$ and $x_2(0) = 0$.

MATLAB will compute the numerical solution of a system of first-order differential equations with the functions *ode23*, *ode45*, *ode15s* and others (see MATLAB *help* for details). The solver will use a user-defined function that describes the right hand side of the system. For example, the system (1.4) will be written as a MATLAB function *fnonlin.m*:

Listing 1.1: *fnonlin.m*

```
function f=fnonlin(t,x)
f=[x(2); -(1-x(1)^2)*x(2)-x(1)];
```

The system (1.5) will be written as a MATLAB function *flin.m*:

Listing 1.2: *flin.m*

```
function f=flin(t,x)
f=[x(2); -x(1)-x(2)];
```

A main program that solves and compares the solution of (1.4) and (1.5) may look like:

Listing 1.3: *VDP_main_script.m*

```
close all %close all graphic windows
clear all %clear all variables

x0=[0.001;0]; % set the initial conditions

[tn,xn]=ode23(@fnonlin,[0 14], x0); %solve the nonlinear equation
[tl,xl]=ode23(@flin,[0 14], x0);    %solve the linear equation

plot(tn, xn(:,1), 'b-', tl, xl(:,1), 'r*'), grid on     % solution x_1(t)
xlabel('t'), ylabel('x_1'), title('Solution x_1(t)')
legend('x_1 in VDP nonlinear', 'x_1 in VDP linear')

figure,  plot(tn, xn(:,2), 'b-', tl, xl(:,2), 'r*'), grid on    % solution x_2(t)
xlabel('t'), ylabel('x_2'), title('Solution x_2(t)')
legend('x_2 in VDP nonlinear', 'x_2 in VDP linear')
```

[1] MATLAB and Simulink are registered trademarks of The MathWorks, Inc.

The *ode23* function has at least three input arguments:

- @*fnonlin*: function handle that evaluates the right side of the differential equation
- [0 14]: A vector specifying the interval of integration: [initial_time final_time]
- x_0: a vector of initial conditions

The function will return the output arguments:

- *tn*: a column vector of time points between the initial time and final time
- *xn*: the solution array. Each column in *xn* corresponds to the solution at a time returned in the corresponding row of *tn*.

The initial conditions chosen in this example are very close to 0, thus the approximation should be very accurate.

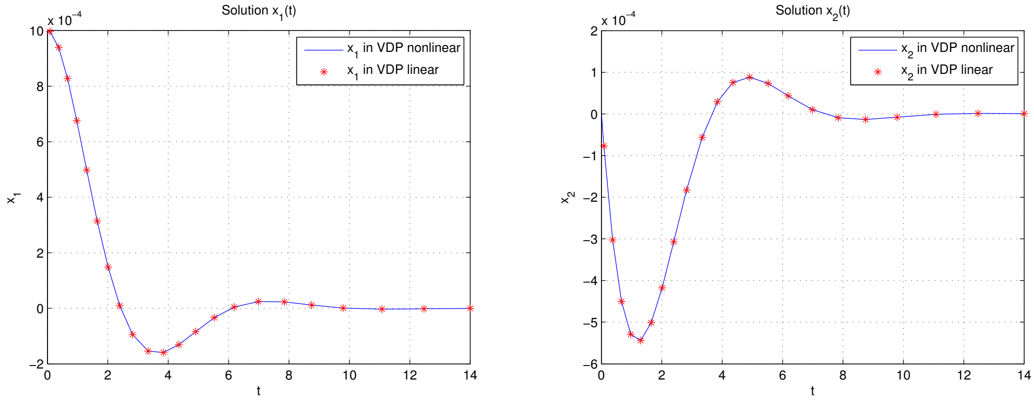


Figure 1.1: *Solutions of the linearized and nonlinear VDP equation*

Modify the initial conditions to $x_1(0) = 0.5$ or $x_1(0) = 1$ and comment the results.

SE 1.2 Magnetic levitation (MagLev), trains are nowadays a promising solution for transportation. They get propulsion force from linear motors and use electromagnets for the suspension system. Two main types of levitation technologies ([22]) will be discussed in this problem:

(I) *electromagnetic suspension* (EMS), that uses magnetic *attractive* force to levitate (Figure 1.2),

(II) *electrodynamic suspension* (EDS), that uses magnetic *repulsive* force for levitation (Figure 1.3).

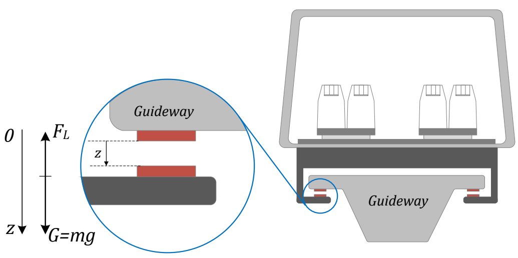


Figure 1.2: *Electromagnetic suspension (magnetic attraction)*

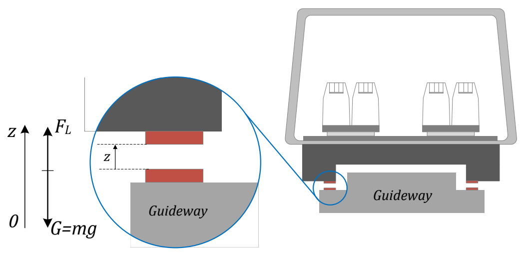


Figure 1.3: *Electrodynamic suspension (magnetic repulsion)*

The levitation force F_L depends on the current $i(t)$ in the levitation coils and the air gap $z(t)$ and may be approximated by, [9]:

$$F_L(t) = k \cdot \frac{i^2(t)}{z^2(t)}$$

where k is a constant. This force is opposed by the gravitational force $G = mg$, where m is the mass of the train and g - the acceleration of gravity.

At equilibrium, the train levitates on an air gap of 1 cm.

Consider the following constants for the model:

- Operating air gap $z_0 = 10^{-2}\ m$
- Mass of the train $m = 10^4\ kg$
- Levitatation force constant $k = 10^{-3}\ Nm^2/A^2$
- Acceleration of gravity $g = 10\ m/s^2$

(a) Determine the nonlinear models of the systems for the levitation system, i.e. the nonlinear relationship between the input current $i(t)$ and the air gap $z(t)$ in both cases.

(b) Obtain a linear approximation of the models around the equilibrium condition, in both cases.

Solution:

(a) In both cases, the dynamical levitation system has the input $i(t)$ - the current through the levitation coils and the output $z(t)$ - the air gap between the train and the guideway (see Figure 1.4).

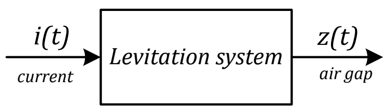


Figure 1.4: *The levitation system: input - current, output - air gap*

A model for this system, or a relationship between $i(t)$ and $z(t)$, can be obtained from the differential equations describing the vertical movement of the train as:

$$mass \times acceleration = \sum forces$$

where *acceleration* refers to the acceleration of the vertical movement obtained as the second derivative of $z(t)$. Considering the orientation of forces and the coordinate system for each case as presented in Figures 1.2 and 1.3, we obtain the following nonlinear models:

(I) EMS:

$$m\ddot{z}(t) = mg - k\frac{i^2(t)}{z^2(t)} \tag{1.6}$$

(II) EDS:

$$m\ddot{z}(t) = k\frac{i^2(t)}{z^2(t)} - mg \tag{1.7}$$

(b) In order to obtain a linear approximation of the models, we have to obtain first the equilibrium conditions. If the train is at equilibrium for the nominal levitation distance $z_0 = 10^{-2}m$, the acceleration is zero $\ddot{z}_0 = 0$. The equilibrium current i_0 can be determined by replacing z_0 and $\ddot{z}_0$ into (1.6) or (1.7):

$$mg - k\frac{i_0^2}{z_0^2} = 0$$

and we obtain:

$$i_0 = z_0\sqrt{\frac{mg}{k}} \tag{1.8}$$

We will determine a linear approximation of (1.6) and (1.7) around the operating point $(z_0,\ \ddot{z}_0,\ i_0)$ using Taylor series.

(I) EMS. Write the truncated Taylor series for the nonlinear function:

$$f_1(z, \ddot{z}, i) = m\ddot{z}(t) - mg + k\frac{i^2(t)}{z^2(t)} = 0$$

as:

$$\begin{aligned} f_1(z, \ddot{z}, i) &= f_1(z_0, \ddot{z}_0, i_0) + \frac{\partial f_1}{\partial z}|_{(z_0,\ddot{z}_0,i_0)} \cdot (z(t) - z_0) + \frac{\partial f_1}{\partial \ddot{z}}|_{(z_0,\ddot{z}_0,i_0)} \cdot (\ddot{z}(t) - z_0) \\ &+ \frac{\partial f_1}{\partial i}|_{(z_0,\ddot{z}_0,i_0)} \cdot (i(t) - i_0) = 0 \end{aligned}$$

We compute now the partial derivatives of f_1 with respect to z, $\ddot{z}$ and i, define a set of variables $\Delta z(t) = z(t) - z_0,\ \Delta i(t) = i(t) - i_0,\ \Delta\ddot{z}(t) = \ddot{z}(t) - \ddot{z}_0$ and obtain:

$$0 - \frac{2ki^2}{z^3}|_{(z_0,\ddot{z}_0,i_0)}\Delta z(t) + m\Delta\ddot{z}(t) + \frac{2ki}{z^2}|_{(z_0,\ddot{z}_0,i_0)}\Delta i(t) = 0$$

or

$$-\frac{2ki_0^2}{z_0^3}\Delta z(t) + m\Delta\ddot{z}(t) + \frac{2ki_0}{z_0^2}\Delta i(t) = 0$$

By replacing i_0 from (1.8) and rearranging the terms we obtain:

$$m\Delta\ddot{z}(t) - \frac{2mg}{z_0}\Delta z(t) + \frac{2\sqrt{mgk}}{z_0}\Delta i(t) = 0 \tag{1.9}$$

(II) EDS. Proceeding in a similar manner for the second case the linearized model is:

$$m\Delta\ddot{z}(t) + \frac{2mg}{z_0}\Delta z(t) - \frac{2\sqrt{mgk}}{z_0}\Delta i(t) = 0 \tag{1.10}$$

SE 1.3 Epidemiological models are used for predicting the evolution of a disease for a specific population dynamics. An example of a simple model used for this purpose is the following, [18]:

$$\begin{cases} S(k+1) &= \mu + S(k)e^{-\beta I(k)} \\ E(k+1) &= S(k) - S(k)e^{-\beta I(k)} \\ I(k+1) &= E(k). \end{cases} \tag{1.11}$$

where S represents the population of individuals susceptible to an infection (in number of individuals), E - the number of individuals exposed to the infection, and I - the number of individuals infected. The discrete time k is the index of the current week of the year. The exponential term is the probability of not contracting the infection. The parameter μ denotes the weekly number of births per person.

Consider the parameter values: $\mu = 0.01$ and $\beta = 0.1$.

(a) Determine the equilibrium values.

(b) Simulate the system's unforced response for initial conditions $S(0) = 50$, $E(0) = 10$, $I(0) = 0$.

Solution:

(a) The equilibrium values are found by setting $S(k+1) = S(k) = S_e$, $E(k+1) = E(k) = E_e$ and $I(k+1) = I(k) = I_e$ and replacing into (1.11):

$$\begin{cases} S_e &= \mu + S_e e^{-\beta I_e} \\ E_e &= S_e - S_e e^{-\beta I_e} \\ I_e &= E_e. \end{cases} \tag{1.12}$$

After some calculations we arrive at:

$$S_e = \frac{\mu}{1 - e^{-\beta\mu}}, \quad E_e = \mu, \quad I_e = \mu \tag{1.13}$$

(b) In order to obtain the systems unforced response to the given initial conditions, we must solve the difference equations iteratively in MATLAB using a simple *for* loop. The script is presented in Listing 1.4 and the results are shown in Figure 1.5.

Listing 1.4: *epidemiological_model.m*

```
close all
clear all
clc
% simulation of the discrete epidemiological model

mu = 0.01;  % constant mu
beta = 0.1; % constant beta

S0=50; E0=10; I0=0;     % initial conditions for the variables
number_of_weeks = 15;   % final discrete time
time = 1:number_of_weeks;   % discrete time vector required for plotting the solutions

% first elements in solution vectors are the initial conditions
S(1)=S0;
E(1)=E0;
I(1)=I0;

% iterate to compute the next solutions
for k = 1:number_of_weeks-1
    S(k+1) = mu+S(k)*exp(-beta*I(k));
    E(k+1) = S(k)-S(k)*exp(-beta*I(k));
    I(k+1) = E(k);
end;
%plot the solution
plot(time, S, time, E, time, I, 'LineWidth', 2), grid on
xlabel('time (weeks)'), ylabel('population (number of individuals)')
legend('S (susceptible)', 'E (exposed)', 'I (infected)')
```

Interpret the results: are we dealing with an epidemic or is the disease constrained?

1.2 Proposed exercises

PE 1.1 Consider a mechanical pendulum as shown in Figure 1.6.

The pendulum consists of a small-diameter ball with mass m suspended on a massless rigid rod of length l. The rod can rotate only in two dimensions, thus the ball will trace an arc of a circle in the vertical plane. The position of the ball is determined only by the pendulum angle from vertical, denoted as x in Figure 1.6.

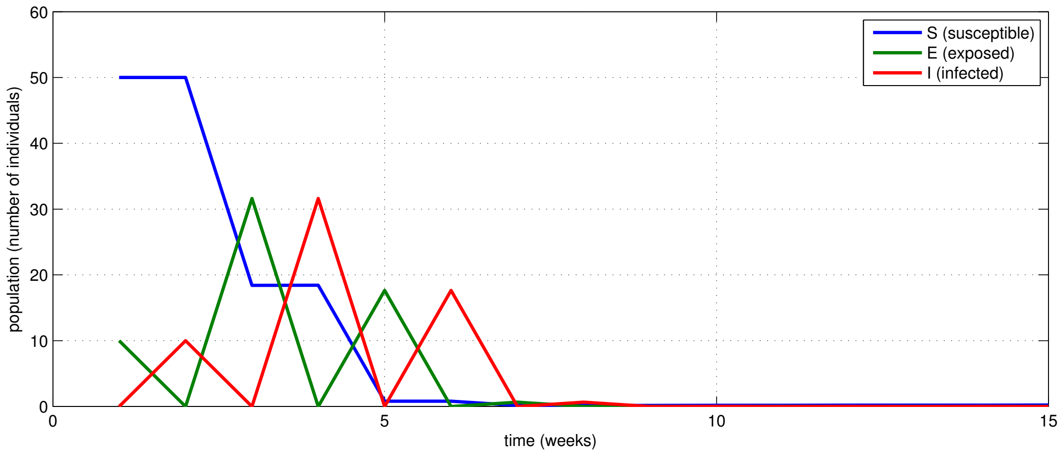


Figure 1.5: *Disease evolution*

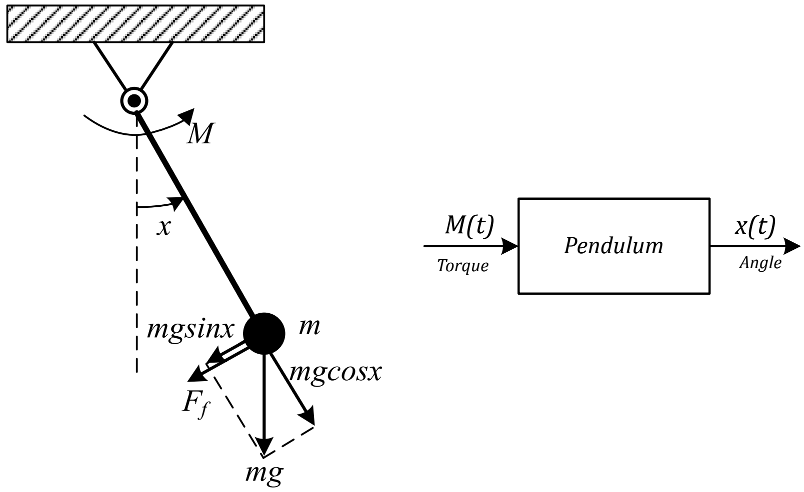


Figure 1.6: *A simple pendulum*

The forces acting on the ball are: the gravity mg, the viscous friction F_f, and the moment of external forces applied to the axis of rotation $M(t)$.

A differential equation describing the motion of the ball can be derived from Newton's second law for rotational systems:

$$moment\_of\_inertia \times angular\_acceleration = \sum moments$$

For the simple pendulum, the moment of inertia is $I = ml^2$, the angular acceleration is the second derivative of the angle position $\ddot{x}(t)$, the gravity component in the direction of the movement is $G_x(t) = mg \sin x(t)$ and the viscous friction is proportional to the angular velocity $F_f(t) = b\dot{x}(t)$. Then we obtain:

$$ml^2\ddot{x}(t) = M(t) - mgl \sin x(t) - bl\dot{x}(t) \tag{1.14}$$

where

- m is the mass of the ball, $m = 0.5$ kg
- l is the length of the rod, $l = 1$ m
- g is the acceleration of gravity, $g = 9.8\ m/s^2$
- b is the viscous friction coefficient, $b = 0.5$

Consider the pendulum as a system with the input $M(t)$ and the output $x(t)$ and solve the following problems:

(a) Obtain a linear approximation of the equation (1.14) around $x_0 = 0$, $\dot{x}_0 = 0$, $\ddot{x}_0 = 0$ and for $M_0 = 0$.

(b) Use the same procedure as in exercise SE 1.1 to obtain and plot the numerical solution for the nonlinear equation and the linear approximation, for $M(t) = 0$ and the initial conditions:

(a) $x(0) = \pi/4$, $\dot{x}(0) = 0$

(b) $x(0) = \pi/2$, $\dot{x}(0) = 0$

Analyze and comment the results.

(c) Obtain and plot the numerical solution for the nonlinear and linear equations, when the initial conditions are zero and $M(t) = 1$.

PE 1.2 Consider a liquid-level tank system [24], where the input is the flow rate $q(t)$ and the output - the liquid level $h(t)$, as shown in Figure 1.7. The differential equation describing

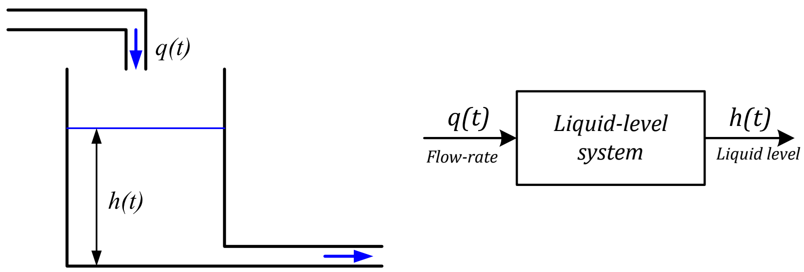


Figure 1.7: *Liquid level tank system*

the systems dynamics in terms of change in liquid level (derived from mass conservation) is:

$$\dot{h}(t) = \frac{1}{C}q(t) - \frac{k}{C}\sqrt{h(t)}. \tag{1.15}$$

where C (the capacitance of the tank) and k (a flow constant) are constants.

Find a linear approximation of (1.15) around the equilibrium point (q_0, h_0) , with $q_0 = 10ml/sec$.

Hint. The equilibrium point can be obtained for a constant value q_0 by solving

$$0 = \frac{1}{C}q_0 - \frac{k}{C}\sqrt{h_0}. \tag{1.16}$$

in respect with h_0.

PE 1.3 Consider a thermal system composed of an isolated vessel that contains a specific liquid and a heater element as in Figure 1.8 ([7]). The presence of a mixer ensures a uniform temperature of the liquid. The system is also referred in the literature as a batch process. In the design of a batch process it is important to know how long it takes for the liquid to reach a desired temperature.

As input signal, we have the energy rate of the heating element $Q_i(t)$. As outputs we have the temperature of the heater $T_h(t)$ and the temperature of the liquid $T_l(t)$. We will consider the following parameters:

- heater capacitance : $C_h = 20 \cdot 10^3 J/K$

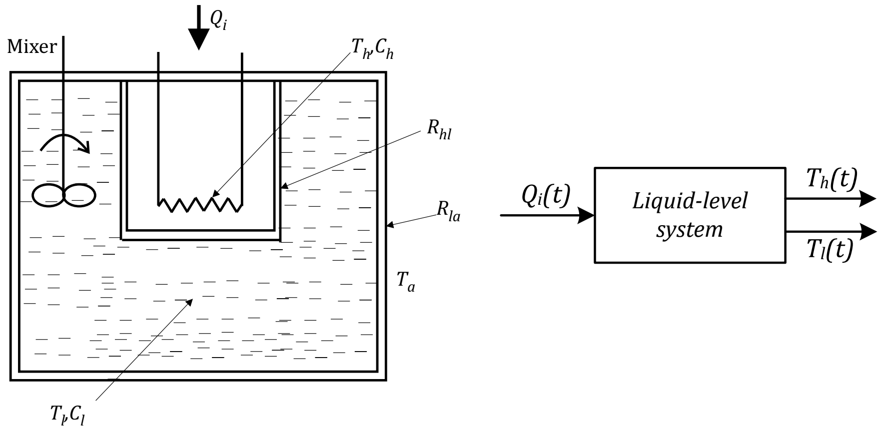


Figure 1.8: *Thermal systems (vessel with heater). Adapted from [7]*

- liquid capacitance: $C_l = 1 \cdot 10^6 J/K$
- heater-liquid resistance: $R_{hl} = 1 \cdot 10^{-3} sK/J$
- liquid-ambient resistance: $R_{la} = 5 \cdot 10^{-3} sK/J$
- ambient temperature $T_a = 300K$
- desired temperature of the liquid: $T_{ld} = 365K$

The energy stored in the system is characterized by the following equations:

$$\begin{cases} \dot{T}_h(t) = \dfrac{1}{C_h}\left[Q_i(t) - \dfrac{T_h(t) - T_l(t)}{R_{hl}}\right] \\ \dot{T}_l(t) = \dfrac{1}{C_l}\left[\dfrac{T_h(t) - T_l(t)}{R_{hl}} - \dfrac{T_l(t) - T_a}{R_{la}}\right] \end{cases} \tag{1.17}$$

with the initial conditions $T_h(0) = T_l(0) = T_a$.

(a) Explain why the system is not linear and obtain a linear approximation of the model.

(b) For a step variation of Q_i with amplitude $1.3 \cdot 10^4 W$ $(t > 0)$ determine the response of the system through simulation and identify on the plots the time needed to reach the desired temperature.

(c) Simulate the response of the linearized system and compare it with that of the original nonlinear system.

PE 1.4 A thermistor is a resistor with a temperature (T) dependent resistance (R). The dependence is usually expressed as an exponential function:

$$R(T) = R_0 e^{-bT}. \tag{1.18}$$

However, most often the thermistor is used in the context of a small range of temperature variation, and thus a linear approximation is used in practice.

(a) Determine the linear model for a temperature operating point of 20 degrees Celsius, and the parameters: $R_0 = 10\ \Omega$ and $b = 0.2$.

(b) Analyze graphically the error of approximation on the temperature domain (10, 30) degrees Celsius.

PE 1.5 Consider a simplified model for tumor growth [10]:

$$\begin{cases} \dot{x}_1(t) = -\lambda_1 x_1(t) \ln\left(\dfrac{x_1(t)}{x_2(t)}\right) \\ \dot{x}_2(t) = b x_1(t) - d x_1(t)^{2/3} x_2(t) - e x_2(t) x_3(t) \\ \dot{x}_3(t) = -\lambda_3 x_3(t) + u(t) \end{cases} \tag{1.19}$$

where $x_1(t)$ is the tumor volume, $x_2(t)$ is the endothelial cell volume, $x_3(t)$ is the administrated inhibitor concentration, $u(t)$ is the inhibitor administration rate. The parameters are, [10]:

- tumor growth rate $\lambda_1 = 0.192/\ln 10$ $(1/day)$
- vascular birth rate $b = 5.85$ $(1/day)$
- vascular death rate $d = 0.00873$ $(1/(mm^{2/3} \cdot day))$
- drug killing parameter $e = 0.66$ $(kg/(mg \cdot day))$
- drug clearance $\lambda_3 = 1.7$ $(1/day)$

(a) Simulate the system response (evolutions of x_1 and x_2) to initial conditions $x_1(0) = 100\ mm^3$, $x_2(0) = 100\ mm^3$, $x_3(0) = 0\ mg/kg/day$, and input $u = 0$. Consider a time period of 120 days. How much does the tumor volume increases? Does it reach a steady state?

(b) Determine a linear approximation of the model for the working (operating) point $x_{10} = 100\ mm^3$, $x_{20} = 100\ mm^3$ and $x_{30} = 0\ mg/kg/day$ and $u = 0$. Compare and discuss the simulation results of the linear and nonlinear models.

PE 1.6 Consider the electrical circuits from Figure 1.9. For each electrical system, a mathematical model in the form of differential equations can be derived using the Kirchhoff's laws.

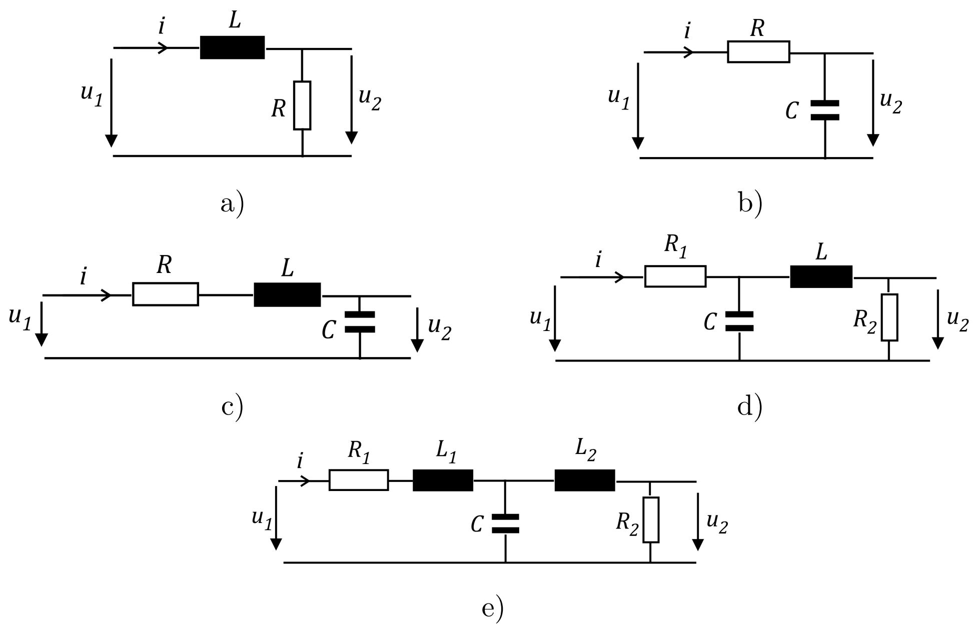


Figure 1.9: *Passive electrical circuits*

(a) Determine the mathematical model for the circuits in Figure 1.9 a) and b) in the general form:

$$\dot{x}(t) = a \cdot x(t) + b \cdot u(t)$$

where the state variable $x(t)$ is the inductor current i_L, for figure a) and the capacitor voltage u_C for figure b). The input $u(t)$ is the voltage u_1.

(b) Determine the mathematical model for the circuits in Figure 1.9 c) and d) in the general form:

$$\dot{x}_1(t) = a_{11} \cdot x_1(t) + a_{12} \cdot x_2(t) + b_1 \cdot u(t)$$
$$\dot{x}_2(t) = a_{21} \cdot x_1(t) + a_{22} \cdot x_2(t) + b_2 \cdot u(t)$$

where the state variables $x_1(t)$, $x_2(t)$ are the inductor current i_L and the capacitor voltage u_C, respectively. The input $u(t)$ is the voltage u_1.

(c) Determine the mathematical model for the circuit in Figure 1.9 e) in the general form:

$$\dot{x}_1(t) = a_{11} \cdot x_1(t) + a_{12} \cdot x_2(t) + a_{13} \cdot x_3(t) + b_1 \cdot u(t)$$
$$\dot{x}_2(t) = a_{21} \cdot x_1(t) + a_{22} \cdot x_2(t) + a_{23} \cdot x_3(t) + b_2 \cdot u(t)$$
$$\dot{x}_3(t) = a_{31} \cdot x_1(t) + a_{32} \cdot x_2(t) + a_{33} \cdot x_3(t) + b_3 \cdot u(t)$$

where the state variables $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the inductor currents i_{L1}, i_{L2} and the capacitor voltage u_C, respectively. The input $u(t)$ is the voltage u_1.

(d) Simulate the electrical system from Figure 1.9 d), for the parameter values $R_1 = R_2 = 1\,\Omega$, $C = 0.01\,F$, $L = 0.02\,H$, on a time interval $t \in [0,\ 0.2]$ seconds, in the following scenarios:

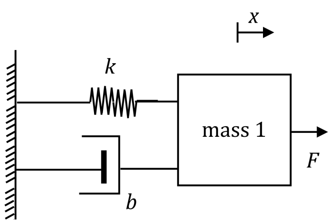
(i) Input $u = 0$ and initial conditions: $x_1(0) = 0.2$, $x_2(0) = 0.3$.

(ii) Input $u = 2$ and initial conditions: $x_1(0) = x_2(0) = 0$.

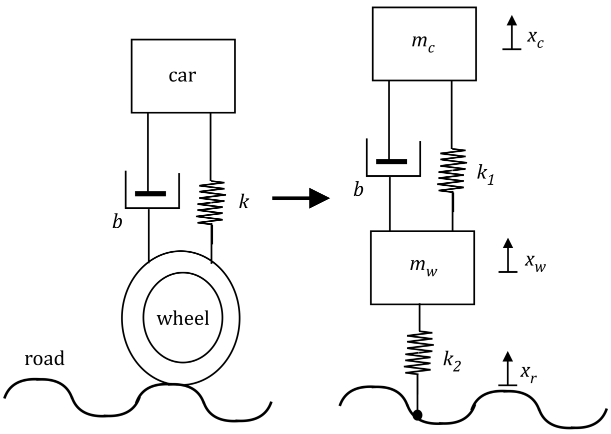
(iii) Input $u = \sin(100\,t)$ and initial conditions: $x_1(0) = x_2(0) = 0$.

Plot and interpret the results for each scenario.

PE 1.7 Consider the spring-mass damper and the car suspension mechanical systems from Figure 1.10 a) and b).



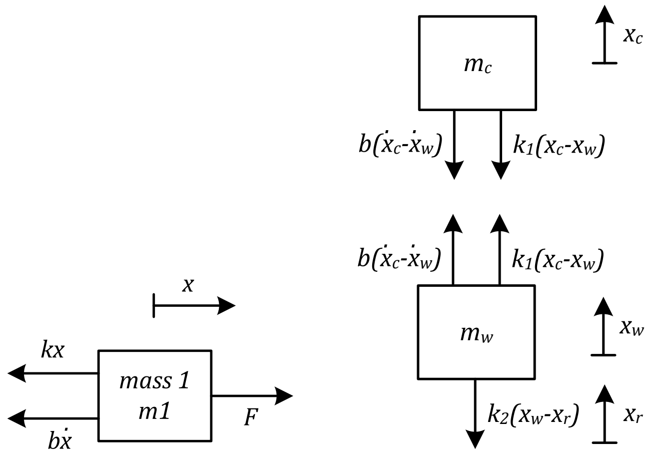
a) *Spring-mass-damper*



b) *Car suspension*

Figure 1.10: *Mechanical systems*

(a) Determine the mathematical model of the spring mass damper system based on Newton's laws with the force $F(t)$ as the input signal and the displacement of the mass $x(t)$ as the output. The variables and constants in Figure 1.10 a) are:



a) Spring-mass-damper b) Car suspension

Figure 1.11: *Free body diagrams of the mechanical systems*

- $F = F(t)$ - the external input force
- $x = x(t)$ - displacement of the mass
- k - spring constant
- b - damping coefficient

(b) For the spring-mass-damper system, choose the states as $x_1(t) = x(t)$ (displacement of the mass), $x_2(t) = \dot{x}(t)$ (velocity of the mass) and the input $u(t) = F(t)$ and determine the state equations of the form:

$$\dot{x}_1(t) = a_{11} \cdot x_1(t) + a_{12} \cdot x_2(t) + b_1 \cdot u(t)$$
$$\dot{x}_2(t) = a_{21} \cdot x_1(t) + a_{22} \cdot x_2(t) + b_2 \cdot u(t)$$

(c) Determine a mathematical model of the car suspension system based on Newton's laws (ignoring gravity). The variables and constants in Figure 1.10 b) are:

- $x_c = x_c(t)$ - the displacement of the car
- $x_w = x_w(t)$ - the displacement of the wheel
- $x_r = x_r(t)$ - a function modeling the shape of the road
- b - suspension damping coefficient
- m_c - the mass of the car
- m_w - the mass of the wheel and tire (notice that they are modeled as a body of mass m_w and a spring with the constant k_2)
- k_1 - spring constant of suspension
- k_2 - spring constant of wheel and tire

Consider the input $x_r(t)$ and the outputs $x_c(t)$ and $x_w(t)$.

(d) For the car suspension system, define the states: $x_1(t) = x_c(t)$, $x_2(t) = \dot{x}_c(t)$, $x_3(t) = x_w(t)$, $x_4(t) = \dot{x}_w(t)$, and the input $u(t) = x_r(t)$ and determine the state equations of the form:

$$\begin{aligned}
\dot{x}_1(t) &= a_{11} \cdot x_1(t) + a_{12} \cdot x_2(t) + a_{13} \cdot x_3(t) + a_{14} \cdot x_4(t) + b_1 \cdot u(t) \\
\dot{x}_2(t) &= a_{21} \cdot x_1(t) + a_{22} \cdot x_2(t) + a_{23} \cdot x_3(t) + a_{24} \cdot x_4(t) + b_2 \cdot u(t) \\
\dot{x}_3(t) &= a_{31} \cdot x_1(t) + a_{32} \cdot x_2(t) + a_{33} \cdot x_3(t) + a_{34} \cdot x_4(t) + b_3 \cdot u(t) \\
\dot{x}_4(t) &= a_{41} \cdot x_1(t) + a_{42} \cdot x_2(t) + a_{43} \cdot x_3(t) + a_{44} \cdot x_4(t) + b_4 \cdot u(t)
\end{aligned}$$

(e) Simulate the car suspension system for the parameter values; $m_c = 10kg$, $m_w = 0.5kg$, $k_1 = 3N/m$, $k_2 = 0.5N/m$, $b = 1Nm/s^{-1}$, on the time interval $t \in [0 \quad 50]$ seconds, zero initial conditions, and input $u(t) = sin(t)$. Plot the time evolution of $x_1(t)$ and interpret the results. *Notice that the gravity is ignored in the model, so the positions are actually deviations from resting equilibrium positions.*

Hint. Use the free body diagrams presented in Figure 1.11 to determine the mathematical models for these systems.

PE 1.8 Consider a server as a software system used for administrating users' e-mails, documents and notes, among other tasks. The central processing unit (CPU) can be overloaded if on a certain time interval there are to many service requests. In order to prevent this, the system administrator limits the maximum number of users ($u = MaxUsers$). We can regard this entire process as a system with the input $MaxUsers$ and output the total number of requests being served ($y = RIS$). A mathematical model that captures the dynamics behavior of the system can be constructed through system identification techniques.

Consider the model determined in [15], for an operating point (equilibrium point) $y_e = 165$ and $u_e = 135$, considering averaged values over a sampling period of 60 seconds:

$$y_l(k+2) - 1.07y_l(k+1) + 0.28y_l(k) = 0.08u_l(k+1) - 0.052u_l(k), \tag{1.20}$$

where $y_l(k) = y(k) - y_e$ and $u_l(k) = u(k) - u_e$.

(a) Simulate in MATLAB the response of the system to discrete impulse and step inputs for $k = \overline{1, 20}$.

(b) Interpret the result for each user request-type scenario.

Hint. Consider, for example, the following values for the discrete impulse input u_l*=(0, 0, 0, 0, 1, 0, ..., 0) and the step input:* u_l*=(0, 0, 0, 0, 1, 1, ..., 1).*

2 Input-Output models

Topics: Laplace transform, Z-transform, transfer functions, poles and zeros, block diagrams.

2.1 Solved exercises

SE 2.1 Consider the electric circuit from Figure 2.1 with the input voltage $u_1(t)$ and output voltage $u_2(t)$.

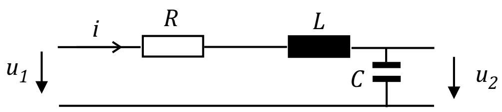


Figure 2.1: *RLC electric circuit*

(a) Write the equations describing the system based on Kirchhoff's circuit laws and calculate the transfer function for zero initial conditions ($i(0) = 0$ and $u_2(0) = 0$).

(b) Determine the poles and zeros of the system.

Solution:

(a) First notice that the same current passes through the resistor, inductor and capacitor. The output voltage u_2 is actually the voltage drop on the capacitor. So Kirchhoff current law (for circuit nodes) is not needed in this case. Kirchhoff voltage law for the loop of our circuit can be written as:

$$u_1(t) = R \cdot i(t) + u_L(t) + u_2(t). \tag{2.1}$$

We also know the current-voltage relation for an inductor:

$$u_L(t) = L \cdot \frac{di(t)}{dt},$$

and a capacitor

$$i(t) = C \cdot \frac{du_2(t)}{dt}.$$

By replacing $u_L(t)$ and $i(t)$ into (2.1) we obtain:

$$u_1(t) = R \cdot C \cdot \frac{du_2(t)}{dt} + L \cdot C \cdot \frac{d^2u_2(t)}{dt^2} + u_2(t).$$

Applying the Laplace transform with zero initial conditions leads to:

$$U_1(s) = RC \cdot s \cdot U_2(s) + LC \cdot s^2 \cdot U_2(s) + U_2(s).$$

The transfer function is:

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{1}{LCs^2 + RCs + 1}. \tag{2.2}$$

(b) The system described by the transfer function (2.2) has no zeros and the poles are the roots of the denominator polynomial:

$$s_{1,2} = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}.$$

SE 2.2 Many motion control systems use a direct current (DC) motor, which converts electric power into mechanical power. A schematic drawing of a DC motor is illustrated in Figure 2.2.

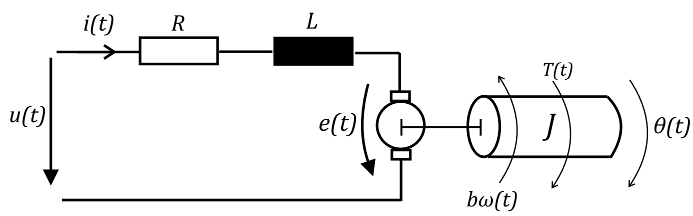


Figure 2.2: *Schematic drawing of a DC motor*

The input signal of the system is the voltage source $u(t)$, and the output signal is the angular position of the shaft $\theta(t)$. The other notations in Figure 2.2 are:

- $e(t) = k_e\omega(t)$ - the back electromotive (emf) voltage,
- k_e - the electromotive force constant
- $\omega(t)$ - the angular speed,
- $T(t) = k \cdot i(t)$ - the applied torque,
- k - the motor torque constant
- $i(t)$ - the current
- J - the moment of inertia.

Determine the transfer function of the system for zero initial conditions ($i(0) = 0$, $\theta(0) = 0$, $\omega(0) = 0$).

Solution:

The equations for the electrical part are given by Kirchhoff voltage law:

$$u(t) = R \cdot i(t) + u_L(t) + e(t), \tag{2.3}$$

where the inductor voltage can be expressed as:

$$u_L(t) = L \cdot \frac{di(t)}{dt},$$

and the back emf voltage is:

$$e(t) = k_e\omega(t) = k_e\frac{d\theta(t)}{dt}.$$

By replacing $e(t)$ and $u_L(t)$ in equation (2.3) we finally obtain:

$$u(t) = R \cdot i(t) + L \cdot \frac{di(t)}{dt} + k_e\frac{d\theta(t)}{dt}. \tag{2.4}$$

The equation for the mechanical part is provided by Netwon's law

$$J\frac{d^2\theta(t)}{dt^2} = -b\frac{d\theta(t)}{dt} + T(t), \tag{2.5}$$

where the torque T is proportional with the current:

$$T(t) = k \cdot i(t).$$

So the mathematical model of our system consists of the following differential equations:

$$\begin{cases} u(t) = R \cdot i(t) + L \cdot \dfrac{di(t)}{dt} + k_e\dfrac{d\theta(t)}{dt} \\ J\dfrac{d^2\theta(t)}{dt^2} = -b\dfrac{d\theta(t)}{dt} + k \cdot i(t), \end{cases} \tag{2.6}$$

We apply the Laplace transform for both equations (2.6):

$$\begin{cases} U(s) = RI(s) + LsI(s) + k_e s\theta(s) \\ Js^2\theta(s) = -bs\theta(s) + kI(s). \end{cases} \tag{2.7}$$

By extracting $I(s)$ from the second equation and replacing into the first one we obtain:

$$\theta(s) = \frac{k}{LJs^3 + (RJ + Lb)s^2 + (kk_e)s}U(s).$$

Thus, the transfer function of the DC motor from the input voltage to the output angular position is:

$$H(s) = \frac{\theta(s)}{U(s)} = \frac{k}{LJs^3 + (RJ + Lb)s^2 + (kk_e)s}. \tag{2.8}$$

SE 2.3 The block diagram of a closed-loop accelerometer is given in Figure 2.3, [5].

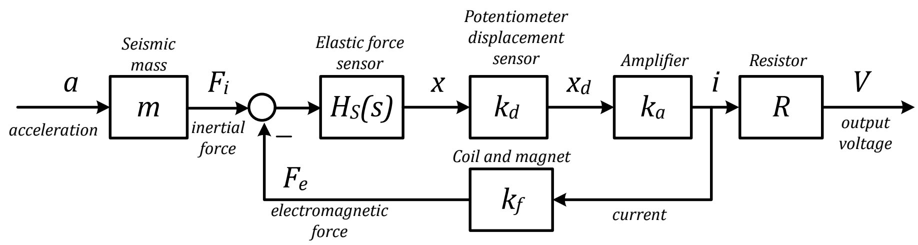


Figure 2.3: *Closed-loop accelerometer*

The transfer function of the elastic force sensor is:

$$H_S(s) = \frac{\frac{1}{k}}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

From the block diagram determine the transfer function $H(s) = \dfrac{V(s)}{a(s)}$ that relates the output voltage V to the input acceleration signal a.

Solution 1: This informational block diagram expresses algebraic relations because all the blocks refer to transfer functions (in Laplace domain). In order to determine the transfer function of the overall scheme, one simply has to write down the algebraic equations that relates different signals of interest from the diagram, be reading in the opposite sense of that indicated by the arrows.

For example, the current i depends on the displacement x through the equation

$$i(s) = k_a \cdot x_d(s) = k_a \cdot k_d \cdot x(s). \tag{2.9}$$

In a similar manner, if we consider the transfer function of the elastic force sensor as $H_s(s)$ and we use the relation for the summing point, then we can write the displacement $x(s)$ as:

$$x(s) = H_s(s) \cdot (F_i(s) - F_e(s)) = H_s(s) \cdot (m \cdot a(s) - k_f \cdot i(s)) \tag{2.10}$$

By replacing $x(s)$ from (2.10) into (2.9):

$$i(s) = k_a \cdot k_d \cdot H_s(s) \cdot (m \cdot a(s) - k_f \cdot i(s))$$

which means the current $i(s)$ can be isolated as:

$$i(s) = \frac{k_a \cdot k_d \cdot H_s(s) \cdot m}{1 + H_s(s) \cdot k_f \cdot k_a \cdot k_d} \cdot a(s)$$

Finally, by adding the output equation

$$V(s) = R \cdot i(s) = R \cdot \frac{k_a \cdot k_d \cdot H_s(s) \cdot m}{1 + H_s(s) \cdot k_f \cdot k_a \cdot k_d} \cdot a(s)$$

the overall transfer function can be calculated as

$$H(s) = \frac{V(s)}{a(s)} = \frac{R \cdot k_a \cdot k_d \cdot H_s(s) \cdot m}{1 + H_s(s) \cdot k_f \cdot k_a \cdot k_d} \tag{2.11}$$

Solution 2: Using the block diagram algebra rules, the overall transfer function from the input $a(s)$ to the output $V(s)$ can be obtained as follows:

- The blocks representing the elastic force sensor, the potentiometer displacement sensor and the amplifier are connected in series. An equivalent transfer function

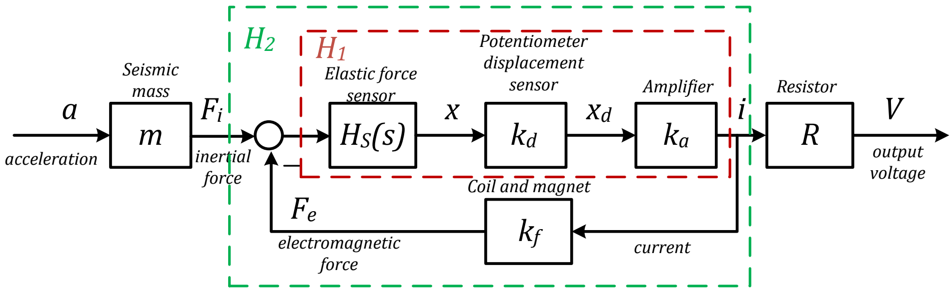


Figure 2.4: *Closed-loop accelerometer*

  is obtained by multiplying together the transfer functions of these blocks (see Figure 2.4) and we obtain:

$$H_1(s) = H_S(s) \cdot k_d \cdot k_a$$

- The new block with the transfer function $H_1(s)$ and the coil and magnet block are in a feedback loop (see Figure 2.4). The equivalent transfer function is then:

$$H_2(s) = \frac{H_1(s)}{1 + H_1 \cdot k_f} = \frac{H_S(s) \cdot k_d \cdot k_a}{1 + H_S(s) \cdot k_d \cdot k_a \cdot k_f}$$

- The seismic mass block, the block with the transfer function $H_2(s)$ and the resistor are in a series connection. The overall transfer function from the input

$a(s)$ to the output $V(s)$ is then:

$$H(s) = m \cdot H_2(s) \cdot R = \frac{m \cdot H_S(s) \cdot k_d \cdot k_a \cdot R}{1 + H_S(s) \cdot k_d \cdot k_a \cdot k_f}, \tag{2.12}$$

the same as (2.11).

+ Replace $H_s(s)$ with the transfer function from the block diagram and determine the poles and zeros of $H(s)$.

2.2 Proposed exercises

PE 2.1 An input $r(t)$ is applied to a system with a transfer function $G(s)$ and the resulting output is $y(t)$. Determine the transfer function $G(s)$ if the input and the output signals are:

(a) $r(t) = t, \quad y(t) = t + e^{-t} - 1, \quad t \geq 0$

(b) $r(t) = \sin t, \quad y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right), \quad t \geq 0$

(c) $r(t) = e^{-t}, \quad y(t) = 1 - e^{-t} + e^{-2t}\cos t, \quad t \geq 0$

PE 2.2 Consider the following linear differential equations as models for some systems with the input $r(t)$ and the output $y(t)$:

$$S1: \frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = r(t)$$
$$S2: \frac{dy(t)}{dt} + 2y(t) = \frac{dr(t)}{dt} + r(t)$$
$$S3: \frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} = 2\frac{dr(t)}{dt} + r(t)$$
$$S4: \frac{d^3y(t)}{dt^3} - \frac{d^2y(t)}{dt^2} - 6\frac{dy(t)}{dt} = 10r(t)$$

For each system:

(a) Determine the transfer function, when all the initial conditions are assumed to be zero.

(b) Determine the poles and the zeros.

PE 2.3 Consider the electric circuit from Figure 2.5 with the input $u_1(t)$ and output $u_2(t)$.

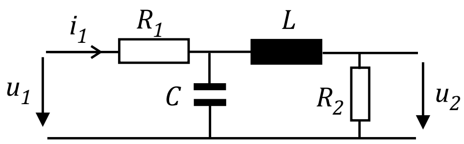


Figure 2.5: *RLC electric circut*

(a) Write the equations of the system based on Kirchhoff's circuit laws and calculate the transfer function for zero initial conditions ($i_L(0) = 0$ and $u_c(0) = 0$).

(b) Determine the poles and zeros of the system.

PE 2.4 Consider the systems given by the block diagrams from Figure 2.6. Determine the overall transfer function for each system from the input $R(s)$ to the output $Y(s)$.

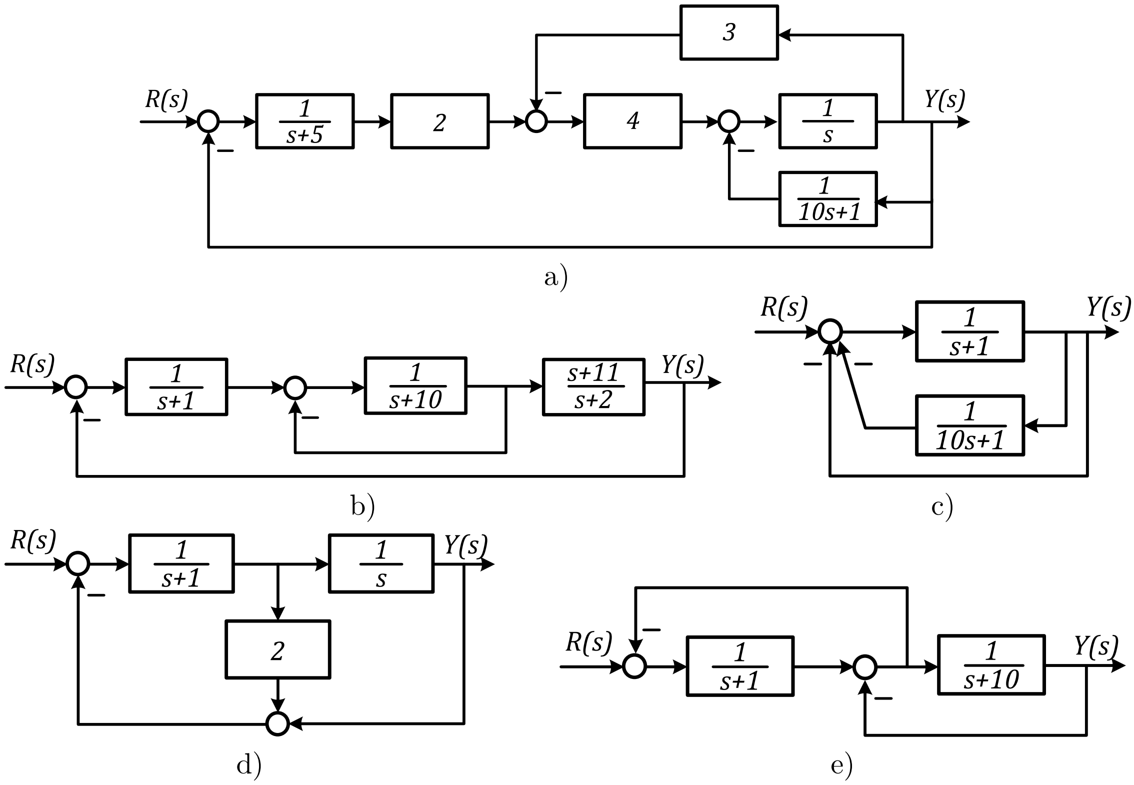


Figure 2.6: *Block diagrams*

PE 2.5 Consider a control system given by the block diagram from Figure 2.7, with the input $W(s)$ and the output $Y(s)$. Determine the equivalent transfer function.

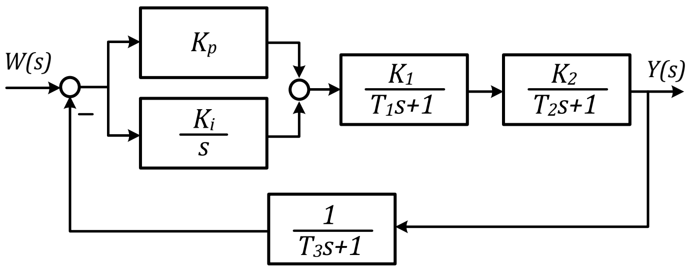


Figure 2.7: *Block diagram of a system*

PE 2.6 A pneumatic actuator of a control system can be regarded as a subsystem with a pneumatic part and a mechanical part [21] (see Figure 2.8 for a diagram representation).

The pneumatic part refers to an air capsule with a flexible membrane and a spring. The force due to the input pressure signal $p_{in}(t)$ pushes on the membrane, which makes the rod move with the position $x(t)$. The shutter at the end of the rod can further influence the flow through a pipe (regarded as a part of the controlled process). For the pneumatic part, the air filling the capsule is described by the first-order equation:

$$T\frac{dp(t)}{dt} + p(t) = p_{in}(t),$$

where T is the time constant, and $p(t)$ is the air pressure inside the capsule.

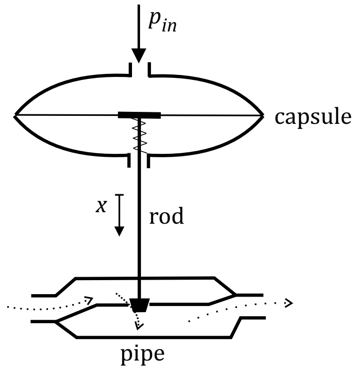


Figure 2.8: *Diagram of a pneumatic actuator (adapted from [21])*

The mechanical part is described by a second-order equation derived from the Newton's second law of motion:

$$m\frac{d^2x(t)}{dt^2} = -b\frac{dx(t)}{dt} - kx(t) + Sp(t),$$

where m is the mass of the moving pieces, b is the coefficient of the viscous friction force, k is the elastic coefficient of the arc, and S is the area of the membrane ($S \cdot p(t)$ equals the input force).

(a) Determine the transfer function that relates the input pressure p_{in} to the rod position x, $H(s) = \dfrac{X(s)}{P_{in}(s)}$.

(b) Calculate the poles and zeros of the system.

PE 2.7 Consider the mathematical model for the motion of the read/write head of a (hard) disk drive ([4], [12]), where the equations are derived from the equivalent mechanical diagram from Figure 2.9:

$$\begin{cases} J_1\dfrac{d^2\theta_1(t)}{dt^2} + b\left(\dfrac{d\theta_1(t)}{dt} - \dfrac{d\theta_2(t)}{dt}\right) + k(\theta_1(t) - \theta_2(t)) = T(t), \\ J_2\dfrac{d^2\theta_2(t)}{dt^2} + b\left(\dfrac{d\theta_2(t)}{dt} - \dfrac{d\theta_1(t)}{dt}\right) + k(\theta_2(t) - \theta_1(t)) = T_d(t), \end{cases} \tag{2.13}$$

where J_1 and J_2 are moments of inertia, b is the friction constant, k is the spring constant, $T(t)$ is the applied torque, $T_d(t)$ is the disturbance torque, $\theta_1(t)$ and $\theta_2(t)$ represent the angular positions of the two masses. Notice that this is a MIMO (Multiple-Input Multiple-Output) system, with two inputs ($T(t)$ and $T_d(t)$) and two outputs ($\theta_1(t)$ and $\theta_2(t)$).

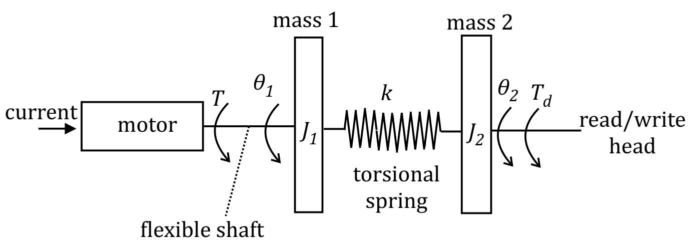


Figure 2.9: *Mechanical diagram for a read/write head of a disk drive*

(a) Determine the two transfer functions that relate the applied input torque $T(t)$ to the angles $\theta_1(t)$ and $\theta_2(t)$.

(b) Determine the two transfer functions that relate the disturbance input torque $T_d(t)$ to the angles $\theta_1(t)$ and $\theta_2(t)$.

Hint. For a linear two-input two-output MIMO system the equations are:

$$\theta_1(s) = H_{11}(s)T(s) + H_{12}(s)T_d(s)$$
$$\theta_2(s) = H_{21}(s)T(s) + H_{22}(s)T_d(s)$$

where $T(s)$, $T_d(s)$ are the input signals and $\theta_1(s)$, $\theta_2(s)$ are the output signals.

PE 2.8 Consider the thermal system from Figure 2.10.

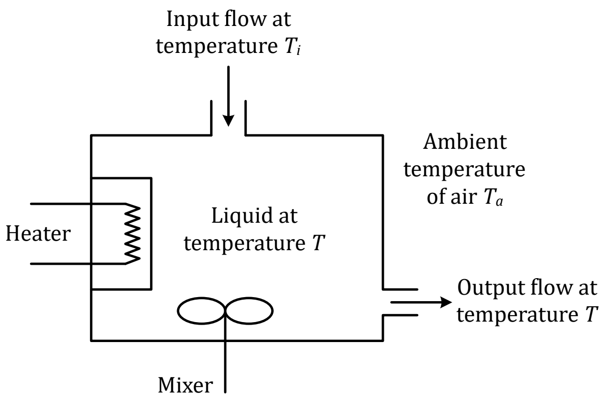


Figure 2.10: *Thermal system (adapted from [25])*

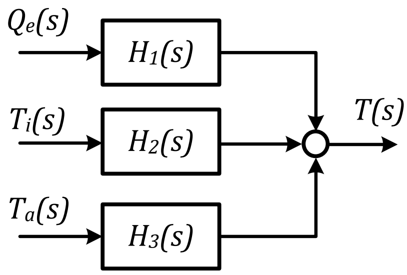


Figure 2.11: *Multiple input single output thermal system*

Based on energy conservation, we can write, [25]:

$$\underbrace{Q_e(t) + Q_i(t)}_{\text{heat in}} = \underbrace{Q_l(t) + Q_o(t) + Q_s(t)}_{\text{heat out}}, \tag{2.14}$$

where

- Q_e is the heat flow from the heater,
- Q_i - the heat from the liquid entering the tank,
- Q_l - the heat flow into liquid,
- Q_o - the heat flow that leaves the thank through the liquid
- Q_s - the output heat flow through the tank.

We can further express the output heat flows as:

$$Q_i(t) = V \cdot H \cdot T_i(t), \tag{2.15}$$
$$Q_l(t) = C\frac{dT(t)}{dt}, \tag{2.16}$$
$$Q_o(t) = V \cdot H \cdot T(t), \tag{2.17}$$
$$Q_s(t) = \frac{T(t) - T_a(t)}{R}, \tag{2.18}$$

with the parameters:

- C - thermal capacity,
- V - flow in and out of the tank (assumed equal),
- H - specific heat of the liquid,

- R - thermal resistance.

Replacing (2.15), (2.16), (2.17) and (2.18) into (2.14) we obtain:

$$Q_e(t) + VHT_i(t) = C\frac{dT(t)}{dt} + VHT(t) + \frac{T(t) - T_a(t)}{R}. \tag{2.19}$$

If we regard the system as having multiple inputs (Q_e, T_i, T_a) and a single output (T), determine the transfer functions $H_1(s)$, $H_2(s)$, $H_3(s)$ from Figure 2.11 of the blocks that connect each input to the output.

PE 2.9 Consider a digital FIR (Finite Impulse Response) filter and a digital IIR (Infinite Impulse Response) filter with the transfer functions $H_1(z)$ and $H_2(z)$, respectively:

$$\text{FIR:} \quad H_1(z) = \frac{Y(z)}{U(z)} = 1 - 2z^{-1} - 0.1z^{-2} + 5z^{-3}$$

$$\text{IIR:} \quad H_2(z) = \frac{Y(z)}{U(z)} = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 2z^{-2}}.$$

For each filter write the corresponding difference equation in time domain in terms of $y(k)$ and $u(k)$, with k as the index of the current sampling period:

(a) in standard form (left hand side contains only terms with the output y, right hand side contains only terms with the input u), used for analysis.

(b) implementation form (the output at current sampling instance $y(k)$ is expressed in terms of previous values of y, u, and current values of u), used in a programming language like C.

PE 2.10 Consider the digital filters represented by the block diagrams from Figure 2.12, with input $u(k)$ and output $y(k)$.

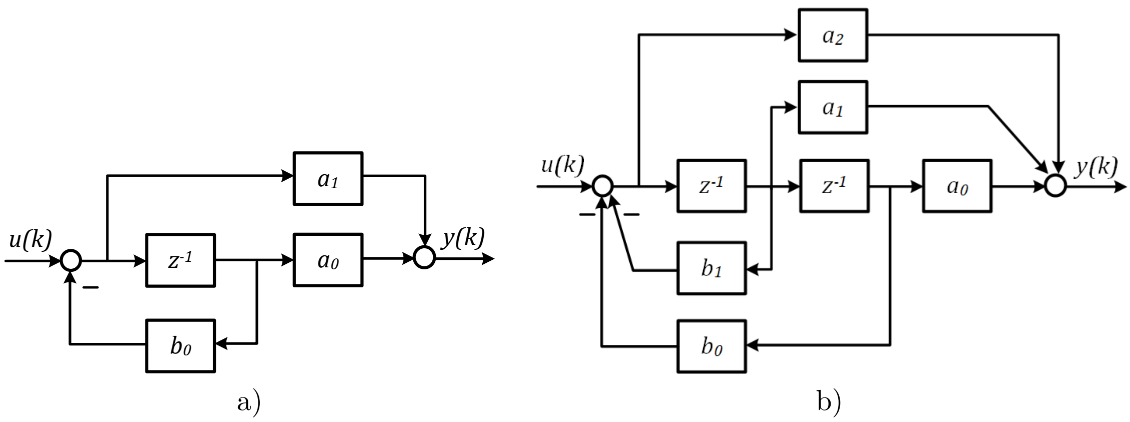


Figure 2.12: *Block diagrams of digital filters*

(a) Determine the difference equation in time domain.

(b) Determine the transfer functions of the filters.

3 State-space models

Topics: State-space models, transfer function to state-space, state-space to transfer function

3.1 Solved exercises

SE 3.1 Consider the input-output model of the disk drive read/write head from exercise PE 2.7:

$$J_1\frac{d^2\theta_1(t)}{dt^2} + b\left(\frac{d\theta_1(t)}{dt} - \frac{d\theta_2(t)}{dt}\right) + k(\theta_1(t) - \theta_2(t)) = T(t), \quad (3.1)$$

$$J_2\frac{d^2\theta_2(t)}{dt^2} + b\left(\frac{d\theta_2(t)}{dt} - \frac{d\theta_1(t)}{dt}\right) + k(\theta_2(t) - \theta_1(t)) = T_d(t), \quad (3.2)$$

where the inputs are the torques $T(t)$ and $T_d(t)$ and the output is the angle $\theta_2(t)$.

Determine the state-space model of the disk drive system.

Solution:

The first step is to choose the states. A standard approach (when there are no derivatives of the input) is:

$$x_1(t) = \theta_1(t) \quad (3.3)$$

$$x_2(t) = \dot{\theta}_1(t) \quad (3.4)$$

$$x_3(t) = \theta_2(t) \quad (3.5)$$

$$x_4(t) = \dot{\theta}_2(t). \quad (3.6)$$

Then, the outputs need to be defined in terms of state variables. In this case, the angle θ_2 is:

$$y(t) = \theta_2(t) = x_3(t). \quad (3.7)$$

For consistency, we also change the notations of the inputs:

$$u_1(t) = T(t),$$

$$u_2(t) = T_d(t).$$

From (3.4) and (3.6):

$$x_2(t) = \dot{\theta}_1(t) = \dot{x}_1(t),$$

$$x_4(t) = \dot{\theta}_2(t) = \dot{x}_3(t).$$

So, two of the state equations are:

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_3(t) = x_4(t).$$

The other two state equations are derived directly from (3.1) and (3.2) by isolating the second-order derivatives and replacing $\ddot{\theta}_1(t)$ with $\dot{x}_2(t)$, and $\ddot{\theta}_2(t)$ with $\dot{x}_4(t)$:

$$\dot{x}_2(t) = \frac{1}{J_1}\left[-b(x_2(t) - x_4(t)) - k(x_1(t) - x_3(t)) + u_1(t)\right], \tag{3.8}$$

$$\dot{x}_4(t) = \frac{1}{J_2}\left[-b(x_4(t) - x_2(t)) - k(x_3(t) - x_1(t)) + u_2(t)\right]. \tag{3.9}$$

The four state equations can be expressed in matrix form as:

$$\underbrace{\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix}}_{\dot{\mathbf{x}}(\mathbf{t})} = \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ -k/J_1 & -b/J_1 & k/J_1 & b/J_1\\ 0 & 0 & 0 & 1\\ k/J_2 & b/J_2 & -k/J_2 & -b/J_2\end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix}}_{\mathbf{x}(\mathbf{t})} + \underbrace{\begin{bmatrix}0 & 0\\ 1/J_1 & 0\\ 0 & 0\\ 0 & 1/J_2\end{bmatrix}}_{\mathbf{B}} \cdot \underbrace{\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}}_{\mathbf{u}(\mathbf{t})}. \tag{3.10}$$

Finally, from (3.7), the output equation in matrix form can be written as:

$$y(t) = \underbrace{\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}}_{\mathbf{C}} \cdot \underbrace{\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix}}_{\mathbf{x}(\mathbf{t})} + \underbrace{\begin{bmatrix}0 & 0\end{bmatrix}}_{\mathbf{D}} \cdot \underbrace{\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}}_{\mathbf{u}(\mathbf{t})}. \tag{3.11}$$

SE 3.2 Compartment models are widely used in engineering, medicine and environmental science for characterizing systems in a more simple manner, as composed out of multiple compartments (each compartment being characterized by the amount of volume, mass or concentration of certain substances), with the purpose of analyzing the exchange of substances between them ([11] - Chapter 7). For biomedical systems, the compartments can be, for example, different parts of the human body, while the transfer between compartments is based on diffusion and mass concentration.

As an application, consider the distribution and monitoring of a drug through the human body (field of *pharmacokinetics*). As a idealization, consider only two compartments: Blood compartment (B) and Tissue compartment (T) - Figure 3.1.

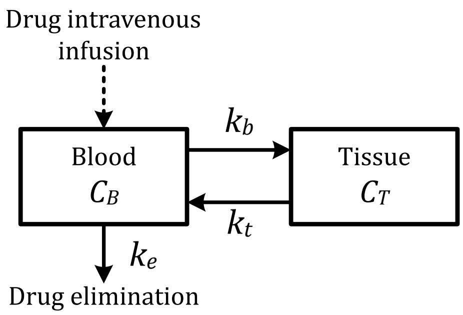


Figure 3.1: *Compartment model*

For this simplified process we can make the following assumptions:

- the volume of substance in each compartment is constant, and the drug passes from one compartment to another through diffusion;
- the drug is removed from a compartment at a rate proportional to the concentration;
- any substance entering a compartment (solute) is mixed instantaneously with the rest of the solution.

As concerns the inputs and the outputs of the system, consider that:

- the drug is initially administrated to the B compartment through intravenous infusion (or Bolus Injection) of a fixed amount (C_0 units). This can be characterized systemically by an impulse input function $\delta(t)$ or as a change in the initial conditions.
- a part of the drug is also eliminated from the B compartment.

The equations for each compartment can be written based on the mass balance:

B. Blood compartment:

$$\underbrace{V_B \dot{C}_B(t)}_{\text{Change in drug quantity in B}} = \underbrace{-q \cdot C_B(t)}_{\text{Drug flow from B to T}} + \underbrace{q \cdot C_T(t)}_{\text{Drug flow from T to B}} - \underbrace{q_e \cdot C_B(t),}_{\text{Drug flow-elimination}} \tag{3.12}$$

T. Tissue compartment:

$$\underbrace{V_T \dot{C}_T(t)}_{\text{Change in drug quantity in T}} = \underbrace{-q \cdot C_T(t)}_{\text{Drug flow from T to B}} + \underbrace{q \cdot C_B(t),}_{\text{Drug flow from B to T}} \tag{3.13}$$

where:

- V_T and V_B are the volumes associated with the two compartments,
- C_B and C_T - the concentration of the drug in compartment B and T, respectively,
- q - the drug flow between the compartments based on the difference of concentrations (diffusion),
- q_e - the elimination drug flow.

By defining the positive coefficients $k_b = q/V_B$, $k_e = q_e/V_B$, $k_t = q/V_T$ (also called *transfer rates*), the equations can be written compactly as state equations:

$$\dot{C}_B(t) = -(k_b + k_e) \cdot C_B(t) + k_b \cdot C_T(t), \tag{3.14}$$

$$\dot{C}_T(t) = k_t \cdot C_B(t) - k_t \cdot C_T(t), \tag{3.15}$$

with the initial conditions $C_B(0) = C_0$, $C_T(0) = 0$.

(a) Write the state-space model of the system in matrix form.

(b) Consider the parameter values $k_b = 0.6hr^{-1}$, $k_e = 0.1hr^{-1}$, $k_t = 0.2hr^{-1}$. Determine through simulation the variation of drug concentration in each compartment for an initial dose of $C_0 = 500$ units. Consider the final simulation time $t_{f1} = 5$hr, and then $t_{f2} = 180$ hr.

(c) For the parameter values given at **(b)** calculate the poles of the system and discuss how they relate to the system response.

(d) Discuss the effect of the parameters on the system response. Can you determine any values of the parameters k_b, k_e, k_t so that the system response oscillates?

Solution:

(a) The drug concentration in the compartments are chosen as the states of the system: $x_1(t) = C_B(t)$, $x_2(t) = C_T(t)$. The system (3.14), (3.15) has no inputs and the outputs can be considered to be also the drug concentrations: $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t)$.

Then, the matrix form of the state space model is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -(k_b + k_e) & k_b \\ k_t & -k_t \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Leftrightarrow \dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) \tag{3.16}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) \tag{3.17}$$

(b) The simulations results for the two scenarios are obtained with the MATLAB code from Listing 3.1 and 3.2 and are shown in Figure 3.2.

Listing 3.1: *blood_tissue.m*

```
close all
clear all
clc
% Simulation of the blood—tissue compartment model

kb = 0.6;   % constant kb
ke = 0.1;   % constant ke
kt = 0.2;   % constant kt
tf1 = 5;    % final time of simulation (first scenario)
tf2 = 180;  % final time of simulation (second scenario)

C0 = 500;  % initial condition C_B(0)

global A
% system matrix A (global constant to be used in function)
A = [—(kb+ke) kb; kt —kt];

[t1,C1] = ode23(@cprime, [0 tf1], [C0; 0]); % first simulation
[t2,C2] = ode23(@cprime, [0 tf2], [C0; 0]); % second simulation

subplot(211), plot(t1,C1, 'LineWidth', 2), grid on
xlabel('time [hr]'), ylabel('Drug concentration [units]')
legend('C_B', 'C_T'), title('Simulation on a short time')
subplot(212), plot(t2,C2, 'LineWidth', 2), grid on
xlabel('time [hr]'), ylabel('Drug concentration [units]')
legend('C_B', 'C_T'), title('Simulation on a long time')
```

Listing 3.2: *cprime.m*

```
function cp = cprime(t,X)
global A
cp = A*X;
```

One can notice that, in the first hours of simulation, as the drug passes in time from the blood compartment to the tissue compartment, the concentration C_B decreases, while C_T increases. However, the simulation on a longer period of time shows that both concentrations will approach zero as the drug will be (almost) eliminated from the system.

(c) The poles of the system are the eigenvalues of the system matrix $\mathbf{A}$. For the parameter values defined in the problem, the matrix $\mathbf{A}$ is constant:

$$\mathbf{A} = \begin{bmatrix} -(k_b + k_e) & k_b \\ k_t & -k_t \end{bmatrix} = \begin{bmatrix} -0.7 & 0.6 \\ 0.2 & -0.2 \end{bmatrix}$$

Through calculation we obtain the system poles or eigenvalues as negative real numbers, equal to: $\lambda_1 = -0.87$, $\lambda_2 = -0.02$.

In this case, the solution of the linear homogeneous system of differential equations (3.16) is a linear combination of exponential functions of the form $e^{\lambda_1 \cdot t}$ or $e^{\lambda_2 \cdot t}$. Since $\lambda_1 < 0$ and $\lambda_2 < 0$, all exponential functions will approach zero and the system response decays exponentially towards zero as time approaches infinity.

(d) The system response, or the solution of (3.16), is oscillatory only when the eigenvalues

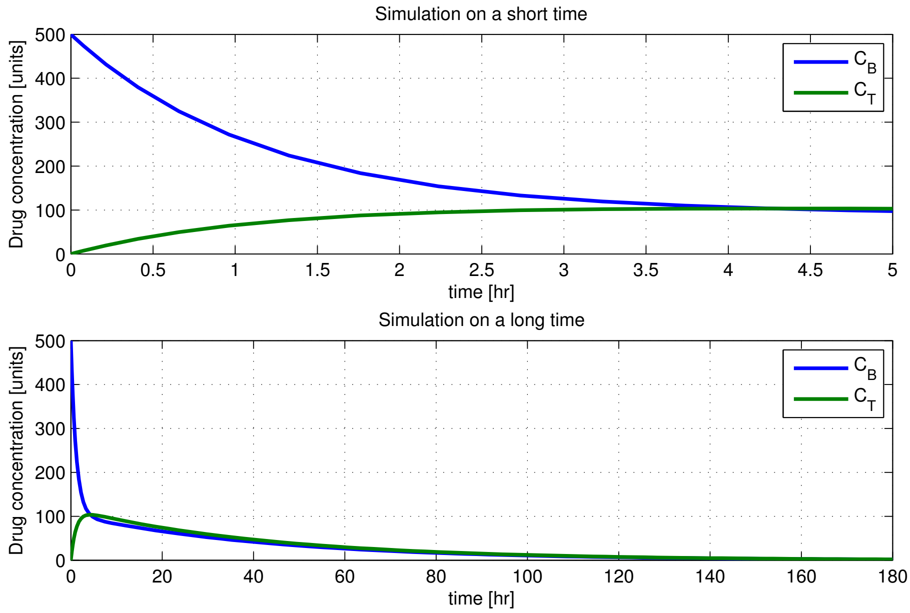


Figure 3.2: *Systems response to initial conditions, for a final simulation time of 5 and 180 hours*

of matrix $\mathbf{A}$ are complex. The eigenvalues of matrix $\mathbf{A}$ are obtained from:

$$\det(s\mathbf{I}-\mathbf{A}) = \det\begin{bmatrix} s+(k_b+k_e) & -k_b \\ -k_t & s+k_t \end{bmatrix} = 0$$

or

$$s^2 + (k_b+k_e+k_t)s + k_e k_t = 0 \tag{3.18}$$

The discriminant of the quadratic equation (3.18) is computed as:

$$\Delta = (k_b+k_e+k_t)^2 - 4k_e k_t = k_b^2 + 2k_b(k_e+k_t) + \underbrace{(k_e+k_t)^2 - 4k_e k_t}_{(k_e-k_t)^2}$$

$$\Delta = k_b^2 + 2k_b(k_e+k_t) + (k_e-k_t)^2 \tag{3.19}$$

Because the parameters k_b, k_e, k_t are always positive (due to their physical meaning), the discriminant (3.19) is always positive, i.e. the roots (3.18) are real numbers. Since the eigenvalues cannot have complex values, the system response cannot exhibit oscillatory behavior.

3.2 Proposed exercises

PE 3.1 Consider the following linear differential equations describing two systems with the input $u(t)$ and the output $y(t)$:

$$4\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} = 10u(t) \tag{3.20}$$

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} + y(t) = \frac{du(t)}{dt} + u(t) \tag{3.21}$$

(a) Choose the state variables as phase variables: $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$, $x_3(t) = \ddot{y}(t)$ and determine the state-space model for the system described by the differential equation (3.20).

(b) Choose the state variables as phase variables: $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$ and determine the state-space model for the system described by the differential equation (3.21).
Hint. One approach can be to obtain first the transfer function and then to convert it into a spate-space model.

(c) For equation (3.21) choose the state variables as $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t) - u(t)$ and determine a new state-space model. Compare this new model with the one obtained at (b).

PE 3.2 Determine a state space model for each of the following transfer function:

(a) $H_1(s) = \dfrac{2}{s^2 + 2s + 3}$

(b) $H_2(s) = \dfrac{s+2}{2s^2 + s + 1}$

(c) $H_3(s) = \dfrac{5}{s^3 + 2s^2 + 4s + 7}$

PE 3.3 Determine a transfer function for each of the following state space models:

(a)
$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

(b)
$$\dot{x} = \begin{bmatrix} 0 & 1 & 1 \\ 4 & 8 & 4 \\ -5 & -8 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

(c)
$$\dot{x} = \begin{bmatrix} -2.5 & -2 & -0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

PE 3.4 Determine the state-space models for the systems given in Figure 3.3.

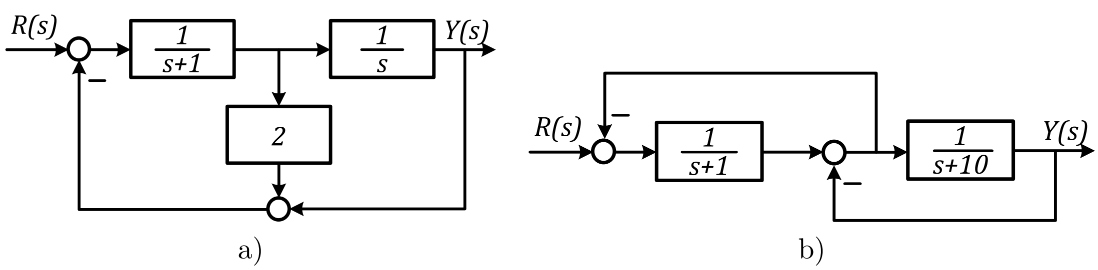


Figure 3.3: *Block diagrams*

PE 3.5 Determine a state space-model for the system given in Figure 3.4.

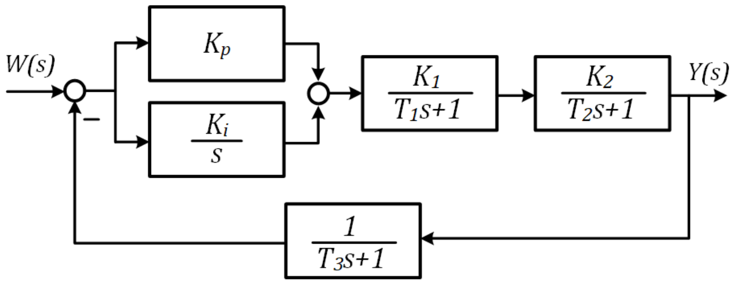


Figure 3.4: *Block diagram*

PE 3.6 For the electric RLC circuit from Figure 3.5.

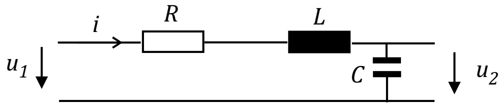


Figure 3.5: *RLC circuit*

(a) Determine the state space model starting from the equations written based on Kirchhoff's circuit laws. Choose the states as the inductor current $x_1(t) = i_L(t)$, and the capacitor voltage $x_2(t) = u_C(t)$.

(b) Determine the state space model starting from the transfer function of the circuit determined in SE 2.1:

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{1}{LCs^2 + RCs + 1}.$$

PE 3.7 For the linearized model of the pendulum from exercise PE 1.1 determine a state space model when the state variables are: $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$, the input is $u(t) = M(t)$ and the output is $y(t) = x_1(t)$. Write the state-space model in the standard form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

Create the state-space model in MATLAB and plot the impulse response of the system. Use the MATLAB functions *ss* and *impulse*.

PE 3.8 Using the linear approximations (1.9) and (1.10) describing the dynamics of the MagLev train from SE 1.2, write the state space models for both cases in the standard form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

The state variables are: $x_1(t) = \Delta z(t)$, $x_2(t) = \Delta \dot{z}(t)$, the input in the system is the current $u(t) = \Delta i(t)$ and the output is the vertical position $y(t) = \Delta z(t)$. Create the state-space models in MATLAB using *ss* and plot the impulse response.

PE 3.9 Consider a nonlinear compartment model that describes the metabolism of alcohol in the body, [4]. We have two compartments: body and liver. The body compartment is described by the equation

$$V_b \frac{C_b(t)}{dt} = q \cdot C_l(t) - q \cdot C_b(t) + q_{iv}(t), \tag{3.22}$$

while the equation for the liver compartment is:

$$V_l \frac{C_l(t)}{dt} = q \cdot C_b(t) - q \cdot C_l(t) - q_{max} \cdot \frac{C_l(t)}{C_0 + C_l(t)} + q_{gi}(t). \tag{3.23}$$

- C_b and C_l are the alcohol concentrations in the two compartments,
- V_b and V_l are the water volumes,
- q is the total hepatic flow,
- q_{iv} and q_{gi} are the injection rates for intravenous and gastrointestinal intake (system inputs).

(a) Determine through simulation the evolution of the concentrations C_b and C_l for the following numerical values: $V_b = 48\ l$, $V_l = 0.6\ l$, $q = 1.5\ l/min$, $q_{max} = 2.75\ mmol/min$, $C_0 = 0.1\ mmol \cdot l$, for a total simulation time $t_{fin} = 3$ min and the input signals (in grams):

$$q_{iv}(t) = \begin{cases} 10, & 0 \le t \le 0.1 \text{ min} \\ 0, & 0.1 < t \le t_{fin} \end{cases}, \qquad q_{gi}(t) = \begin{cases} 5, & 0 \le t \le 0.1 \text{ min} \\ 0, & 0.1 < t \le t_{fin} \end{cases} \tag{3.24}$$

(b) Determine the linear approximation of the state-space model in the equilibrium point obtained for zero inputs. Compare through simulation the response of the linear and nonlinear models for the inputs (3.24). Assess if the linear model is a good approximation for the nonlinear system.

4 Time response

Topics: System response, transient response, steady state analysis, standard input signals, inverse Laplace transform

4.1 Solved exercises

SE 4.1 Consider the RLC circuit from Figure 4.1, described by the differential equation:

$$L \cdot C \cdot \frac{d^2y(t)}{dt^2} + R \cdot C\frac{dy(t)}{dt} + y(t) = r(t),$$

with the input $r(t) = u_1(t)$ and the output $y(t) = u_2(t)$.

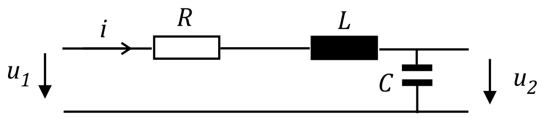


Figure 4.1: *RLC electrical circuit*

The initial conditions are considered to be zero and the transfer function of the system is:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{LCs^2 + RCs + 1}.$$

For the parameter values $R = 50\Omega$, $L = 5mH$, $C = 2\mu F$:

(a) Determine the impulse response of the system $y_i(t)$.

(b) Determine the unit step response of the system $y_s(t)$.

(c) Use the MATLAB functions *impulse* and *step* to plot the impulse and step response for this system. Compare the results to the plot of $y_i(t)$ and $y_s(t)$, over a time interval $t \in [0,\ 1.4 \cdot 10^{-3}]$ sec.

(d) Use the MATLAB function *lsim* to compute and plot the system response for a sinusoidal input signal $u_1(t) = \sin(5000t)$, for a time interval $t \in [0,\ 0.01]$sec.

Solution:

(a) For the values of the parameters given in the problem: $R = 50\Omega$, $L = 5mH = 5 \cdot 10^{-3}H$, $C = 2\mu F = 2 \cdot 10^{-6}F$, the transfer function is:

$$H(s) = \frac{1}{LCs^2 + RCs + 1} = \frac{1}{10^{-8}s^2 + 10^{-4}s + 1} = \frac{10^8}{s^2 + 10^4 s + 10^8}.$$

The impulse response can be found by using the inverse Laplace transform:

$$y_i(t) = \mathscr{L}^{-1}\{H(s)R(s)\}$$

with $R(s) = \mathscr{L}\{\delta(t)\} = 1$.

By replacing the transfer function we obtain:

$$y_i(t) = \mathscr{L}^{-1}\left\{\frac{10^8}{s^2 + 10^4 s + 10^8}\right\}.$$

From the Laplace transform table of Appendix A we use the transformation:

$$\mathscr{L}^{-1}\left\{\frac{b}{(s+a)^2 + b^2}\right\} = e^{-at}\sin bt \tag{4.1}$$

and then, we write the impulse response as:

$$y_i(t) = \mathscr{L}^{-1}\left\{\frac{10^8}{\left(s + \frac{10^4}{2}\right)^2 + 10^8 - \left(\frac{10^4}{2}\right)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{10^8}{\left(s + \frac{10^4}{2}\right)^2 + \frac{3}{4}\cdot 10^8}\right\}$$

$$y_i(t) = \mathscr{L}^{-1}\left\{\frac{10^8 \cdot \frac{\sqrt{3}\cdot 10^4}{2}}{\left(s + \frac{10^4}{2}\right)^2 + \left(\frac{\sqrt{3}\cdot 10^4}{2}\right)^2} \cdot \frac{1}{\frac{\sqrt{3}\cdot 10^4}{2}}\right\} = \mathscr{L}^{-1}\left\{\frac{2\cdot 10^4}{\sqrt{3}}\frac{\frac{\sqrt{3}\cdot 10^4}{2}}{\left(s + \frac{10^4}{2}\right)^2 + \left(\frac{\sqrt{3}\cdot 10^4}{2}\right)^2}\right\}$$

or:

$$y_i(t) = \frac{2\cdot 10^4}{\sqrt{3}}\mathscr{L}^{-1}\left\{\frac{5000\sqrt{3}}{(s+5000)^2 + \left(5000\sqrt{3}\right)^2}\right\} = \frac{2\cdot 10^4}{\sqrt{3}}e^{-5000t}\sin\left(5000\sqrt{3}t\right).$$

(b) If the input is a unit step, $R(s) = \mathscr{L}\{1\} = \frac{1}{s}$ and the step response can be found by using the inverse Laplace transform:

$$y_s(t) = \mathscr{L}^{-1}\{H(s)R(s)\} = \mathscr{L}^{-1}\left\{\frac{10^8}{s(s^2 + 10^4 s + 10^8)}\right\}$$

Thorough partial fraction expansion we further obtain:

$$\begin{aligned} y_s(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{s + 10^4}{s^2 + 10^4 s + 10^8}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{s + \frac{10^4}{2} + \frac{10^4}{2}}{\left(s + \frac{10^4}{2}\right)^2 + \left(\frac{\sqrt{3}\cdot 10^4}{2}\right)^2}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{s + \frac{10^4}{2}}{\left(s + \frac{10^4}{2}\right)^2 + \left(\frac{\sqrt{3}\cdot 10^4}{2}\right)^2} - \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}\cdot 10^4}{2}}{\left(s + \frac{10^4}{2}\right)^2 + \left(\frac{\sqrt{3}\cdot 10^4}{2}\right)^2}\right\}. \end{aligned}$$

We use now the transformation (4.1) and also, extract from the Laplace transform table of Appendix A:

$$\mathscr{L}^{-1}\left\{\frac{s+a}{(s+a)^2 + b^2}\right\} = e^{-at}\cos bt \tag{4.2}$$

The step response of the system will be:

$$y_s(t) = 1 - e^{-5000t}\cos\left(5000\sqrt{3}t\right) - \frac{1}{\sqrt{3}}e^{-5000t}\sin\left(5000\sqrt{3}t\right).$$

(c) Using the MATLAB functions *tf*, along with *impulse* and *step* (see, for example the code from Listing 4.1), one can obtain the results from Figure 4.2.

The figure illustrates the transient response the the output voltage of the circuit ($u_2(t)$) to an impulse, respectively a step, input voltage ($u_1(t)$).

Listing 4.1: *RLC_response.m*

```
close all
clear all
clc
% plot the impulse and step response of a system

% input transfer function of the system
RLC_system = tf(1e8, [1 1e4 1e8]); % transfer function H(s) = 10^8/(s^2+10^4s+10^8)

tfinal = 1.4e-3;            % final time
t=0:tfinal/50:tfinal;   % create the time vector for plotting y_i and y_s

% define anonymous functions for y_i and y_s:
y_i = @(t) 2e4/sqrt(3)*exp(-5000*t).*sin(5000*sqrt(3)*t);
y_s = @(t) 1-exp(-5000*t).*cos(5000*sqrt(3)*t)-1/sqrt(3)*exp(-5000*t).*sin(5000*sqrt(3)*t);

subplot(211)
impulse(RLC_system, tfinal), grid on  % compute and plot the impulse response for t =[0, tfinal]
hold on, plot(t, y_i(t), 'r.')        % plot y_i(t) on the same axes
legend('Matlab impulse response', 'y_i(t)')

subplot(212)
step(RLC_system, tfinal), grid on   % compute and plot the step response for t =[0, tfinal]
hold on, plot(t, y_s(t), 'r.')      % plot y_s(t) on the same axes
legend('Matlab step response', 'y_s(t)')
```

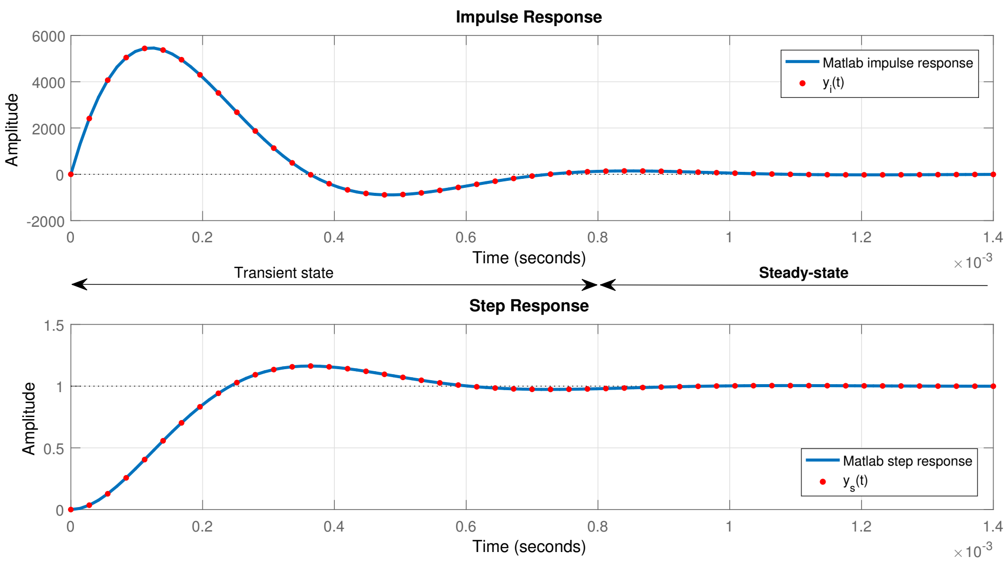


Figure 4.2: *Response of the RLC electrical circuit to impulse and step inputs*

(d) The simulation of the system response for any input signal can be obtained in MATLAB with the function *lsim*. The RLC response to a sinusoidal input $u_1(t) = \sin(5000t)$ can be obtained, for example, with the code from Listing 4.2 and the results are shown in Figure 4.3.

Listing 4.2: *RLC_sin_response.m*

```
close all
clear all
clc
% plot the impulse and step response of a system

% input transfer function of the system
RLC_system = tf(1e8, [1 1e4 1e8]); % transfer function H(s) = 10^8/(s^2+10^4s+10^8)
```

```

tfinal = 0.01;           % final time
t=0:tfinal/150:tfinal;   % create the time vector for plotting y_i and y_s

% create input vector u1(t) = sin(5000*t)
u1 = sin(5000*t);

% compute the system response to the input u1, over the time t
y_sin=lsim(RLC_system, u1, t);

% plot the input signal and the system response on the same axes
plot(t, u1, t, y_sin, 'LineWidth', 2), grid on
xlabel('time (sec)'), ylabel('Amplitude'), title('System response to a sinusoidal input')
legend('input: sin(5000t)', 'system response')
```

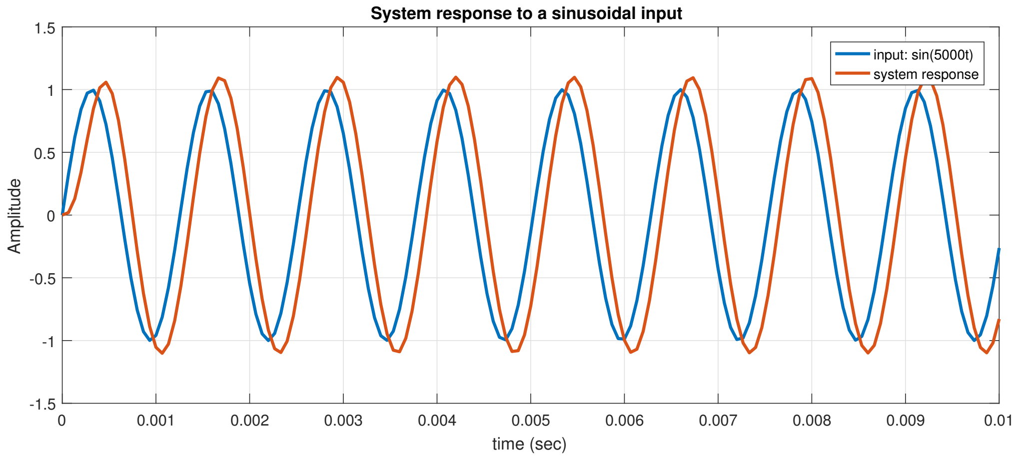


Figure 4.3: *Response of the RLC electrical circuit to a sinusoidal input*

SE 4.2 Consider the control of a hard disk drive using a PD (proportional-derivative) controller, as shown in the block diagram from Figure 4.4 ([9]-p.958). The inputs are the desired position of the disk drive head $r(t)$ and a disturbance $d(t)$ while the output is the actual disk drive head position $y(t)$.

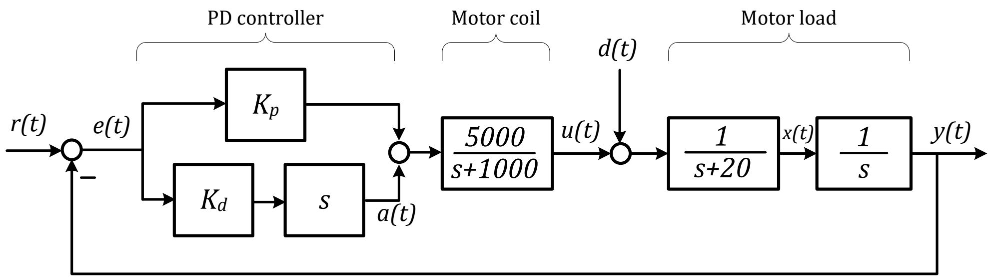


Figure 4.4: *PD control of a HDD*

Perform a steady-state analysis of the control system, for a step reference input $r(t) = 10,\ t \geq 0$ and a constant disturbance $d(t) = 5,\ t \geq 0$. Determine the steady-state value of the output (y_{ss}), the steady-state error (e_{ss}), and the steady-state value of the control signal (u_{ss}).

Solution:

We will show two methods for doing steady state analysis: one that uses the equations derived from the block diagram, and one that uses directly the block diagram to perform the analysis.

Method 1 - *Equation based steady-state analysis*

Using the rules of block diagram algebra, the block diagram of the control system can be simplified to the one presented in Figure 4.5. The Laplace transforms of the signals of interest are denoted on the block diagram as $E(s)$, $U(s)$ and $Y(s)$ and the inputs are $R(s)$ and $D(s)$. The equivalent transfer function of the PD controller and

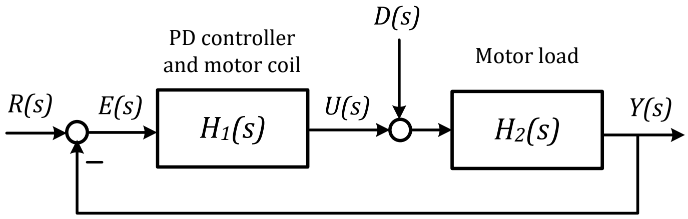


Figure 4.5: *PD control of a HDD. Simplified block diagram*

motor coil resulted as:

$$H_1(s) = \frac{5000(K_p + K_d s)}{s + 1000} \tag{4.3}$$

and for the motor load the equivalent transfer function is:

$$H_2(s) = \frac{1}{s(s + 20)}. \tag{4.4}$$

The steady-state values of the signals will be computed using the final value theorem (see Appendix A). For example, the final value of the output is:

$$y_{ss=} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

The first step is to determine the expression of $Y(s)$ as a function of $H_1(s)$, $H_2(s)$ and also the inputs $R(s)$ and $D(s)$. From the block diagram in Figure 4.5, we know that:

$$\begin{aligned} Y(s) &= H_2(s)\left(D(s) + U(s)\right) = H_2(s)\left(D(s) + H_1(s)E(s)\right) \\ &= H_2(s)\left(D(s) + H_1(s)(R(s) - Y(s))\right). \end{aligned}$$

If we further extract $Y(s)$, we obtain:

$$Y(s) = \frac{H_1(s)H_2(s)}{1 + H_1(s)H_2(s)}R(s) + \frac{H_2(s)}{1 + H_1(s)H_2(s)}D(s) \tag{4.5}$$

By replacing (4.3) and (4.4), the relation (4.5) becomes:

$$\begin{aligned} Y(s) &= \frac{5000(K_p + K_d s)}{s(s+20)(s+1000) + 5000(K_p + K_d s)}R(s) + \\ &+ \frac{s + 1000}{s(s+20)(s+1000) + 5000(K_p + K_d s)}D(s) \end{aligned}$$

Next, we replace $R(s) = \mathscr{L}\{10\} = \frac{10}{s}$ and $D(s) = \mathscr{L}\{5\} = \frac{5}{s}$ and use the final value theorem to determine the steady-state value of the output:

$$y_{ss} = \lim_{s\to 0} sY(s) = 10 + \frac{1}{K_p}$$

We determine now the steady-state error e_{ss}, starting with the expression of $E(s)$.

From the block diagram, the Laplace transform of the error signal is:

$$E(s) = R(s) - Y(s) = R(s) - H_2(s)(D(s) + U(s)) = R(s) - H_2(s)(D(s) + H_1(s)E(s)).$$

Hence

$$E(s) = \frac{1}{1 + H_1(s)H_2(s)} R(s) - \frac{H_2(s)}{1 + H_1(s)H_2(s)} D(s). \tag{4.6}$$

We replace now $H_1(s)$, $H_2(s)$, $R(s) = \frac{10}{s}$ and $D(s) = \frac{5}{s}$ into (4.6) and compute e_{ss} with the final value theorem:

$$e_{ss} = \lim_{s \to 0} sE(s) = -\frac{1}{K_p}.$$

So due to a *persistent* disturbance, the steady-state error is not zero, but it can still be made very small by designing the K_p parameter of the controller.
In order to obtain the steady-state value of the signal u, we compute $U(s)$ from the block diagram:

$$U(s) = H_1(s)E(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} R(s) - \frac{H_1(s)H_2(s)}{1 + H_1(s)H_2(s)} D(s), \tag{4.7}$$

replace $H_1(s)$, $H_2(s)$, $R(s)$ and $D(s)$ into (4.7), apply the final value theorem and obtain:

$$u_{ss} = \lim_{s \to 0} sU(s) = -5.$$

Method 2 - *Block diagram based steady-state analysis*
Steady-state analysis if often done in engineering directly on the block diagram, which is more intuitive and permits to determine simultaneously the steady-state values of all the signals of interest in the system.

1. The first step is to decompose each block into simple/elementary transfer functions (like pure gain, integrator, derivative, first-order element-real poles/zeros, second order element-pair of complex poles/zeros). In our particular case, the system from Figure 4.4 is already decomposed into elementary transfer functions.
2. The second step is to replace each transfer element with its steady-state equivalent:
   - the integrator block becomes an "open-circuit" with zero steady-state input,
   - the derivative block becomes an "open-circuit" with zero steady-state output,
   - all other blocks become simple gains (all "s" are set to zero).

   The rules listed above are derived from the fact that in steady-state, for constant inputs, all signals on the block diagram should be constant. In particular, for an integrator block this means that the input is zero while for a derivative block - the output is zero (see Figure 4.6).

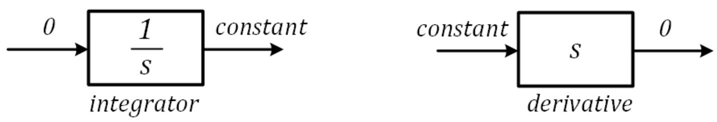


Figure 4.6: *Steady-state of an integrator block and a derivative block*

Figure 4.7 shows the resulting steady-state block diagram.
The steady state values of all the signals involved can be determined directly on the

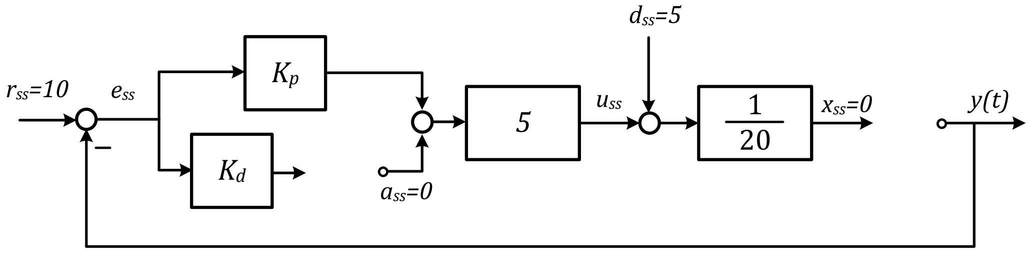


Figure 4.7: *Steady-state diagram of PD control of a HDD*

block diagram:

$$x_{ss} = 0 \implies u_{ss} + d_{ss} = 0 \implies u_{ss} = -5,$$
$$u_{ss} = 5k_p e_{ss} \implies e_{ss} = -\frac{1}{k_p},$$
$$r_{ss} - y_{ss} = e_{ss} \implies y_{ss} = 10 + \frac{1}{k_p},$$

The power of this analysis method comes from the fact that one can change the steady state values of the reference inputs and disturbance inputs, and tune the controller parameter K_p and K_d such that some steady-state performance requirements are met for an entire range of values (operating regime).

4.2 Proposed exercises

PE 4.1 Calculate the system response to a unit step input signal, when the transfer function is:

(a) $H(s) = \dfrac{s}{s^2 - 1}$

(b) $H(s) = \dfrac{1}{s^2 + 2s + 2}$

PE 4.2 Calculate the systems response to an ideal impulse signal, when the transfer function is:

(a) $H(s) = \dfrac{s}{s^2 - 1}$

(b) $H(s) = \dfrac{s + 1}{s^2 + 1}$

PE 4.3 Consider a first-order system with the transfer function:

$$H(s) = \frac{K}{Ts + 1}$$

(a) Use the MATLAB functions *tf* and *step* or Simulink to plot the step response of the system when:

(i) $K = 1,\ T = 1$

(ii) $K = 3,\ T = 1$

(iii) $K = 1,\ T = 3$

(iv) $K = 1,\ T = 6$

(b) Compare the plots and discuss the influence of the gain K and the time constant T on the system response.

(c) Determine the settling time for all cases.

PE 4.4 Consider a second-order system with the transfer function:

$$H(s) = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(a) Use the MATLAB functions *tf* and *step* or Simulink to plot the step response of the system when:

(i) $K = 1,\ \omega_n = 1,\ \zeta = 0$

(ii) $K = 1,\ \omega_n = 3,\ \zeta = 0$

(iii) $K = 1,\ \omega_n = 1,\ \zeta = 0.1$

(iv) $K = 1,\ \omega_n = 1,\ \zeta = 0.6$

(v) $K = 1,\ \omega_n = 1,\ \zeta = 1$

(vi) $K = 1,\ \omega_n = 1,\ \zeta = 2$

(vii) $K = 3,\ \omega_n = 1,\ \zeta = 0.6$

(b) Compare the plots (i) and (ii) and discuss the influence of the natural frequency ω_n on the system response.

(c) Compare the plots (iii), (iv), (v) and (vi) and discuss the influence of the damping factor ζ on the system response.

(d) Compare the plots (iv) and (vii) and discuss the influence of the gain K on the system response. Determine the settling time from the plot.

PE 4.5 Consider the plots in Figure 4.8 representing the unit step responses of eight systems. Match the step response plots to the following transfer functions:

$$H_1(s) = \frac{0.5}{s+0.5}, \qquad H_2(s) = \frac{2}{s+2}, \qquad H_3(s) = \frac{4}{s+2}$$

$$H_4(s) = \frac{1}{s^2+1}, \qquad H_5(s) = \frac{9}{s^2+9}$$

$$H_6(s) = \frac{9}{s^2+0.9s+9}, \quad H_7(s) = \frac{9}{s^2+3s+9}, \quad H_8(s) = \frac{18}{s^2+3s+9}$$

PE 4.6 Consider the the linearized model of the pendulum from PE 1.1 (see Figure 1.6), obtained for small variations of the angle $x(t)$ around the equilibrium point $x_0 = 0$:

$$ml^2\ddot{x}(t) = M(t) - mglx(t) - bl\dot{x}(t) \tag{4.8}$$

where:

- $x(t)$ is the angle position of the pendulum (output signal)
- $M(t)$ is the moment of force (torque) at the pivot point (input signal)
- m is the mass of the ball, $m = 0.5$ kg
- l is the length of the rod, $l = 1$ m
- g is the acceleration of gravity, $g = 9.8\ m/s^2$
- b is the viscous friction coefficient, $b = 0.5$

(a) Obtain the transfer function between the input $M(t)$ and the output $x(t)$.

(b) Plot the step response of the system using the MATLAB function *step*.

PE 4.7 Consider the linearized models of the MagLev train from SE 1.2:

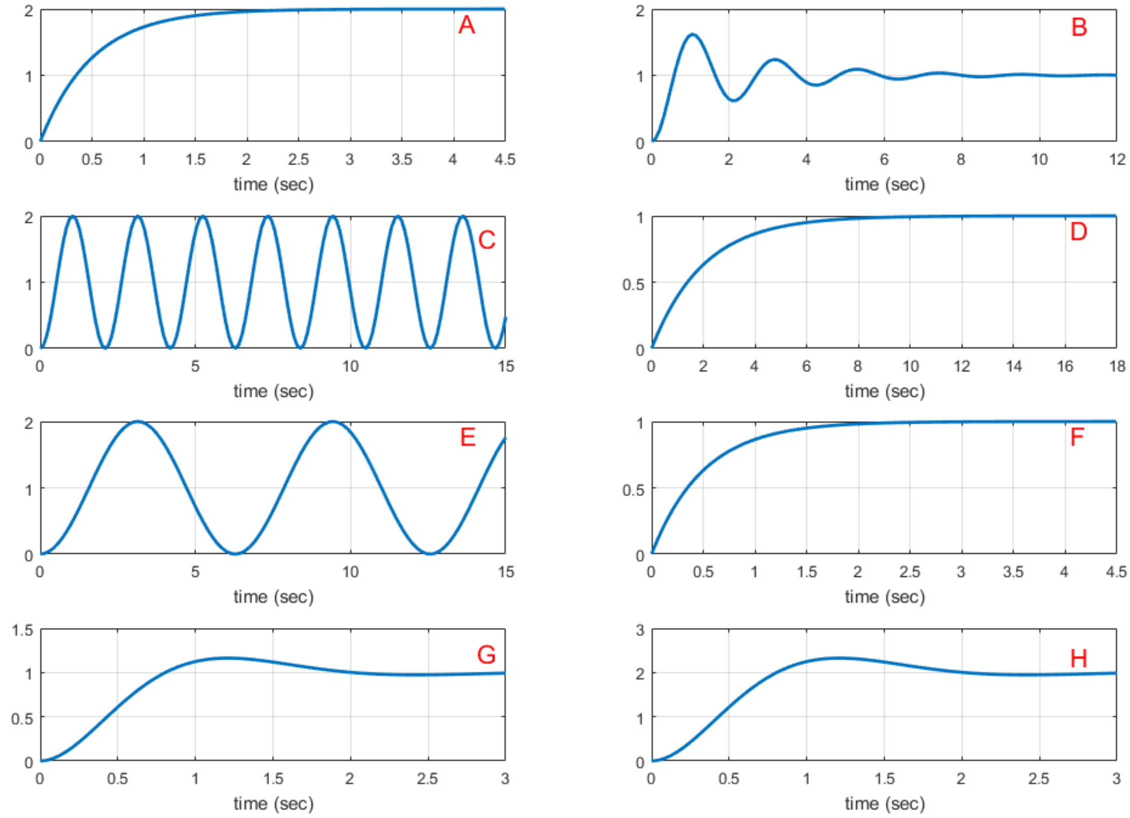


Figure 4.8: *Step responses*

(I) Electromagnetic suspension (EMS):

$$m\Delta\ddot{z}(t) - \frac{2mg}{z_0}\Delta z(t) + \frac{2\sqrt{mgk}}{z_0}\Delta i(t) = 0 \tag{4.9}$$

(II) Electrodynamic suspension (EDS):

$$m\Delta\ddot{z}(t) + \frac{2mg}{z_0}\Delta z(t) - \frac{2\sqrt{mgk}}{z_0}\Delta i(t) = 0 \tag{4.10}$$

where:

- $\Delta i(t)$ is the variation of the current around the equilibrium value (the input signal),
- $\Delta z(t)$ - the variation of the vertical position of the train around the equilibrium value z_0 (the output signal),
- $m = 10^4\ kg$ (the mass of the train),
- $g = 10\ m/s^2$ (the acceleration of gravity),
- $k = 10^{-3}\ Nm^2/A^2$ (the levitation force constant),
- $z_0 = 10^{-2}\ m$ (the operating air-gap).

(a) Determine the transfer function from the input current $\Delta i(t)$ to the output position $\Delta z(t)$, for both cases.

(b) Plot the impulse response of the open-loop systems for a period of time of 0.1 seconds (case I) and 1 second (case II). Use the MATLAB function *impulse*.

(c) Analyze and explain the results.

PE 4.8 For the feedback control systems shown in Figure 4.9 and a unit step input $r(t) = 1,\ t \geq 0$:

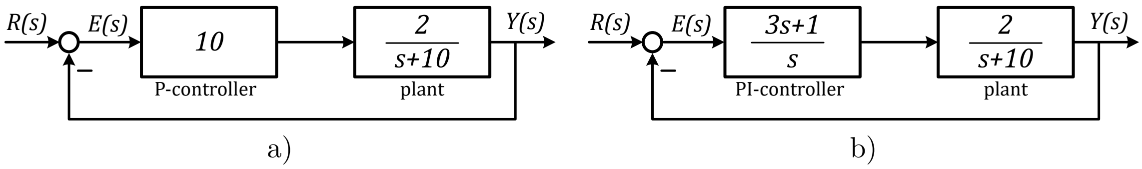


Figure 4.9: *Closed-loop control systems*

(a) Compute the steady-state value of the output.

(b) Compute the steady-state error.

(c) Plot the unit step response and determine the steady-state error from the plot.

PE 4.9 For the systems represented by the block diagrams shown in Figure 4.10.

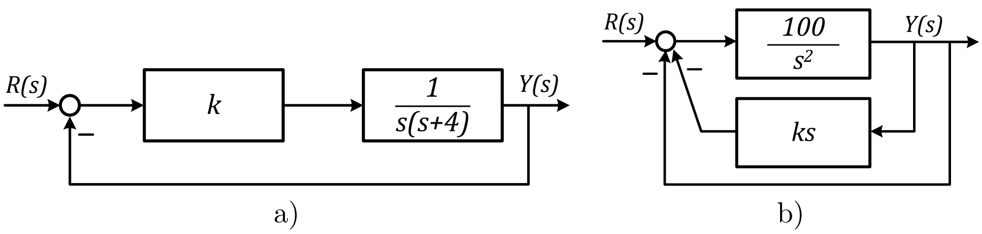


Figure 4.10: *Closed-loop systems*

(a) Calculate the steady-state error for a ramp input, $r(t) = t,\ t \geq 0$

(b) Determine the range of values for k, $(k > 0)$ for which the step response is overdamped (with no overshoot).

PE 4.10 Consider the control system shown in the Figure 4.11, with $k > 0$.

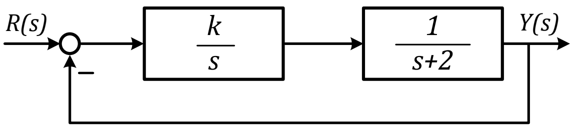


Figure 4.11: *Block diagram of a feedback system*

(a) Determine the steady state error for a step signal $r(t) = 3,\ t \geq 0$.

(b) Determine the steady state error for a ramp signal $r(t) = t,\ t \geq 0$.

(c) Determine the values of k so that the step response of the closed-loop system is overdamped.

PE 4.11 A position control system is described by the block diagram shown in Figure 4.12, where $K_p > 0$.

(a) Determine the steady-state error for a step input $r(t) = 2,\ t \geq 0$.

(b) Determine the value of K_p that will result in a steady-state error $e_{ss} = 0.1$ for a ramp input $r(t) = 100t,\ t \geq 0$.

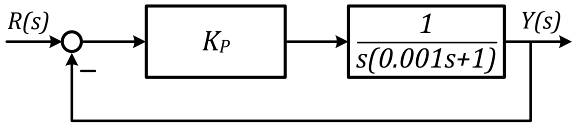


Figure 4.12: *Block diagram of a feedback system*

PE 4.12 Consider a general second-order system with the transfer function:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Show the area where the poles can be located in the complex plane, if the specifications for a step input are:

(a) Damping factor $\zeta \geq \frac{\sqrt{3}}{2}$.

(b) Damping factor $\zeta \geq \frac{\sqrt{2}}{2}$ and settling time $t_s < 4$ sec.

(c) Damping factor $\zeta \leq \frac{\sqrt{2}}{2}$, settling time $t_s < 4$ sec and peak time $t_p = \frac{\pi}{3}$ sec.

PE 4.13 The predator-prey problem, [4], refers to an ecological system in which we have two species, one of which feeds on the other. This type of system has been studied for decades and is known to exhibit interesting dynamics. A simple model for this situation can be constructed by keeping track of the rate of births and deaths of each species. Let $H(t)$ represent the number of hares (prey) and let $L(t)$ represent the number of lynxes (predator). The input u corresponds to the growth rate for hares, which we might modulate by controlling a food source for the hares. The dynamics of the system are modeled as:

$$\begin{aligned} \frac{dH(t)}{dt} &= (1.6 + u(t))H(t)\left(1 - \frac{H(t)}{125}\right) - \frac{3.2H(t)L(t)}{50 + H(t)}, \quad H \geq 0, \\ \frac{dL(t)}{dt} &= 0.6\frac{3.2H(t)L(t)}{50 + H(t)} - 0.56L(t), \quad L \geq 0 \end{aligned}$$

We first linearize the system around the equilibrium point of the system $(H_e,\ L_e,\ u_e)$ which can be determined numerically to be $H_e = 20.6$, $L_e = 29.5$ for $u_e = 0$. This yields a linear dynamical system:

$$\frac{dz_1(t)}{dt} = 0.13z_1(t) - 0.93z_2(t) + 17.2u(t) \quad (4.11)$$

$$\frac{dz_2(t)}{dt} = 0.57z_1(t) \quad (4.12)$$

where $z_1(t) = H(t) - H_e$ and $z_2(t) = L(t) - L_e$ (i.e. the variation of the number of hares and lynxes around the equilibrium values).

The block diagram for this system is shown in Figure 4.13, where $Z_1(s) = \mathscr{L}\{z_1(t)\}$, $Z_2(s) = \mathscr{L}\{z_2(t)\}$, $U(s) = \mathscr{L}\{u(t)\}$.

(a) Apply the Laplace transform of relations (4.11) and (4.12) and determine the transfer functions $G_1(s)$, $G_2(s)$ and $G_3(s)$, as shown in Figure 4.13.

(b) Build the block diagram in Simulink.

(c) Simulate the system for a step input, plot the evolution of $z_1(t)$ and $z_2(t)$ and explain the result.

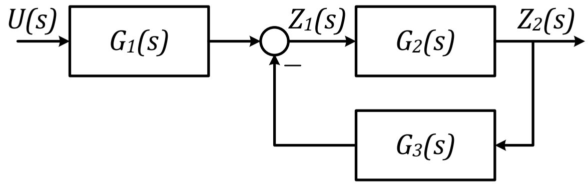


Figure 4.13: *Block diagram of the predator-prey system*

PE 4.14 Consider the electrical circuit from Figure 4.14, representing an RC filter, with parameter values $R = 100\Omega$, $C = 100\mu F$.

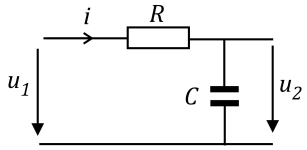


Figure 4.14: *RC filter*

(a) Determine the transfer function $H(s) = \dfrac{U_2(s)}{U_1(s)}$.

(b) Using the inverse Laplace transform, calculate the response of the system $u_2(t)$, to a sinusoidal input $u_1(t) = sin(\omega t)$, with $\omega = 100 rad\, sec^{-1}$.

(c) Simulate in MATLAB the system's response for $u_1(t) = sin(\omega t)$ on the interval 0-0.5 seconds using the *lsim* function.

(d) Plot and compare the results from points **(b)** and **(c)**.

(e) Plot the input $u_1(t)$ and output $u_2(t)$ on the same figure. Analyze the difference between the input and output signals in terms of amplitude (peak-to-peak) and peak time moments (phase shift).

5 Stability analysis

Topics: Stability, Routh-Hurwitz method

5.1 Solved exercises

SE 5.1 Consider the problem of Pogo vibrations for powerful liquid rocket vehicles, [30, 31]. These self-excited vibrations are caused by instabilities arising from the interaction between the vehicle structure and the propulsion system - Figure 5.1 ([31]).

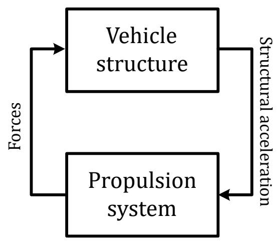


Figure 5.1: *Closed loop representation of Pogo vibration dynamics, [31]*

Let the parameter k denote the coupling strength between the subsystems. In [30], a linear model is developed with different parameter configurations. The characteristic equation of the system is:

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

. Consider the following parameter configuration: $a_3 = 1.212$, $a_2 = 2.014 - k$, $a_1 = 1.212$, $a_0 = 1$.

(a) Analyze the stability of the system for $k = 1$.

(b) Determine the range of k for which the system is stable.

Solution:

(a) For $k = 1$ the characteristic equation becomes:

$$s^4 + 1.212s^3 + 1.014s^2 + 1.212s + 1 = 0. \tag{5.1}$$

In order to asses the stability we will use the Routh-Hurwitz method. First we check if all the coefficients are positive and non-zero (necessary condition). For (5.1) this holds.

Next, we construct the Routh array:

$$\begin{array}{lccc} s^4: & 1 & 1.014 & 1 \\ s^3: & 1.212 & 1.212 & \\ s^2: & \alpha_1 & \alpha_2 & \\ s^1: & \alpha_3 & \alpha_4 & \\ s^0: & \alpha_5 & & \end{array}$$

The first line is given by the odd coefficients, while the second line by the even ones. The coefficients of the subsequent lines are calculated as follows:

$$3^{rd} \; line: \; \alpha_1 = -\frac{\begin{vmatrix} 1 & 1.014 \\ 1.212 & 1.212 \end{vmatrix}}{1.212} = 0.014, \quad \alpha_2 = -\frac{\begin{vmatrix} 1 & 1 \\ 1.212 & 0 \end{vmatrix}}{1.212} = 1$$

$$4^{th} \; line: \; \alpha_3 = -\frac{\begin{vmatrix} 1.212 & 1.212 \\ \alpha_1 & \alpha_2 \end{vmatrix}}{\alpha_1} = -\frac{\begin{vmatrix} 1.212 & 1.212 \\ 0.014 & 1 \end{vmatrix}}{0.014} = -85.35, \; \alpha_4 = -\frac{\begin{vmatrix} 1.212 & 0 \\ \alpha_1 & 0 \end{vmatrix}}{\alpha_1} = 0$$

$$5^{th} \; line: \; \alpha_5 = -\frac{\begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{vmatrix}}{\alpha_3} = -\frac{\begin{vmatrix} 0.014 & 1 \\ -85.35 & 0 \end{vmatrix}}{-85.35} = 1$$

After all these calculations the Routh array becomes:

$$\begin{array}{l|c|cc} s^4: & 1 & 1.014 & 1 \\ s^3: & 1.212 & 1.212 & \\ s^2: & 0.014 & 1 & \\ s^1: & \text{-85.35} & 0 & \\ s^0: & 1 & & \end{array}$$

Finally, in order to determine stability, we look at the signs of the elements from the first column. For the system to be stable, all elements must have the same sign (or, in this case, they must be positive). Because we have one negative element -85.35 the system is unstable. Two signs changes on the first column, from the third to the fourth line, and from the fourth line to the fifth, means that we have two unstable poles (i.e. two poles in the right half-plane).

Indeed, if we calculate numerically the roots of the polynomial (for example using the *roots* function in MATLAB) we obtain: $r_{1,2} = 0.2786 \pm 0.9604i$, $r_{2,3} = -0.8846 \pm 0.4663i$. The first pair of complex roots has a positive real part.

The strength of the analytical Routh-Hurwitz method here relies on the fact the we assessed the stability without having to calculate explicitly all the poles of the system.

(b) The condition that all coefficients must be positive and non-zero implies that:

$$2.014 - k > 0 \Rightarrow k \in (-\infty, 2.014). \tag{5.2}$$

We further construct the Routh array:

$$\begin{array}{lccc} s^4: & 1 & \text{2.014-k} & 1 \\ s^3: & 1.212 & 1.212 & \\ s^2: & \alpha_1 & \alpha_2 & \\ s^1: & \alpha_3 & \alpha_4 & \\ s^0: & \alpha_5 & & \end{array}$$

The unknown elements are calculated as follows:

$$3^{rd}\ line\text{:}\ \alpha_1 = -\frac{\begin{vmatrix} 1 & 2.014-k \\ 1.212 & 1.212 \end{vmatrix}}{1.212} = 1.014 - k, \quad \alpha_2 = -\frac{\begin{vmatrix} 1 & 1 \\ 1.212 & 0 \end{vmatrix}}{1.212} = 1.$$

$$4^{th}\ line\text{:}\ \alpha_3 = -\frac{\begin{vmatrix} 1.212 & 1.212 \\ \alpha_1 & \alpha_2 \end{vmatrix}}{\alpha_1} = \frac{1.212(k-0.014)}{k-1.014}, \quad \alpha_4 = -\frac{\begin{vmatrix} 1.212 & 0 \\ \alpha_1 & 0 \end{vmatrix}}{\alpha_1} = 0.$$

$$5^{th}\ line\text{:}\ \alpha_5 = -\frac{\begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{vmatrix}}{\alpha_3} = 1.$$

Now the table becomes:

s^4 :	1	2.014-k	1
s^3 :	1.212	1.212	
s^2 :	1.014-k	1	
s^1 :	$\frac{1.212(k-0.014)}{k-1.014}$	0	
s^0 :	1		

Finally, we impose that all the elements of the first column (marked in a rectangle) are strictly positive:

$$1.014 - k > 0 \Rightarrow k \in (-\infty, 1.014), \tag{5.3}$$

$$\frac{1.212(k-0.014)}{k-1.014} > 0 \Rightarrow k \in (-\infty, 0.014) \cup (1.014, \infty). \tag{5.4}$$

The range of k for which the system is stable is given by the set that respects all three conditions (5.2), (5.3) and (5.4) simultaneously (intersection) $\Rightarrow k \in (-\infty, 0.014)$.

The strength of the Routh-Hurwitz method lies in the possibility to determine analytically the range of a parameter that preserves stability. The alternative method to compute the roots of a fourth-order polynomial with k as a parameter, would require very complicated calculations. Numerically, this could be done by computing the roots exhaustively for every possible value of k.

5.2 Proposed exercises

PE 5.1 Determine the poles, plot the impulse response and the step response and comment on the stability of the systems with the following transfer functions:

$$H_1(s) = \frac{4}{s^2+5s+4}, \quad H_2(s) = \frac{4}{s^2+s+4}, \quad H_3(s) = \frac{4}{s^2-4}, \quad H_4(s) = \frac{4}{s^2-s+4},$$

$$H_5(s) = \frac{4}{s^2+4}, \quad H_6(s) = \frac{4}{s(s+4)}, \quad H_7(s) = \frac{4}{(s^2+4)^2}$$

Hint. Use the MATLAB functions impulse *and* step *and place both responses for each transfer function in the same figure (use* subplot*). For $H_5(s)$ and $H_7(s)$ set the final time of simulation at* 30 *sec.*

PE 5.2 Determine the stability of the following characteristic polynomials using the Routh-Hurwitz criterion:

(a) $s^2 + 4s + 1$

(b) $s^3 + 2s^2 + 5s + 8$
(c) $s^3 + 2s^2 - 5s + 8$
(d) $s^4 + 2s^3 + 3s^2 + 4s + 5$
(e) $s^5 + 2s^4 + 3s^3 + 4s^2 + 6$

PE 5.3 Use the Routh-Hurwitz criterion to determine the range of k for a stable system, if the characteristic polynomial is:

(a) $s^3 + s^2 + s + k$
(b) $s^4 + 2s^3 + 3s^2 + 4s + k$

PE 5.4 Consider the closed-loop system shown in Figure 5.2.

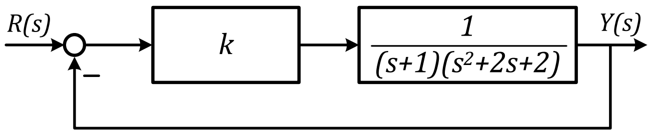


Figure 5.2: *Closed-loop system*

(a) Using the Routh-Hurwitz criterion find the range of k to ensure stability of the closed-loop system.
(b) Plot the roots of the characteristic equation for $k \in [0,\ 15]$ and discuss the location of the closed-loop poles and system stability.
(c) Find the value of k that makes the system's step response oscillate and determine the frequency and period of oscillation. Plot the step response of the closed-loop system for this value of k and compare the period of oscillation from the plot with the one calculated.

PE 5.5 Consider the simple pendulum system modeled as in PE 1.1. The nonlinear differential equation describing this system (when all constants have been replaced) is:

$$\ddot{x}(t) = 2M(t) - 9.8 \sin x(t) - \dot{x}(t)$$

where $M(t)$ is the torque at the pivot point and $x(t)$ is the angular position of the pendulum.

Prove that the systems has two equilibrium points, one stable and one unstable: $(0,\ 0)$ and $(\pi,\ 0)$.

Hint. Linearize the system in each equilibrium point and asses the stability by calculating the poles. A stable/unstable linearized system means a stable/unstable equilibrium point.

PE 5.6 Consider the linearized models of the Maglev trains from SE 1.2 built based on magnetic attraction (EMS) or magnetic repulsion (EDS). If all parameters in the model are replaced by their constant values, the transfer functions from the input (current) to the output (vertical position) are:

$$H_{EMS}(s) = -\frac{0.2}{s^2 - 2000}, \qquad H_{EDS}(s) = \frac{0.2}{s^2 + 2000}$$

(a) Plot the impulse response for each case and comment on the system stability.
(b) Compute the poles of the transfer functions and analyze the stability.

PE 5.7 Consider the linearized predator-prey model from PE 4.13:

$$\frac{dz_1(t)}{dt} = 0.13z_1(t) - 0.93z_2(t) + 17.2u(t) \tag{5.5}$$

$$\frac{dz_2(t)}{dt} = 0.57z_1(t) \tag{5.6}$$

where $z_1(t)$ and $z_2(t)$ are the variation of the number of hares and lynxes, respectively, around the equilibrium values. The input $u(t)$ is the growth rate for hares, which may be considered proportional with the food source for the hares.

(a) Prove that the system is unstable.

(b) Consider a feedback law of the form $u(t) = -k \cdot z_1(t)$. Find a gain k for which the system is stable.

PE 5.8 Consider the control loop for a simplified model of an aircraft as in Figure 5.3, [9]. The setpoint $r(t)$ is the desired orientation and the output $y(t)$ is the actual orientation. The controller has a positive gain k ($k > 0$), a negative zero at -1 and a negative pole ($p > 0$).

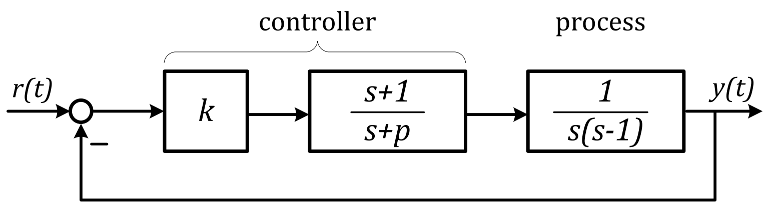


Figure 5.3: *Simplified aircraft control loop*

Use the Routh-Hurwitz criterion to determine the stability conditions for the parameters k and p of the controller.

PE 5.9 Consider the closed-loop system from Figure 5.4.

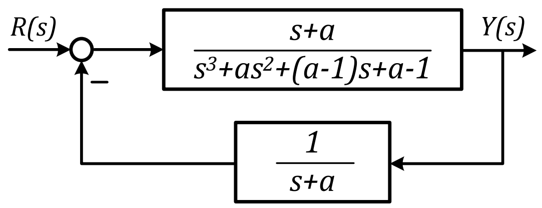


Figure 5.4: *Closed-loop system*

(a) Determine the range of the parameter a such that the closed-loop system is stable.

(b) Determine the location of the closed-loop poles when the system is marginally stable.

PE 5.10 Consider the pupillary light reflex as an example of a biological feedback system. A block diagram of the linearized dynamics is shown in Figure 5.5, [20]. The input is the (change in) intensity of ambient light ($I(t)$) and the output is the (change in) pupil area ($A(t)$). An increase of light received by the retina is *sensed* and *fed back* through a neural pathway and, in the end, the iris muscle is commanded to contract/relax, thus reducing the pupil area.

Consider the following power series approximation for the time delay:

$$e^{-sD} = 1 - Ds + \frac{D^2s^2}{2} - \frac{D^3s^3}{6}. \tag{5.7}$$

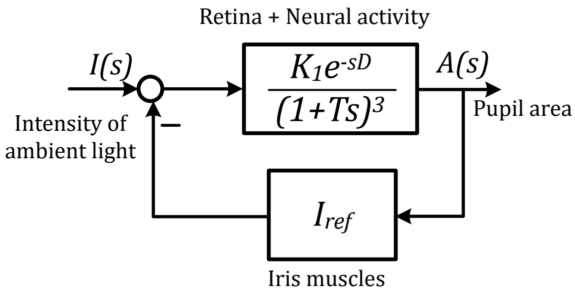


Figure 5.5: *Block diagram for the linearized dynamics of the pupillary control system*

The characteristic equation of the system can be calculated as:

$$(T^3 - \frac{KD^3}{6})s^3 + (3T^3 + \frac{KD^2}{2})s^2 + (3T - KD)s + (1 + K) = 0, \tag{5.8}$$

with $K = K_1 I_{ref}$.

For the parameter values $D = 0.18$s and $T = 0.1$s, determine the range of K for which the system is stable using the Routh-Hurwitz method.

6 Root Locus

Topics: Root locus analysis, open/closed-loop poles, stability analysis, transient response

6.1 Solved exercises

SE 6.1 Phase-locked loops are widely used in computers, telecommunications and electronic applications, but are encountered also in biology as synchronization mechanisms. Consider the application to a precision motor speed control system - Figure 6.1, [14].

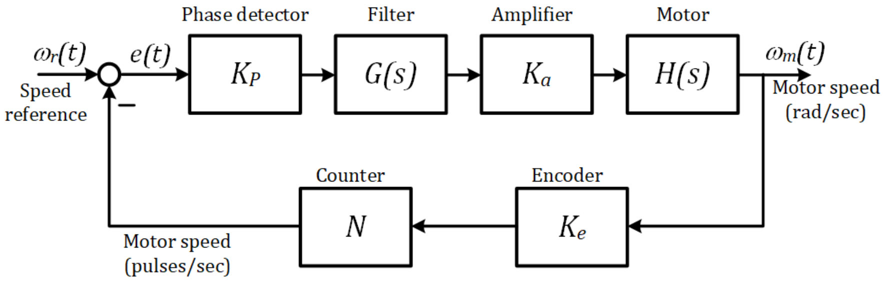


Figure 6.1: *Phase-lock loop for a motor speed control system (adapted from [14])*

The digital encoder on the feedback generates a train of impulses, with a frequency related to the motor speed. The input reference signal is also a sequence of impulses. The phase detector element has the role of detecting the phase difference between these two signals. For the analysis presented here, as a simplification, we consider as feedback and reference signals directly the frequency (in pulses/sec), and the phase detector becomes a simple gain K_p.

The performance of the phase-locked loop can be imposed through the *filter circuit*, which acts as a controller, minimizing the synchronization error e. (For an example of filter design and discussion see the data sheet from National Semiconductor [19]). Here we will use the simple filter:

$$G(s) = \frac{1 + R_2 C s}{R_1 C s}. \tag{6.1}$$

The transfer function of the motor is:

$$H(s) = \frac{10}{s(1 + 0.05s)}. \tag{6.2}$$

Consider the following values for the parameters: $K_p = 0.06$, $K_a = 20$, $K_e = 5.73$ pulse/rad, $N = 1$, $R_1 = 2 \cdot 10^6\ \Omega$, $C = 1\ \mu F = 10^{-6} F$.

(a) Sketch the root locus for all values of the parameter $R_2 \in [0,\ \infty)$.

(b) Analyze the closed-loop system stability and transient behavior for $R_2 \in [0,\ \infty)$.

Solution:

(a) Root locus

- The open loop transfer function of the system from Figure 6.1 is the product of all transfer functions around the loop:

$$G_0(s) = K_p K_a K_e N G(s) H(s) = \frac{34.38(1 + 10^{-6} R_2 s)}{s^2(1 + 0.05s)}. \tag{6.3}$$

- The characteristic equation for the system is:

$$1 + G_0(s) = 0, \quad \text{or} \quad 1 + \frac{34.38(1 + 10^{-6} R_2 s)}{s^2(1 + 0.05s)} = 0 \tag{6.4}$$

- We have to rearrange the characteristic equation so that the parameter of interest appears as a multiplying factor, in the general form:

$$1 + K \cdot P(s) = 0$$

Therefore, we rewrite (6.4) as follows (common denominator):

$$\frac{s^2(1 + 0.05s) + 34.38(1 + 10^{-6} R_2 s)}{s^2(1 + 0.05s)} = 0, \quad \text{or} \quad s^2(1+0.05s)+34.38(1+10^{-6} R_2 s) = 0$$

After some calculations, the characteristic equation becomes:

$$s^3 + 20s^2 + 678.6 \cdot 10^{-6} R_2 s + 687.6 = 0.$$

We further isolate the term that includes R_2 (the parameter of interest):

$$(s^3 + 20s^2 + 687.6) + 6.786 \cdot 10^{-4} R_2 s = 0$$

and divide the entire equation with the term written between parentheses:

$$1 + R_2 \frac{678.6 \cdot 10^{-6} s}{s^3 + 20s^2 + 687.6} = 0.$$

To simplify the procedure, we may introduce the notation $K = R_2 \cdot 678.6 \cdot 10^{-6}$ and the equation becomes:

$$1 + K \frac{s}{s^3 + 20s^2 + 687.6} = 0. \tag{6.5}$$

- We will sketch the root locus for the new equivalent characteristic equation (6.5), where the new "open-loop" transfer function is $P(s) = \dfrac{s}{s^3 + 20s^2 + 687.6}$.

- $P(s)$ has:
  - one open-loop zero $z_1 = 0$
  - three open-loop poles $p_1 = -21.49$, $p_2 = 0.75 + 5.6i$, $p_3 = 0.75 - 5.6i$.

  The number of open-loop zeros and open-loop poles are $n_z = 1$ and $n_p = 3$, respectively.

- The root locus will have $n_p = 3$ branches, starting (when $K = 0$) from the open-loop poles. As K increases towards infinity, one branch will approach the zero z_1 and the other two will approach two asymptotes, symmetrically about the real axis. Indeed, the number of asymptotes is equal to $n_p - n_z = 2$.

- The center of the asymptotes is:

$$\sigma = \frac{\sum poles - \sum zeros}{n_p - n_z} = \frac{-21.48 + 0.75 + 5.6i + 0.75 - 5.6i - 0}{3 - 1} = -10$$

The angles of the two asymptotes with respect to positive real axis are:

$$\Phi_q = \frac{2q+1}{n_p - n_z} \cdot 180^\circ, \quad q = \overline{0, n_p - n_z - 1}$$

$$\Phi_1 = \frac{2 \cdot 0 + 1}{2} \cdot 180^\circ = 90^\circ, \quad \Phi_2 = \frac{2 \cdot 1 + 1}{2} \cdot 180^\circ = 270^\circ.$$

- Figure 6.2 shows a sketch with the open-loop poles and zero, the two asymptotes and the segment on real axis which is part of the root locus.
The root locus lies on the real axis to the left of an odd number of open-loop poles and zeros: in our case, the segment between the zero z_1 and the real pole p_1.

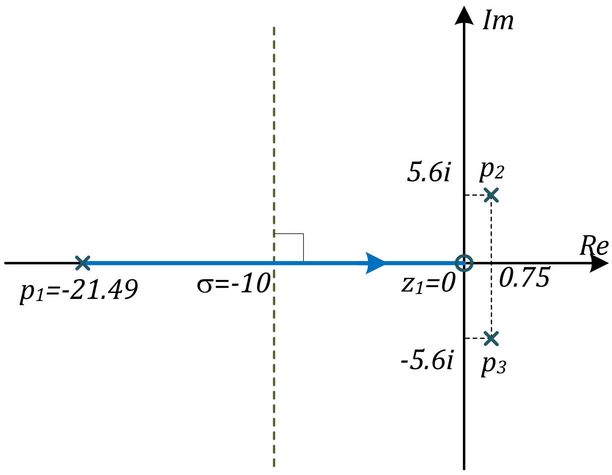


Figure 6.2: *Root locus sketch - Open-loop poles, zero and the asymptotes*

- We can further deduce that as K goes from 0 to ∞, the real pole moves towards the zero, and the complex conjugate poles will move towards the asymptotes (symmetrical in respect with the real axis). The new sketch of the root locus is shown in Figure 6.3.

- We see that as K increases (that is R_2 increases), the closed-loop complex poles move from the right half-plane to the left half-plane , which means that the system is unstable for small values of K. Then, for larger values of K, the closed-loop system becomes stable.
It is of practical interest to determine the value of K for which the closed-loop poles are on the imaginary axis (when the system is marginally stable) and the location of these poles. From the characteristic equation (6.5) written as:

$$s^3 + 20s^2 + Ks + 687.6 = 0$$

we may calculate the value of K that makes the system marginally stable, using the Routh-Hurwitz method.
We build the Routh array as given below:

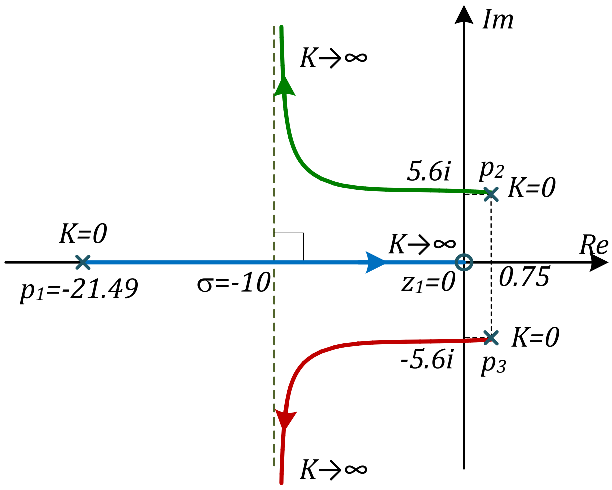


Figure 6.3: *Root locus sketch*

s^3 :	1	K
s^2 :	20	687.6
s^1 :	$\dfrac{20K - 687.6}{20}$	
s^0 :	687.6	

The system is marginally stable when the first column has a zero element. This means that

$$\frac{20K - 687.6}{20} = 0 \quad \Rightarrow \quad K = 34.38 \tag{6.6}$$

The values of the closed-loop poles on the imaginary axis can be found by simply replacing the value of K in the characteristic equation:

$$s^3 + 20s^2 + 34.38s + 687.6 = 0$$

and computing the roots:

$$s^2(s + 20) + 34.38(s + 20) = 0 \quad \Rightarrow \quad (s^2 + 20)(s + 34.38) = 0$$

$$s_{1,2} = \pm 5.86i, \quad s_3 = -34.38$$

The complex (imaginary) roots $s_{1,2}$ are the intersection of the root locus with the imaginary axis.

(b) The root locus from Figure 6.3 shows the location of the closed-loop poles of the system as the parameter K is varied between 0 and ∞. For our problem, the parameter of interest is R_2, given by: $R_2 = \dfrac{K \cdot 10^6}{678.6}$. When K increases from 0 to ∞, R_2 will increase also between the same limits. If $K = 34.38$, the parameter $R_2 = \dfrac{34.38 \cdot 10^6}{678.6} = 50 \cdot 10^3\ \Omega$.

From the root locus plot we may summarize the following observations related to the closed-loop system stability and transient behavior:

- For any value of $K \in [0,\ \infty)$ (or $R_2 \in [0,\ \infty)$), the closed-loop system has three poles and two of them are always complex conjugate. This means that the system response is oscillatory, for any positive K (or R_2).
- As K increases from 0 to 34.38 (or R_2 increases from 0 to $50 \cdot 10^3$), the closed-loop system is unstable because the complex poles move from the location of the

open-loop poles p_2 and p_3 towards the imaginary axis, but the paths are located in the right half-plane. The third pole (real negative pole) moves in the same time in the left half-plane, but two out of three poles of the closed-loop system have positive real parts, which makes the closed-loop system unstable.
The system response will exhibit oscillations growing in amplitude.

- For $K = 34.38$ or $R_2 = 50 \cdot 10^3$ the closed-loop system is marginally stable because it has two imaginary roots located at $s_{1,2} = \pm 5.86i$. The system response will oscillate with constant amplitude.
- As K increases over the value 34.38 (or R_2 over $50 \cdot 10^3$), the complex conjugate closed-loop poles move from the imaginary axis towards the asymptotes, along to their paths located in the left half-plane (they have negative real parts). The third pole is also on the left half-plane, moving on the negative real axis between -21.49 and 0. The closed-loop system is stable and, although the system response is underdamped, the oscillations decrease in amplitude.

SE 6.2 The heart electrical conduction system, which is responsible for commanding the heart's pumping action (muscle electrical stimulation leads to contraction), may suffer from different abnormalities or can be even blocked. For such cardiac problems a device called pacemaker is implanted in the human body, which transmits the necessary electrical stimulus that approximates the normal function of the system. Rate responsive pacemakers have control strategies such that the pacemaker changes the electrical stimulus transmitted to the heart in non-resting scenarios like exercise. One practical control variable for rate-responsive pacing is the oxygen saturation in the venous blood (SO_2). Consider the simplified (and linearized) open-loop dynamics of a SO_2 pacemaker cardiovascular system, [16]:

$$G(s) = \frac{Measured\ SO_2}{Prescribed\ SO_2} = K_C \cdot e^{-T_d s} \cdot \frac{A \cdot OXC}{1 + Ts} \tag{6.7}$$

where K_C is the pacemaker controller gain, T_d - the time delay, T - the time constant, A - a parameter depending on the exercise level and OXC - the oxygen consumption rate. The block diagram of the closed-loop system is presented in Figure 6.5.

Figure 6.4: *Pacemaker*

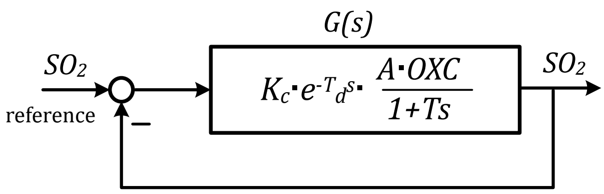


Figure 6.5: *Closed-loop SO_2 pacemaker cardiovascular system, [16]*

The time delay element can be approximated as (2/1 Padé approximation):

$$e^{-T_d s} \approx \frac{1 - \frac{2}{3}T_d s + \frac{1}{6}T_d^2 s^2}{1 + \frac{1}{3}T_d s}. \tag{6.8}$$

For an exercise level of $25W$ and the parameter values $A = 0.00183$, $T_d = 8.79$,

$OXC = 0.0841$, $T = 15$, the transfer function (6.7) becomes:

$$G(s) = K_C \frac{0.00183 \cdot 0.0841 \cdot (1 - 5.86s + 12.87s^2)}{(1 + 2.93s)(1 + 15s)} = $$
$$= K_C \cdot 1.53 \cdot 10^{-4} \frac{12.87s^2 - 5.86s + 1}{43.95s^2 + 17.93s + 1}. \quad (6.9)$$

(a) Draw the root locus of the closed-loop system for $K_C \in [0, \infty)$.

(b) Analyze the closed-loop stability and transient behavior for all positive values of K.

Solution:

(a) In order to sketch the root locus we rewrite the equation (6.9) as

$$G(s) = K \frac{s^2 - 0.46s + 0.08}{s^2 + 0.41s + 0.02} \quad (6.10)$$

where

$$K = K_C \cdot 1.53 \cdot 10^{-4} \frac{12.87}{43.95} = K_C \cdot 4.5 \cdot 10^{-5}$$

The open-loop transfer function from (6.10) can be factored in terms of two poles and two zeros: $G(s) = K \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$, since the numerator and denominator are both second-order polynomials.

The characteristic equation (see Figure 6.5) is:

$$1 + G(s) = 1 + K \cdot P(s) = 0, \quad \text{or} \quad 1 + K \frac{s^2 - 0.46s + 0.08}{s^2 + 0.41s + 0.02} = 0 \quad (6.11)$$

- First, we determine $n_z = 2$ open-loop zeros and $n_p = 2$ open-loop poles as the roots of the numerator and denominator polynomials of $G(s)$:

  zeros: $z_{1,2} = 0.23 \pm 0.16i$, poles: $p_1 = -0.34$, $p_2 = -0.07$

- Because the number of poles equals the number of zeros, there are no asymptotes. Each closed-loop pole will move from one open-loop pole towards an open-loop zero as the gain K increases from zero to infinity.
  Figure 6.6 shows a sketch with the open-loop poles and zeros, and the highlighted segment on the real axis which is part of the root locus (segment to the left of an odd number of poles and zeros).

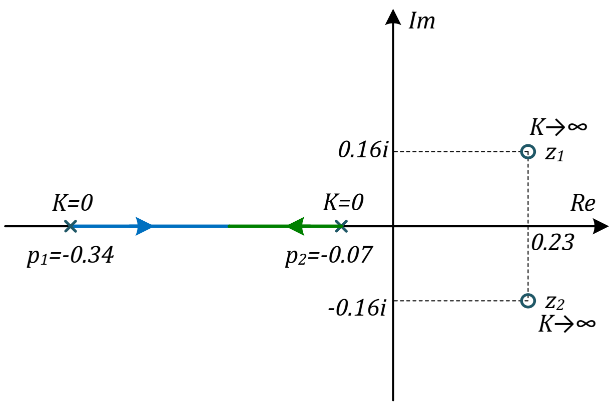


Figure 6.6: *Root locus sketch - Open-loop poles and zeros*

- We can further deduce that as K goes from 0 to ∞, the closed-loop poles must move from the real negative values p_1 and p_2 and towards the complex open-loop zeros z_1, z_2 located in the right half-plane and the paths must include the highlighted segment on the real axis. All these are possible only if:
  - there is a break-away point between p_1 and p_2,
  - the root locus plot crosses the imaginary axis since it has to pass from the left to the right half-plane, i.e. there is a value of K for which the system is marginally stable.
- The breakaway point can be calculated starting from the the characteristic equation (6.11). We write $K = -\dfrac{1}{P(s)} = p(s)$. Based on (6.11) we obtain

$$p(s) = \frac{s^2 + 0.41s + 0.02}{s^2 - 0.46s + 0.08}. \tag{6.12}$$

The break-away point is given by the solution of the equation $\dfrac{dp(s)}{ds} = 0$. Thus, after calculating the derivative of $p(s)$ from (6.12) and setting the numerator equal to zero, we obtain:

$$-0.87s^2 + 0.12s + 0.042 = 0, \tag{6.13}$$

which has the solutions $s_1 = 0.3$ and $s_2 = -0.16$. The first solution is not valid because is not in our interval of interest $(-0.34, -0.07)$, so the break-away point is:

$$s_x = s_2 = -0.16.$$

- The intersections with the imaginary axis as K increases can be calculated using the Routh-Hurwitz method. We use the characteristic equation (6.11) and compute the value of K that makes the closed-loop system marginally stable. The characteristic equation is written as:

$$s^2 + 0.41s + 0.02 + K(s^2 - 0.46s + 0.08) = 0$$
$$(K + 1)s^2 + (0.41 - 0.46K)s + 0.02 + 0.08K = 0$$

The Routh array is:

s^2 :	$K + 1$	$0.02 + 0.08K$
s^1 :	$0.41 - 0.46K$	
s^0 :	$0.02 + 0.08K$	

Since $K \geq 0$, the only element from the first column that can become zero so that the system is marginally stable is the one located on the second row:

$$0.41 - 0.46K = 0 \qquad \Rightarrow \qquad K = 0.89$$

The intersection with the imaginary axis is obtained by replacing this value in the characteristic equation, and we have:

$$1.89s^2 + 0.091 = 0, \quad \text{with the roots} \;\; s_{1,2} = \pm 0.22i$$

The root locus can further be sketched as in Figure 6.7.

- We may also compute the value of K at the break-away point, which will be useful for the analysis of the transient behavior of the closed-loop system. It is one of the properties of the points located on the root locus that the absolute value of the open-loop transfer function, computed at a specific point belonging to the root locus (the break-away point s_x in this case), is equal to 1. This

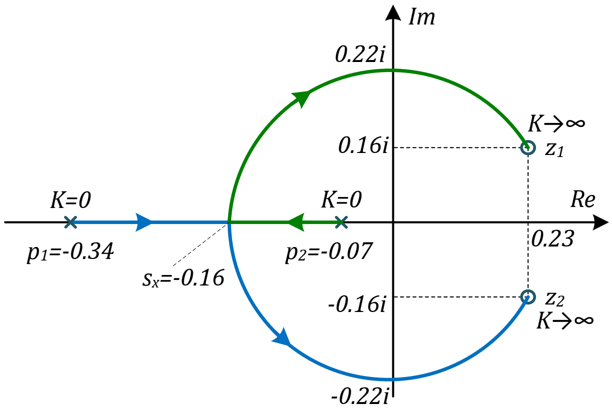


Figure 6.7: *Root locus sketch - Pacemaker Cardiovascular System*

condition can be written as:

$$|G(s)|_{s=s_x} = 1 \quad \Rightarrow \quad K_x = \frac{|s-p_1||s-p_2|}{|s-z_1||s-z_2|}|_{s=s_x} \text{ or } K_x = \left|\frac{s^2+0.41s+0.02}{s^2-0.46s+0.08}\right|_{s=s_x}$$

and the value of K at the break-away point $s_x = -0.16$ is: $K_x = 0.11$.

(b) From the final plot of the root locus from Figure 6.7 the behavior of the closed-loop system for $K \in [0,\ \infty)$ will be as follows:

- For $K \in [0,\ 0.11)$, the closed-loop poles move on the negative real axis from the open-loop poles p_1 and p_2 towards the breakaway point $s_x = -0.16$. In this range of K, both poles are real and negative, thus the system will be stable and overdamped. The system response has no oscillations (the damping factor $\zeta > 1$).
- For $K = 0.11$, the closed-loop poles are equal and also equal to s_x (the break-away point). The system is critically damped - the system response still has no oscillations.
- For $K \in (0.11,\ 0.89)$ the closed-loop poles become complex with negative real part. They are located in the left half-plane, therefore the closed-loop system is stable and underdamped. The system response will exhibit decaying oscillations.
- For $K = 0.89$, the closed-loop poles are located on the imaginary axis ($\pm 0.22i$). The system is marginally stable and the response will oscillate continuously.
- For $K > 0.89$, the closed-loop poles are complex and move in the right half-plane towards the open-loop zeros as K increases towards infinity. The real part of the closed-loop poles is positive so the system is unstable. The response will be exhibit oscillations growing in amplitude.

6.2 Proposed exercises

PE 6.1 Consider nine closed-loop systems with the open-loop pole-zero configurations as shown in Figure 6.8 (a-i).

(a) Sketch the root loci using the following rules:

- Root locus is symmetric about the real axis.
- The number of branches equals the number of open-loop poles.
- Root locus lies on the real axis to the left of an odd number of open-loop poles

and zeros.

- Root locus starts at the open-loop poles and ends at the open-loop zeros or infinity (along the asymptotes).

(b) Assign numerical values to each pole and zero and check the root locus in MATLAB using the *zpk* and *rlocus* functions.

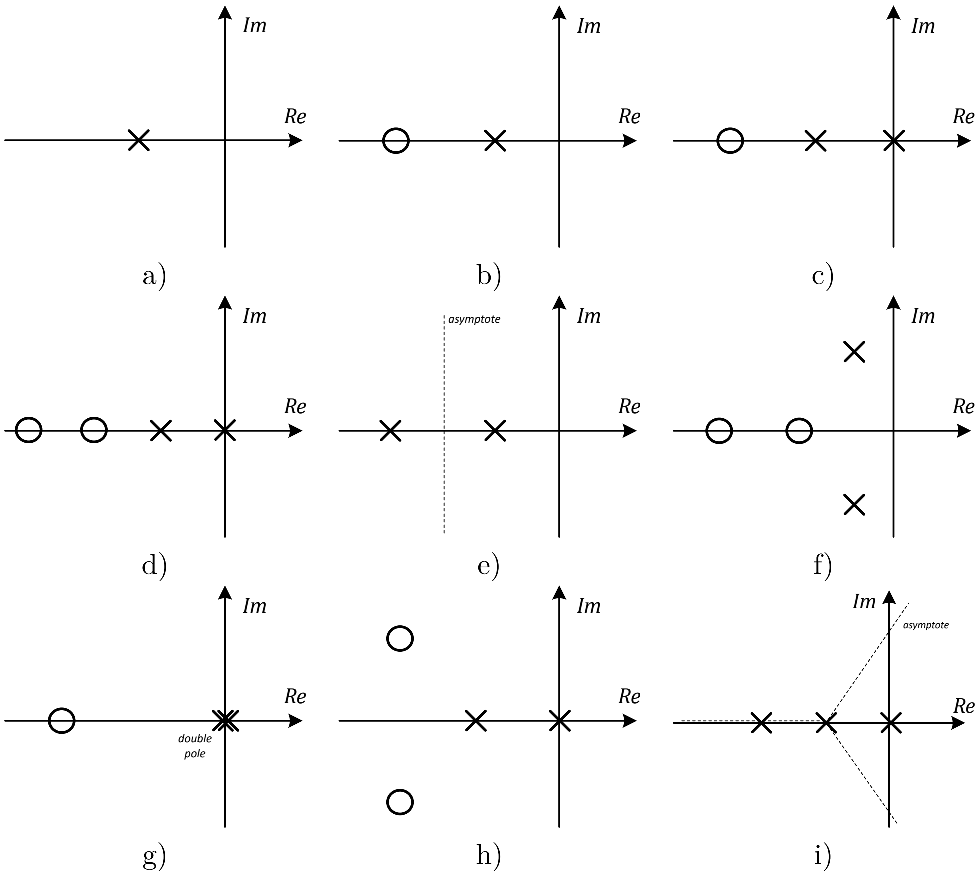


Figure 6.8: *Pole-zero maps*

PE 6.2 Consider the closed-loop systems with the following characteristic equations:

(1) $1 + k \cdot \dfrac{s+2}{s^2} = 0$ (2) $1 + k \cdot \dfrac{s+4}{s(s+3)} = 0$

(3) $1 + k \cdot \dfrac{s+1}{s^2-2s+2} = 0$ (4) $1 + k \cdot \dfrac{s-1}{(s+1)(s^2+1)} = 0$

(5) $1 + k \cdot \dfrac{1}{s(s^2+2s+2)} = 0$ (6) $1 + k \cdot \dfrac{s}{s^2+2s+2} = 0$

(7) $1 + k \cdot \dfrac{s+4}{s(s^2+4)} = 0$ (8) $1 + k \cdot \dfrac{s}{s(s^4-16)} = 0$

(a) Sketch the root locus for $k \in [0, \infty)$.

(b) Analyze the closed-loop system stability and transient behavior for all positive values of k.

(c) Verify the root loci in MATLAB using *rlocus* or *rltool*.

PE 6.3 Consider the closed-loop control system with the characteristic equation:

$$1 + \frac{ks(s+2)}{s^2+4s+8} = 0$$

(a) Sketch the root locus for $k \in [0,\ \infty)$.

(b) Find the gain where the closed-loop poles are equal.

(c) On the root locus plot, place the symbols $\square$ to indicate the complex closed-loop poles for which the system has a settling time $t_s = 8/3 = 2.66$sec.

PE 6.4 A closed-loop system has an open-loop transfer function:

$$G(s)H(s) = \frac{k}{s(s+1)^2}, \qquad k \geq 0$$

(a) Sketch the root locus for $k \in [0,\ \infty)$.

(b) Analyze the closed-loop system stability for all positive values of k.

(c) On the root locus plot, place the symbols $\square$ to indicate the complex closed-loop poles for which the system has a damping factor $\zeta = \dfrac{\sqrt{3}}{2}$.

PE 6.5 A unity feedback system has an open-loop transfer function

$$G(s) = \frac{k(s+1)}{s(s-1)(s+6)}$$

(a) Draw the root locus for $k \in [0,\ \infty)$.

(b) Analyze the stability of the closed-loop system for $0 \leq k < \infty$.

(c) On the root locus plot show the closed-loop complex poles with a maximum damping factor.

PE 6.6 Draw the root locus (including the asymptotes, intersection with imaginary axis, break-away point) of a system with the following open-loop transfer function :

$$G(s) = \frac{k}{s(s+1)(s^2+4s+13)}.$$

PE 6.7 The open-loop transfer function of a missile launching a satellite into space is, [9]:

$$G(s) = \frac{k(s^2+18)(s+2)}{(s^2-2)(s+12)}.$$

Figure 6.9: *Rocket launch*

(a) Draw the root locus.

(b) Use the root locus plot to analyze the stability of the closed-loop system for $k \in [0, \infty)$.

PE 6.8 Consider the feedback control system shown in Figure 6.10, where the controller has a variable parameter, z.

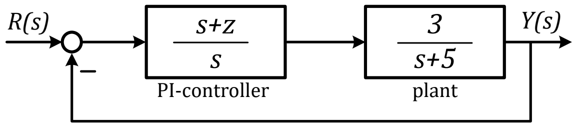


Figure 6.10: *Closed-loop control system*

(a) Sketch the root locus for $0 \leq z < \infty$.
(b) Determine the value of z such that the damping factor of the dominant closed-loop poles is $\zeta = \frac{\sqrt{2}}{2}$.
(c) Compute the overshoot of the closed-loop step response for $\zeta = \frac{\sqrt{2}}{2}$ and use *rltool* in MATLAB to plot the step response and check the result.

PE 6.9 Consider a closed-loop control system for a motorcycle as shown in Figure 6.12, [9]. The motorcycle moves in a straight line at a constant speed and the controller is a robot. The goal is to maintain the vertical position equal to the setpoint P_d (the angle with respect to the vertical).

Figure 6.11: *Motorcycle*

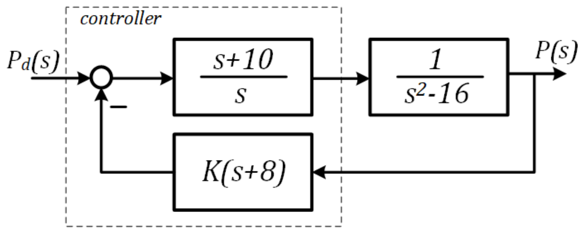


Figure 6.12: *Robot controlled motorcycle - block diagram, (adapted from [9])*

(a) Determine the value of the constant gain K, ($K > 0$) such that the closed-loop system is stable.
(b) Draw the root locus using *rltool* from MATLAB.
(c) Verify the result obtained at (a) using *rltool*.
(d) Find the gain K for which the settling time for a unit step input is $t_s = 1$ sec, using *rltool*.
(e) Analyze how the step response changes when the closed-loop poles move in the complex plane.

PE 6.10 Consider the closed-loop control for a positioning system, as shown in the block diagram from Figure 6.13.

The transfer function of the process to be controlled is $G(s) = \frac{1}{12s^2}$ and the controller has the transfer function $C(s)$.

The requirements for the closed-loop response are:

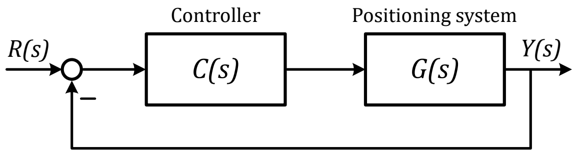


Figure 6.13: *Control of a positioning system*

- Closed-loop system is stable.
- Steady-state error equal to zero for a unit step input.
- Settling time less than 4 seconds.

Plot the root locus for $K > 0$ and discuss the requirements if the controller is:

(a) a proportional controller: $C(s) = K$. Can you find a value for K so that the requirements are fulfilled? Please argue the answer. Discuss the influence of K on the system response.

(b) a lead compensator: $C(s) = \dfrac{K \cdot (s+1)}{s+4}$. Analyze the location of the closed-loop poles as K increases from 0 to ∞ and discuss the requirements.

Hint. Use the MATLAB SISO Design tool - rltool*. Plot the root locus and the closed-loop system step response, change the location of the closed-loop poles and analyze the results.*

7 Frequency Response

Topics: Bode diagrams, filters, signal processing, phase margin

7.1 Solved exercises

SE 7.1 Consider an audio amplifier [2], connected to a woofer and a tweeter speakers as in Figure 7.1. The corresponding electrical circuit is shown in Figure 7.2, where we have a high pass RC filter in parallel with a low pass RL filter.

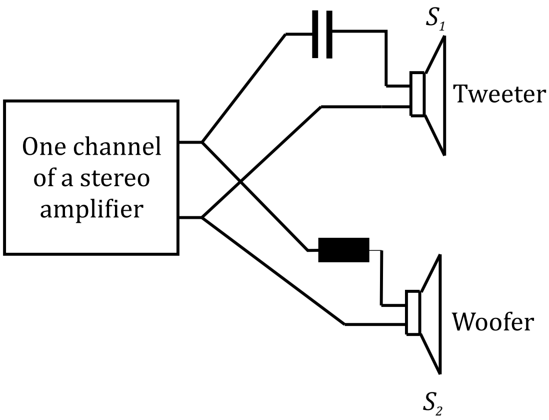


Figure 7.1: *Loudspeakers - adapted from [2]*

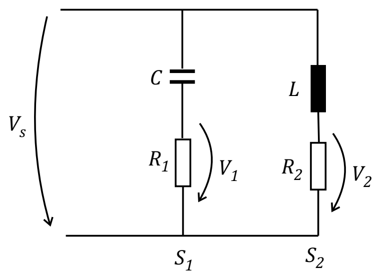


Figure 7.2: *Loudspeakers equivalent circuit -adapted from [2]*

(a) Determine the transfer function of each filter:

- $H_1(s)$ - from the input voltage V_s to the output voltage V_1,
- $H_2(s)$ - from the input voltage V_s to the output voltage V_2.

(b) Determine the values of the resistors R_1 and R_2, capacitance C and inductance L, such that the woofer reproduces sounds with frequencies less than 3kHz, while the tweeter reproduces sounds with frequencies lager than 3kHz.

Solution:

(a) The transfer functions H_1 for the RC circuit branch corresponding to the tweeter (S_1), and H_2 for the RL circuit branch corresponding to the woofer (S_2), can be determined using Kirchhoff laws and Laplace transform.

- For S_1 we can write:

$$
\begin{aligned}
V_s(t) &= V_c(t) + V_1(t) \quad \Rightarrow \quad V_C(t) = V_s(t) - V_1(t) \\
i_C(t) &= i_{R1}(t) \quad \text{or} \quad C\frac{dV_c(t)}{dt} = \frac{V_1(t)}{R_1}
\end{aligned}
$$

where i_C is the current through the capacitor (or resistor R_1) and V_C is the voltage drop on the capacitor. Then, we combine these relations, apply the Laplace transform with the initial condition $V_C(0) = 0$ and obtain:

$$Cs\,(V_s(s) - V_1(s)) = \frac{V_1(s)}{R_1} \quad \Rightarrow \quad R_1 C s V_s(s) = (sR_1C + 1)\,V_1(s)$$

The transfer function for the subsystem S_1 is:

$$H_1(s) = \frac{V_1(s)}{V_s(s)} = \frac{R_1Cs}{R_1Cs+1}, \tag{7.1}$$

- For S_2, we have:

$$\begin{aligned} V_s(t) &= V_L(t) + V_2(t) \quad \Rightarrow V_s(t) = L\frac{di_L(t)}{dt} + V_2(t) \\ i_L(t) &= i_{R2}(t) = \frac{V_2(t)}{R_2} \end{aligned}$$

where i_L and V_L are the current and the voltage drop on the inductor, respectively. Then, we combine the two relations above, apply the Laplace transform with the initial condition $i_L(0) = 0$ and obtain:

$$V_s(s) = \frac{L}{R_2} s V_2(s) + V_2(s).$$

The resulting transfer function is:

$$H_2(s) = \frac{V_2(s)}{V_s(s)} = \frac{1}{\frac{L}{R_2}s + 1}, \tag{7.2}$$

(b) We will determine the combination of values for L, R_2 (for S_2) and C, R_1 (for S_1) such that the cutoff frequency is 3 kHz ($\omega_c = 2\pi \cdot 3000$ rad/sec) in both cases. Also, for S_2, all sinusoidal components with the frequency $\omega < \omega_c$ will have the same magnitude as the input (low-pass filter), while for S_1, the magnitude of all components with the frequency $\omega > \omega_c$ will be the same as the input (high-pass filter).

- **The woofer (S_2).** The transfer function (7.2) has a Bode magnitude plot as presented in Figure 7.3.

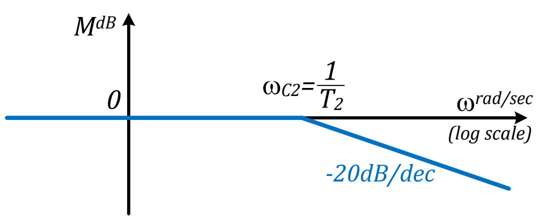


Figure 7.3: *Bode magnitude plot of the low-pass filter*

For the low-pass filter given by $H_2(s)$ the cutoff frequency is $\omega_{c2} = 2\pi \cdot 3000 rad/sec$. This means that the time constant is $T_2 = 1/\omega_{c2}$. From the transfer function we know that $T_2 = L/R_2$. So, if we choose a resistance of 1Ω, then $L = 1/(6000\pi)H$.
The Bode magnitude plot of the filter can be drawn directly as in Figure 7.4.

- **The tweeter (S_1).** For the Bode plot, we write the transfer function as a

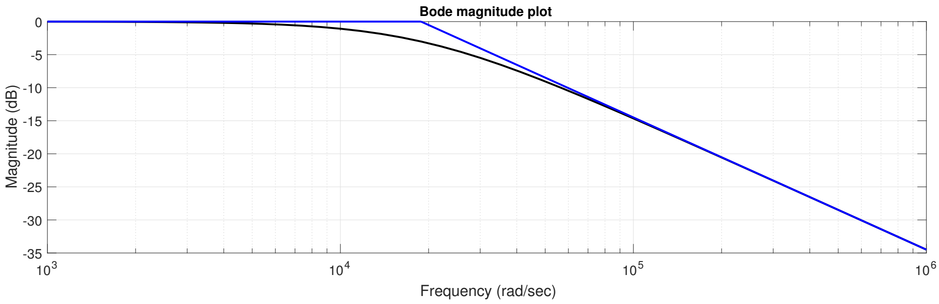


Figure 7.4: *Bode magnitude diagram of the low pass filter H_2*

product of two factors:

$$H_1(s) = H_{11}(s) \cdot H_{12}(s) \text{ where } H_{11}(s) = R_1 C s = K s, \quad H_{12}(s) = \frac{1}{sR_1C + 1} = \frac{1}{T_1 s + 1}.$$

In general, the Bode magnitude plot for such combination of factors is shown in Figure 7.5 where:

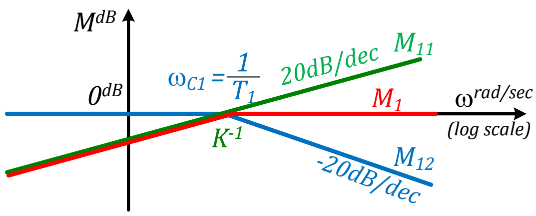


Figure 7.5: *Bode magnitude plot of the high-pass filter*

– The magnitude of H_{11} (M_{11}) has a slope of 20 dB/dec and crosses the ω-axis at K^{-1}.
– The high frequency asymptote of H_{12} (M_{12}) has a slope of -20 dB/dec and the corner frequency $\omega_{c1} = 1/T_1$
– On the right of ω_{c1} the sum of slopes of M_{11} and M_{12} will be 0 dB/dec and the resulting magnitude (M_1) will be 0 dB, because M_{11} crosses the ω-axis at the corner frequency: $\omega_{c1} = K^{-1} = \dfrac{1}{T_1}$.

For the high pass filter we have the same cutoff frequency, $\omega_{c1} = 2\pi \cdot 3000 rad/sec$, while the time constant of $H_2(s)$ is $T_2 = R_1 C$. So if we take again a 1Ω resistor, the capacitance will be $C = 1/(6000\pi)F$.
The Bode plots of H_{11} and H_{12} can be drawn directly as in Figure 7.6 a). By adding point by point the amplitude-frequency plots we obtain the Bode plot of the high-pass filter H_1 as in Figure 7.6 b).

SE 7.2 Consider the RLC tuner circuit for a FM Radio Receiver from Figure 7.7 [2]. A radio channel is selected by setting the resonant peak of the Bode magnitude plot at the corresponding frequency. For a $4\mu H$ coil, a $20k\Omega$ resistor, determine the range of the variable capacitor necessary to cover a frequency range from 88 to $100MHz$.

Solution:

Considering the input of the system to be the current and the output - the voltage

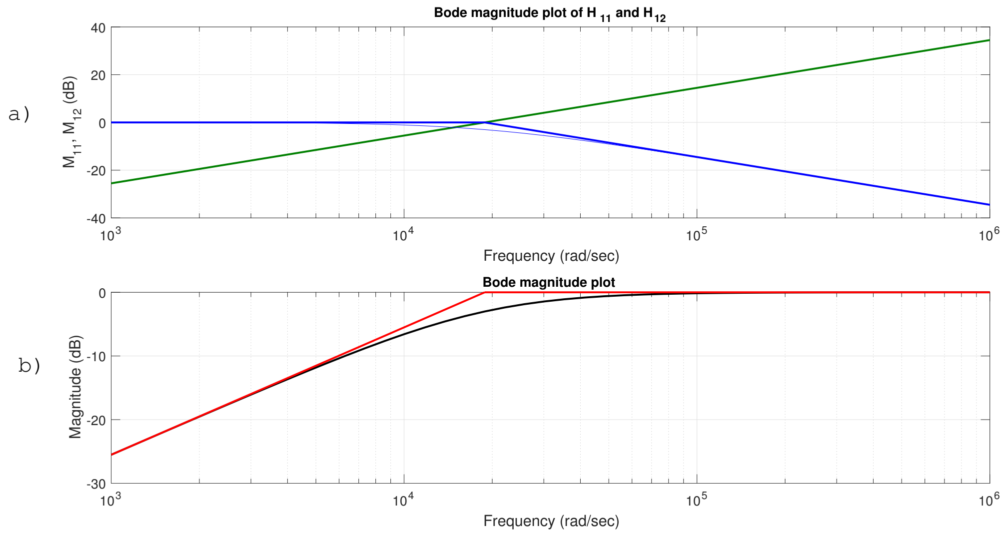


Figure 7.6: *a) Bode plots of H_{11} (green line) and H_{12} (blue line); b) the resulting Bode diagram of the high pass filter H_1*

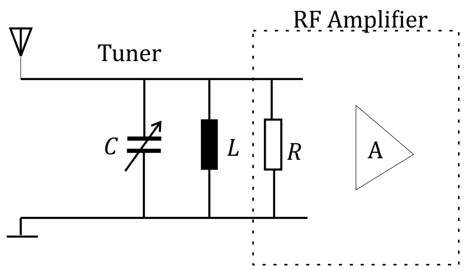


Figure 7.7: *Tuner circuit - adapted from [2]*

drop on the resistor, then by using Kirchhoff laws we can calculate the transfer function as:

$$H(s) = \frac{s}{Cs^2 + \frac{1}{R}s + \frac{1}{L}},$$

which can also be written as

$$H(s) = \frac{Ls}{LCs^2 + \frac{L}{R}s + 1}, \quad \text{or as a product:} \quad H(s) = Ls \cdot \frac{1}{LCs^2 + \dfrac{L}{R}s + 1}$$

A sketch of the Bode magnitude plot for $H_1(s) = Ls$, $H_2(s) = \frac{1}{LCs^2 + \frac{L}{R}s + 1}$ as well as the resultant $H(s) = H_1(s) \cdot H_2(s)$ is shown in Figure 7.8 (M_1, M_2 and M, respectively).

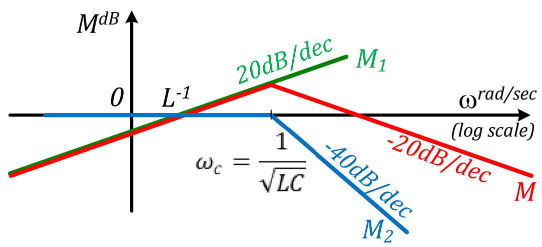


Figure 7.8: *Bode diagram of a bandpass filter*

Thus, we are dealing with a band-pass filter with the resonant frequency

$$\omega_c = \omega_n = \frac{1}{\sqrt{LC}}$$

that can select a specific radio channel.

The goal is to determine the range of C such that the filter can select radio channels within the specified frequency range: $f \in [88,\ 108]$ Mhz. Because the angular frequency is $\omega = 2\pi f$, the range is equivalent to $[553 \cdot 10^6,\ 679 \cdot 10^6]$ rad/sec.

The range of values of C such that:

$$553 \cdot 10^6 \leq \frac{1}{\sqrt{LC}} \leq 679 \cdot 10^6, \quad \text{where} \;\; L = 4\mu H = 4 \cdot 10^{-6} H$$

is $0.541 \cdot 10^{-12}\ F \leq C \leq 0.817 \cdot 10^{-12}\ F$, or $C \in [0.541,\ 0.817]$ pF.

The magnitude plot for three values of $C = \{0.541,\ 0.65,\ 0.817\}$ pF are shown in Figure 7.9.

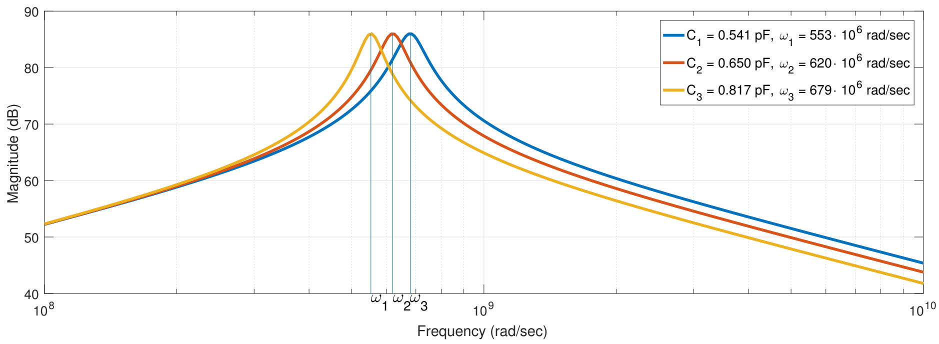


Figure 7.9: *Bode diagram of Tuner filter*

Thus, according to how one tunes the varying capacitor within the given range, a different radio channel is selected through the resonant peak that amplifies that specific signal and transmits it forward to the RF amplifier.

SE 7.3 Consider the attitude control system of an aircraft as shown in Figure 7.10 [14]. The goal is to control the positions of the fins of the aircraft, using DC-motors as actuators. The measured position θ_y should track the reference (prescribed) position θ_r. In order to achieve this goal, multiple control loops are usually employed besides the position feedback: velocity feedback and current feedback.

The parameter values are:

$K_s = 1$, $K_1 = 10$, $K_2 = 0.5$, $K_t = 0$, $R_a = 5$, $L_a = 0.003$,
$K_i = 9$, $K_b = 0.0636$, $N = 0.1$, $J_t = 0.0002$, $B_t = 0.015$,

and consider the following values for the control gain $K \in \{7.248,\ 14.5,\ 181.2,\ 273.57555\}$.

(a) Determine the phase margin of the overall system and asses the stability of the closed-loop system for all values of K.

(b) Determine the overshoot and the settling time of the closed-loop system in each case and correlate them to the corresponding phase margin.

Solution:

(a) In order to use the phase margin method for assessing the stability of the closed-loop system, we need to determine the open-loop transfer function of the system shown in Figure 7.10. From the block diagram, the open-loop transfer function is calculated

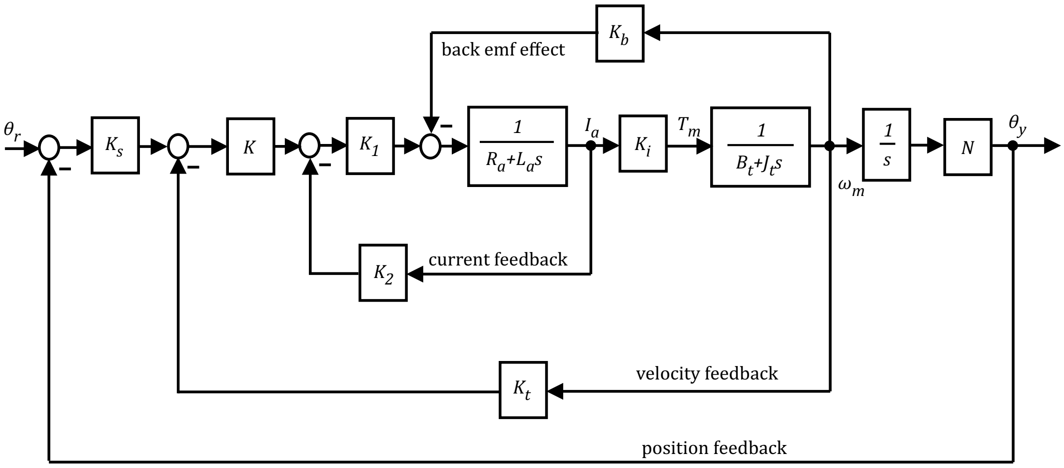


Figure 7.10: *Attitude control system of an aircraft - adapted from [14]*

as:

$$G_0(s) = \frac{K_s K_1 K_i K N}{s[L_a J_t s^2 + (R_a J_t + L_a B_t + K_1 K_2 J_t)s + (R_a B_t + K_1 K_2 B_t + K_i K_b + K K_1 K_t K_i)]} \tag{7.3}$$

and the equivalent representation of the system is given in Figure 7.11.

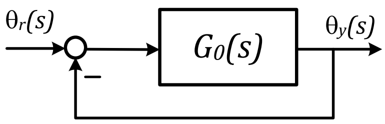


Figure 7.11: *Reduced block diagram of the aircraft attitude control system*

By substituting the parameter values we further obtain:

$$G_0(s) = \frac{1.5 \cdot 10^7 K}{s(s+400.26)(s+3008)}. \tag{7.4}$$

The phase margin allows us to determine the stability of the closed-loop system based on the transfer function of the open-loop system - in our case (7.4).

The bode plot of G_0, for different values of K is given in Figure 7.12.

The phase margin criterion states that the closed-loop system is stable if the phase margin (PM) is positive. The phase margin for each case can be evaluated from the Bode plot or can be computed using the function *margin* in MATLAB. We see in Figure 7.12 that for $K = 273.56$ the phase margin is zero, which means that the closed-loop system is marginally stable. For smaller gains the system is stable.

(b) The step response of the closed loop system for each value of K can be computed in MATLAB using the *step* function. Figure 7.13 shows the step response of the closed-loop system for all values of K and Table 7.1 shows comparatively the frequency domain results (phase margin) and the time domain results (overshoot - σ, settling time - t_s). We see that as the gain decreases, the phase margin increases (the system becomes "more" stable), the overshoot decreases, and the settling time increases. In other words, as the gain decreases the system becomes slower and the step response is more damped.

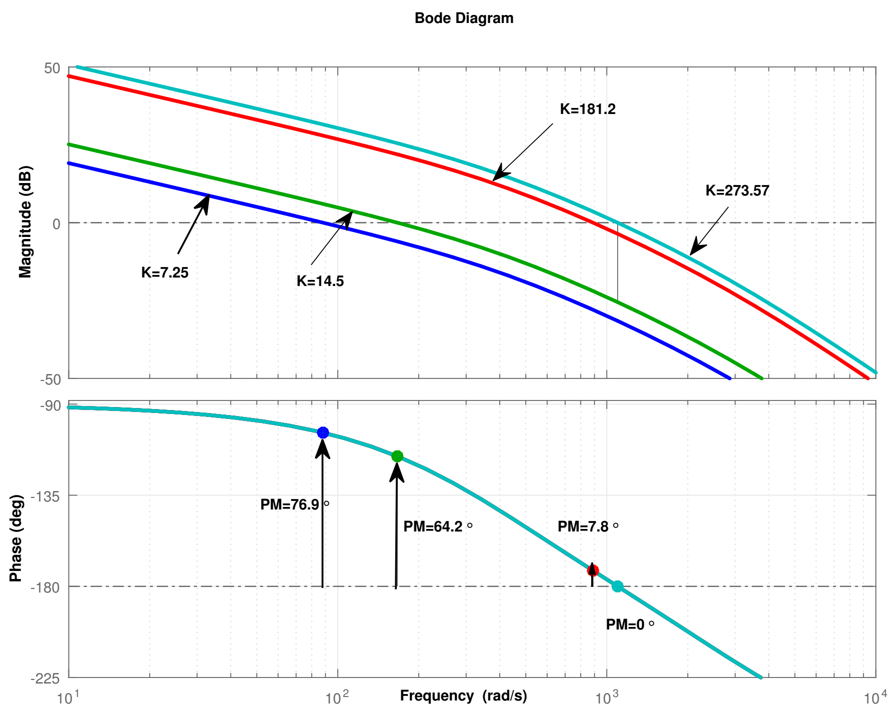


Figure 7.12: *Bode plot for the aircraft attitude control system*

K	$PM(°)$	$\sigma(\%)$	$t_r(s)$
7.25	75.9	0	0.03
14.5	64.2	4.7	0.02
181.2	7.8	78.9	0.06
273.57555	0	100	∞

Table 7.1: *Summary of the results for aircraft control system*

7.2 Proposed exercises

PE 7.1 Consider four systems with the following transfer functions:

$$G_1(s) = \frac{0.1(s+10)}{s+1}, \quad G_2(s) = \frac{10(s+1)}{s+10}, \quad G_3(s) = \frac{10}{s^2+s+1}, \quad G_4(s) = \frac{s^2+s+1}{s^2+s+10}$$

(a) Plot the system response for a sinusoidal input, $r(t) = sin(t)$, using the MATLAB function *lsim* for a time interval $t \in [0,\ 30]$ sec.

(b) For each system, analyze the magnitude and phase angle of the output signal and compare it with the input signal. Determine if the systems have phase lead or phase lag.

(c) Draw the Bode diagrams, using the MATLAB function *bode* and read from the plots the magnitude and the phase angle for each output signal, when the input is $r(t) = sin(t)$.

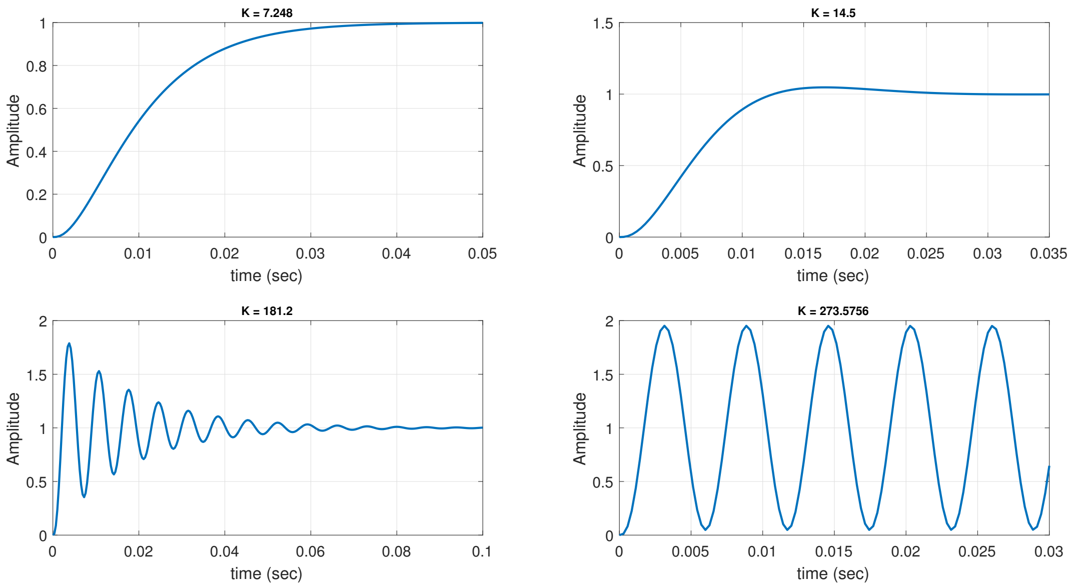


Figure 7.13: *Step response of the closed-loop system for all values of K*

PE 7.2 **(a)** Sketch the Bode diagram for the systems with the following transfer functions:

$$G_1(s) = \frac{s^2}{(10s+1)^2}, \quad G_2(s) = \frac{10s+10^4}{s^2+s+1}, \quad G_3(s) = \frac{10^9 s}{(s+1000)(s+10^7)},$$

$$G_4(s) = \frac{s+10}{10s+1}, \quad G_5(s) = \frac{10(s+10)}{s^2+s+1}, \quad G_6(s) = \frac{10^3 s(10^{-1}s+1)}{(s+1)(10^2 s+1)}.$$

(b) Determine the frequencies for which the system amplifies or attenuates the sinusoidal input signals.

(c) For each system, use the Bode diagram to determine the magnitude of the output signal if the input is:

$$u_1(t) = sin(t),$$
$$u_2(t) = 0.1 sin(10^{-3}t),$$
$$u_3(t) = 3 sin(100t).$$

PE 7.3 Consider a second-order system having the transfer function:

$$G(s) = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} \tag{7.5}$$

Several input signals have been applied to the system and the outputs have been recorded. The inputs are sinusoidal signals $r(t) = \sin \omega t$, where the frequency is $\omega \in \{1,\ 2,\ 5,\ 8,\ 9.5,\ 10,\ 20,\ 100\}$ rad/sec. The outputs, for each input frequency, are shown in Figure 7.14.

Using the frequency response, determine the system parameters: K, ω_n and ζ, by completing the following steps:

(a) Sketch the Bode diagram for the transfer function given by (7.5).

(b) Obtain an experimental Bode magnitude plot using the output signals from Figure 7.14, as follows:

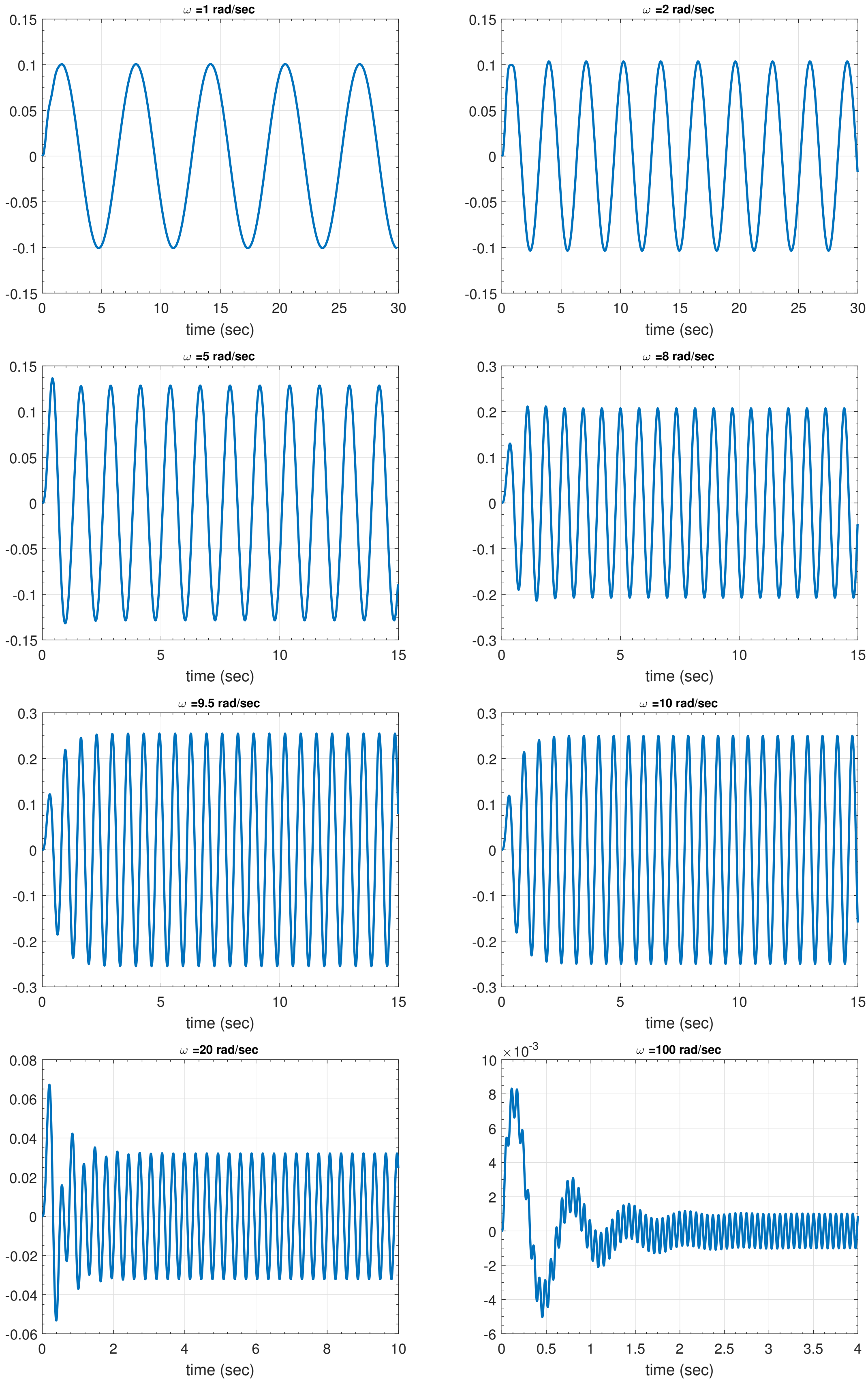


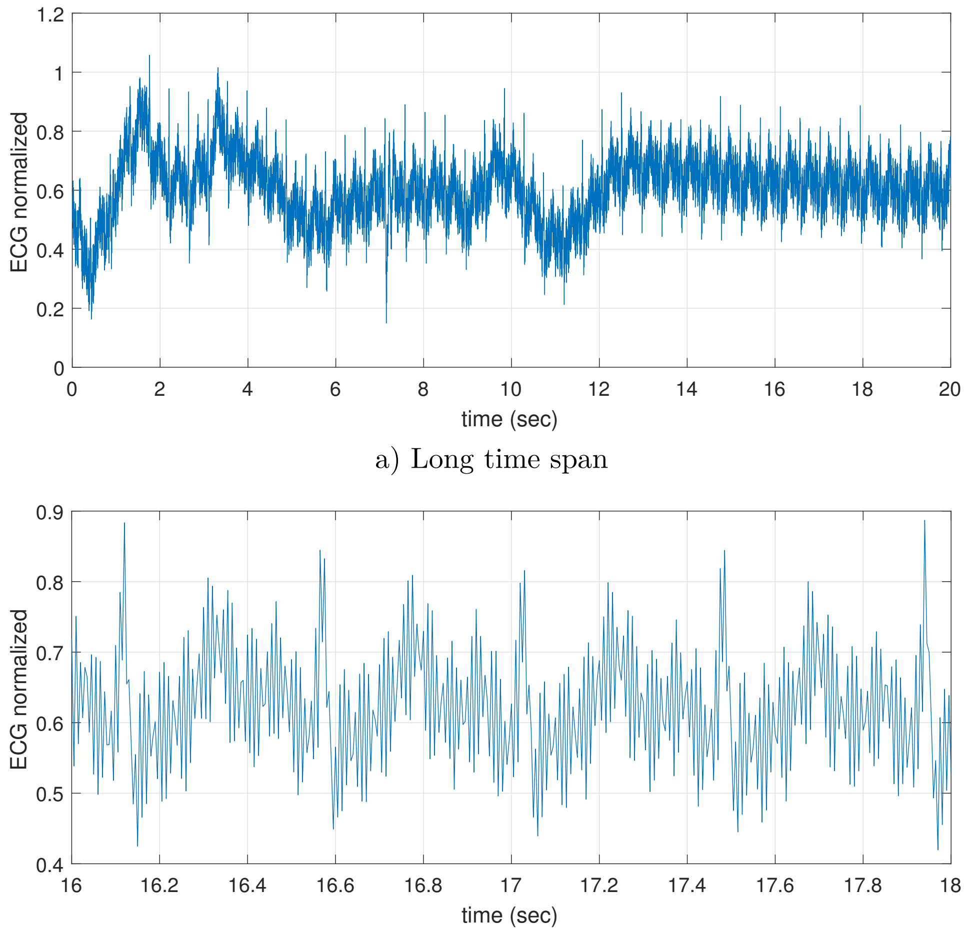
Figure 7.14: *Output signals for various input frequencies*

- Read the magnitude of the output signal at steady-state and divide it to the magnitude of the input. Save all numbers in an array $\mathbf{M} = [M_1, \ \ldots, \ M_8]$.
- Convert the numbers from the array $\mathbf{M}$ into decibels and plot them versus all values of ω on a logarithmic scale (use the MATLAB function *semilogx*).

(c) Compare the Bode diagrams obtained at **(a)** and **(b)** and determine the system parameters from the plots.

PE 7.4 ECG signal processing problem.

Consider the ECG measurements from [26] shown in Figure 7.15. The measurements include low frequency disturbances due to breathing and coughing of the patient, and high frequency noise due to the (low quality) sensors.



a) Long time span

b) Short time span

Figure 7.15: *ECG measurements*

The ideal ECG measurement, taken on a short time span, should look as in Figure 7.16. The 3 peaks (R, T and P - from the largest to the smallest), along with their timing, are very important for medical diagnosis. For example, the period between the R peaks is used for determining the heart rate (reciprocal of the heart period)[1].

It is obvious that medical doctors cannot use the ECG measurements from Figure 7.15 for diagnosis purposes. In order to solve this problem, a band-pass filter (see Figure 7.17)

[1] For details see also: http://www.medicine.mcgill.ca/physio/vlab/cardio/introecg.htm

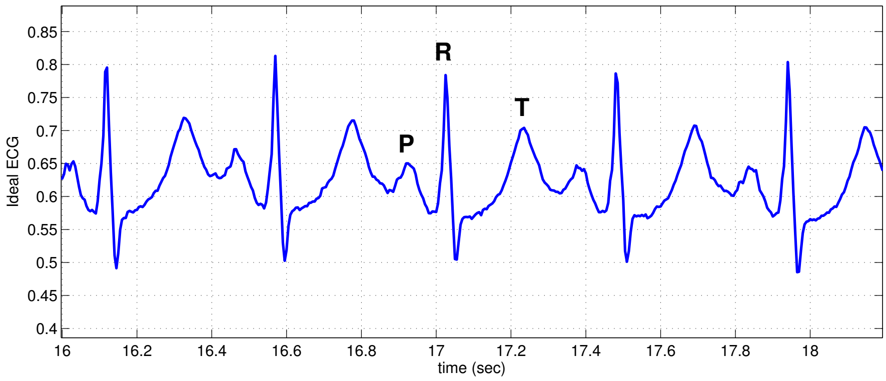


Figure 7.16: *Ideal ECG measurements (short time span)*

must be designed to remove the low and high frequency artifacts. The frequency of interest for ECG measurements is usually between 0.5 Hz and 100 Hz [26].

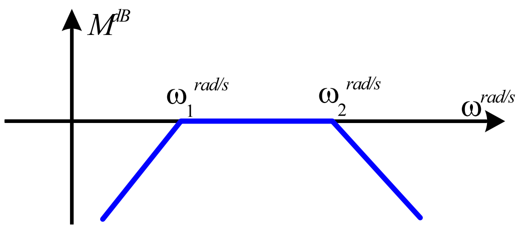


Figure 7.17: *Magnitude plot of a band-pass filter*

The goal of this application is to design a band-pass filter for removing the low and high frequency components and to compare it with a commonly used filter (Butterworth filter). The cut-off frequencies of the filters will be $f_1 = 0.5$Hz and $f_2 = 50$Hz.

(a) Read data from *ECGdata.txt* file. The example code is given in Listing 7.1.

Listing 7.1: *readECGdata.m*

```
close all
clear all
clc
% read data from ECGdata.txt

fileID = fopen('ECGdata.txt', 'r');      % open ECGdata.txt for read access
A = fscanf(fileID, '%f %f',[2,Inf]);     % read data from file in array A
fclose(fileID)                           % close ECGdata.txt
time = A(1,:);                           % save first row of data in variable "time"
necg = A(2,:);                           % save second row of data in variable "necg"
```

The data is organized now in two row vectors: *time* with 4000 values of time between 0 and 20 seconds, and *necg* with the normalized values of the ECG signal for all moments of time. Plot *necg* versus *time* and the figure will be similar to Figure 7.15 (a).

(b) Design a Butterworth filter using the function *butter* with the following specifications:

- Filter type: analog bandpass
- Filter order: 8
- Filter cutoff frequencies: $\omega_1 = 0.5 * 2 * \pi$ rad/s and $\omega_2 = 50 * 2 * \pi$ rad/s;

The code example is:

```
[num,den]=butter(4, [0.5*2*pi 50*2*pi], 'bandpass', 's');
```

Obs. The first input argument in function butter is $n = 4$, but a band-pass filter will have the order $2n = 8$. The function butter returns the numerator and denominator polynomials of the filter transfer function.

(c) Plot the Bode diagram of the filter using the *bode* function.

(d) Determine the filtered ECG signal by using the *lsim* function, with the filter transfer function and the ECG signal as input (see Figure 7.18).

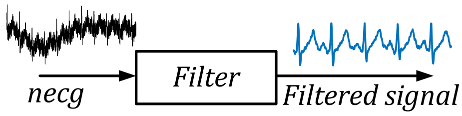


Figure 7.18: *Filter*

Plot the ECG signal and the filtered ECG signal on the same figure and analyze the results. Can you read the three peaks on the plot? Notice the differences between the two signals:

- on the entire time interval,
- on a time interval between 16 and 18 sec,
- on a time interval between 2 and 4 sec.

(e) Design an 8-th order bandpass filter using first-order elements, as close as possible to the previous Butterworth filter.

(i) First try to find the right combination of elements in order to obtain a band-pass Bode diagram (Figure 7.17) with the same cutoff frequencies as for the Butterworth filter and the slopes of the lines are +20dB/dec and −20 dB/dec.

(ii) Determine the parameters (gain and time constants of the filter transfer function) that give the desired cutoff frequencies.

(iii) Multiply the previously obtained filter transfer function by itself iteratively and notice how the Bode diagram changes (use MATLAB) through each iteration. A 4 time multiplication should provide a Bode diagram similar to the Butterworth filter. Plot both Bode diagrams on the same figure and compare them.

(f) Plot the ECG signal, the filtered ECG signal with the Butterworth filter, and the filtered ECG signal with the custom filter (from point (iii)) and compare the results (zoom in).

PE 7.5 Consider a low pass filter implemented as an analog electrical circuit as in Figure 7.19.

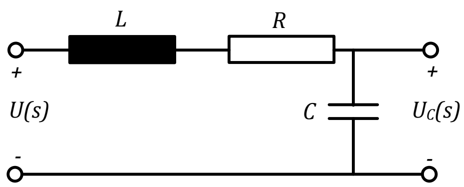


Figure 7.19: *Electrical circuit*

(a) Determine the transfer function between the input voltage U and the output the voltage U_C across the capacitor.

(b) Sketch the Bode diagram of the filter.

(c) Determine the parameters of the electrical circuit such that the filter has a corner frequency ω_c of $0.1 rad/sec$ and a damping factor ζ of 0.7.

Hint. Choose one of the parameters and calculate the other two.

PE 7.6 Consider a Butterworth filter with the electrical circuit from Figure 7.20, where $R = 1\Omega$, $C = 0.2F$, $L = 0.2H$.

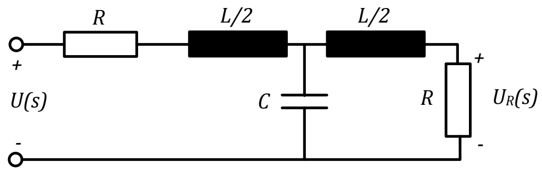


Figure 7.20: *Filter*

(a) Prove that the transfer function between the input voltage U and the resistor voltage U_R is $G(s) = \dfrac{R}{(0.5Ls + R)(0.5s^2LC + CRs + 2)}$ and for $R = \sqrt{\dfrac{L}{C}}$ it can be written as: $G(s) = \dfrac{1}{2\left(\frac{1}{\omega_c}s + 1\right)\left(\frac{1}{\omega_c^2}s^2 + \frac{1}{\omega_c}s + 1\right)}$

(b) Determine the corner frequencies and sketch the Bode diagram of the filter.

(c) Draw the poles of the filter in the complex plane and prove that they are distributed evenly on the left half-plane semi-circle centered at the origin and with a radius equal to ω_c.

PE 7.7 Determine the transfer functions for the systems having the Bode diagrams (magnitude plot) shown in Figure 7.21.

PE 7.8 Consider the eye movement control from Figure 7.22 [20], with the parameter values: $G/J = 14400 \quad rad^2 \cdot s^{-2}$, $B/J = 24 \quad rad \cdot s^{-1}$. Determine the phase margin for $K_v = 0.01$ and $K_v = -0.002$. Analyze the stability of the closed-loop system.

PE 7.9 Consider a control system with negative unity feedback, with the open-loop transfer function $G(s) = \dfrac{a(s+1)}{s^2}$. Determine the value of the parameter a such that the phase margin of the system is 45°. Check the results in MATLAB using the *margin* function.

PE 7.10 A control system with negative unity feedback has the open-loop transfer function

$$G(s) = \frac{k(s+3)}{s(s+1)(s+5)}.$$

Draw the Bode diagram in MATLAB and determine the gain k such that the phase margin is 40°.

PE 7.11 Consider the position control system of a Mars rover (terrain vehicles used by NASA, powered by solar energy and remote controlled from Earth) from Figure 7.23 [9]. Determine the gain K which maximizes the phase margin.

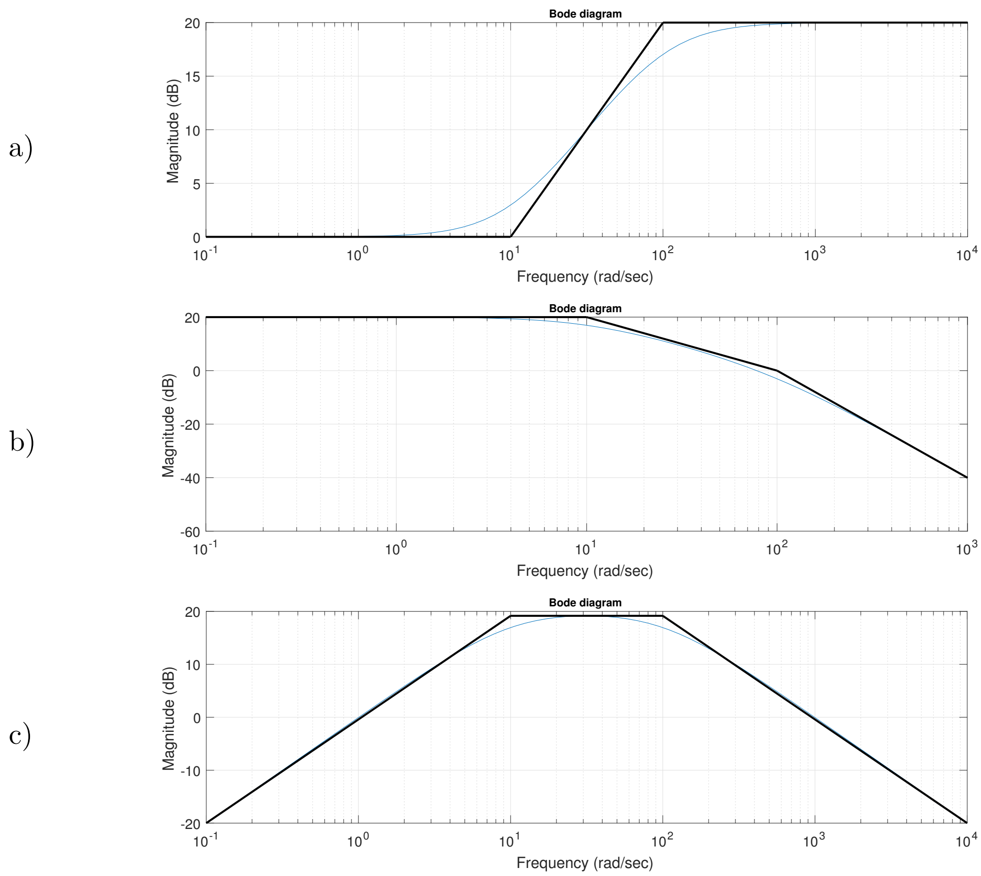


Figure 7.21: *Bode magnitude diagrams*

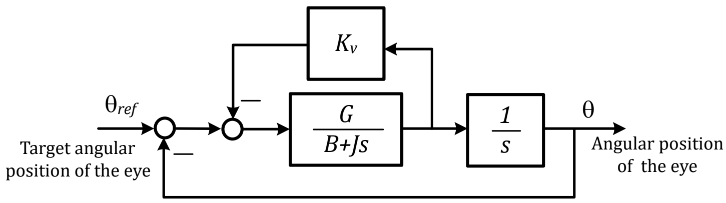


Figure 7.22: *Block diagram of eye movement control*

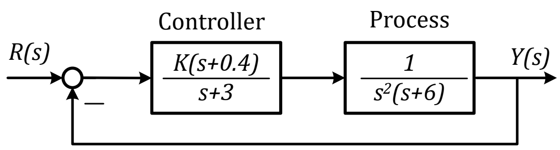


Figure 7.23: *Position control system of a Mars rover*

8 Output feedback control

Topics: Lead-Lag Controllers, PID Controllers, Stabilization, Tracking, Root-locus design, Tuning methods

8.1 Solved exercises

SE 8.1 Unmanned Aerial Vehicles (UAVs) like Quadcopters can be controlled by using multiple feedback loops and PID controllers. The control structure usually consists of a feedback loop for the angular velocities and positions (attitude: pitch, roll, yaw), and a second feedback loop for linear velocities and positions (altitude, lateral/sideway, forward).

Consider the AR Drone Parrot 2.0 [1] shown in Figure 8.1.

Figure 8.1: *AR Drone Parrot 2.0 Quadcopter*

The transfer function that relates the sideway velocity to the control input is [8]:

$$H(s) = \frac{7.081s + 16.82}{s^3 + 4.728s^2 + 14.65s + 5.83}. \tag{8.1}$$

Design a PD controller (Figure 8.2) such that the closed-loop system has two complex conjugate poles that will result in an overshoot of 10% and a settling time of about 0.5 seconds for a second-order system.

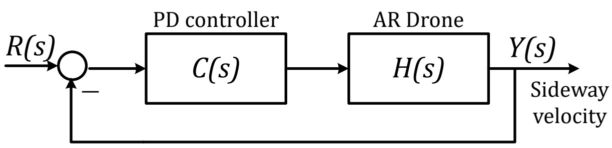


Figure 8.2: *Control system for the sideway velocity*

Solution:

The control performance specifications are $M_p = 10\%$ and $t_s = 0.5sec$. From these specifications we can determine the damping ratio and the natural frequency of a pair of complex poles:

$$M_p = e^{-\pi\zeta/\sqrt{(1-\zeta^2)}} \cdot 100 \quad \Rightarrow \quad \zeta = \sqrt{\frac{ln^2(M_p/100)}{\pi^2 + ln^2(M_p/100)}} = 0.59$$

$$t_s = \frac{4}{\zeta\omega_n} \quad \Rightarrow \quad \omega_n = \frac{4}{\zeta t_s} = 13.53.$$

Then, the closed-loop poles that we need to impose are:

$$r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}j = -8 \pm 10.9j.$$

The process transfer function can be rewritten as:

$$H(s) = 7.81 \cdot \frac{s + 2.37}{(s + 0.46)(s^2 + 4.27s + 12.67)} = K\frac{(s + z_1)}{(s + p_1)(s + p_2)(s + p_3)}, \tag{8.2}$$

where the zero and the poles of the system are: $-z_1 = -2.37$, $-p_1 = -0.46$, $-p_{2,3} = -2.13 \pm 2.85j$.

The PD controller has the transfer function:

$$C(s) = K_d(s + z_d). \tag{8.3}$$

The PD controller parameters are calculated so that the root locus of the compensated system will pass through the desired locations $r_{1,2}$. The phase angle condition that ensures that the poles $r_{1,2}$ are on the root locus of the compensated system is:

$$\angle C(s)H(s)|_{s=r_1} = -180°, \tag{8.4}$$

and will give us the zero of the PD controller.

The gain of the PD controller will be computed from the magnitude condition at the specific pole location r_1:

$$|C(s)H(s)|_{s=r_1} = 1. \tag{8.5}$$

Figure 8.3 shows the open-loop poles and zeros in the complex plane, as well as their angle contribution to the condition (8.4).

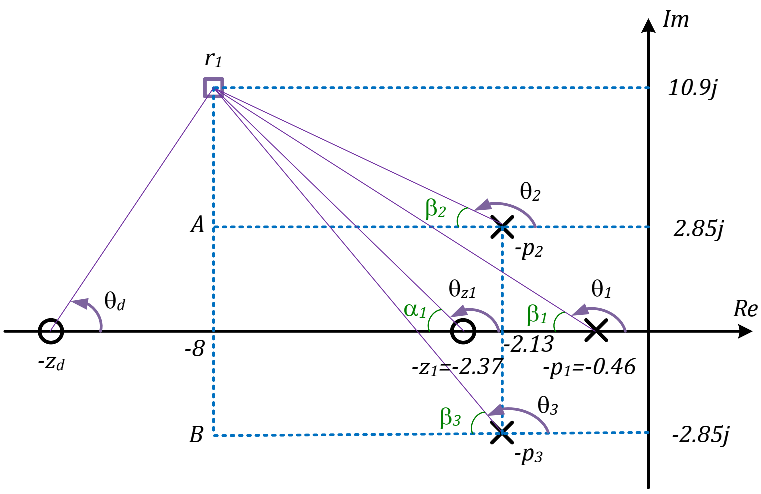


Figure 8.3: *Zeros and poles in the complex plane*

The phase condition (8.4) can be written as:

$$\angle C(s)H(s)|_{s=r_1} = \left(\angle(s+z_1) + \angle(s+z_d) - \angle(s+p_1) - \angle(s+p_2) - \angle(s+p_3)\right)|_{s=r_1} = -180^\circ$$

which becomes (see Figure 8.3):

$$\theta_{z1} + \theta_d - \theta_1 - \theta_2 - \theta_3 = -180^\circ$$

or:

$$(180^\circ - \alpha_1) + \theta_d - (180^\circ - \beta_1) - (180^\circ - \beta_2) - (180^\circ - \beta_3) = -180^\circ$$

The angles α_1, β_1, β_2 and β_3 can be determined as follows:

- α_1 from the right triangle with the vertices r_1, $(-8,0)$, $(-z_1,0)$:

  $$\alpha_1 = \arctan\frac{10.9}{8-2.37} = 62.7^\circ$$

- β_1 from the right triangle with the vertices r_1, $(-8,0)$, $(-p_1,0)$:

  $$\beta_1 = \arctan\frac{10.9}{8-0.46} = 55.3^\circ$$

- β_2 from the right triangle with the vertices r_1, A, $-p_2$:

  $$\beta_2 = \arctan\frac{10.9-2.85}{8-2.13} = 53.9^\circ$$

- β_3 from the right triangle with the vertices r_1, B, $-p_3$:

  $$\beta_3 = \arctan\frac{10.9+2.85}{8-2.13} = 66.9^\circ.$$

It results that $\theta_d = 66.6^\circ$. The parameter z_d is further calculated from the right triangle with the vertices at r_1, $(-z_d,0)$, $(-8,0)$:

$$\tan\theta_d = \frac{10.9}{z_d-8} = 2.31 \quad \Rightarrow \quad z_d = 12.7.$$

The zero of the controller is actually $-z_d = -12.7$ and the transfer function is $C(s) = K_d(s+12.7)$.

Next, the PD gain K_d is calculated from the magnitude condition (8.5):

$$|C(s)H(s)|_{s=r_1} = \left|K_d(s+12.7)\cdot\frac{7.81(s+2.37)}{(s+0.46)(s^2+4.27s+12.67)}\right|_{s=-8+10.9j} = 1$$

and we obtain:

$$K_d = 1.73.$$

The root locus of the compensated system is shown in Figure 8.4. The complex poles r_1 and r_2 are on the root locus but the closed-loop system has a third real negative pole, closer to the origin than the complex ones. The influence of this pole is reduced, in this case, because it is very close to one of the closed-loop system zeros (the same as the process zero).

The unit step response of the closed-loop system is shown in Figure 8.5. Notice that the overshoot and settling time are close to the ones imposed. The differences between the desired performance specifications and the ones obtained in simulation are due to the fact that in design we focused only on the complex closed-loop poles. The zero introduced by the controller has the effect of increasing the overshoot and, in general, influences the overall performance.

Note: The PD controller can be designed in MATLAB using the SISO Design Tool. One

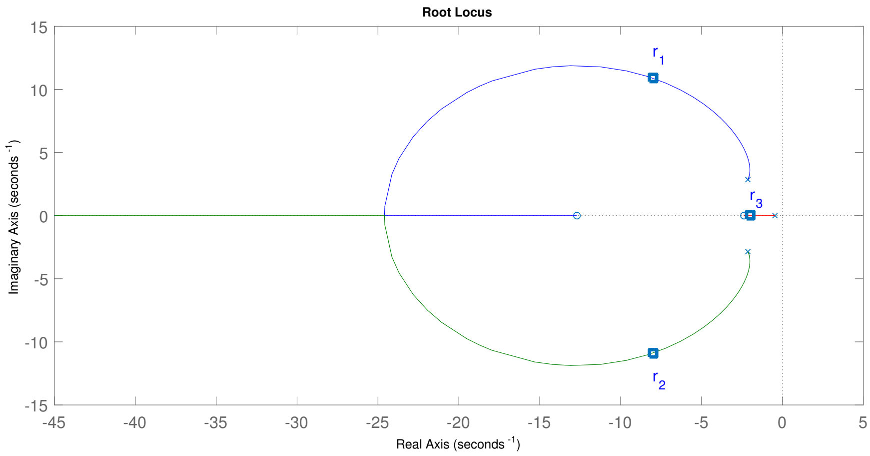


Figure 8.4: *Root locus of the closed-loop system*

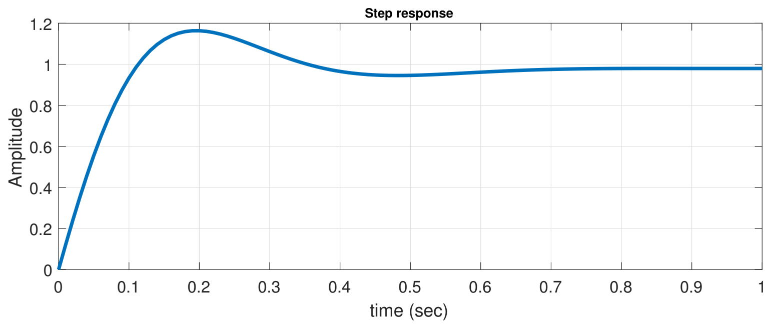


Figure 8.5: *Step respone of the closed loop system - sideway velocity control*

possible approach is:

- Define the transfer function H in MATLAB.
- Call the function *rltool* with H as input argument.
- Define the constraints (in the SISO design window - rightclick - Design requirements): set the settling time and the overshoot to 0.5 sec and 10% respectively. The imposed poles are at the intersection of the constraint lines in the complex plane.
- Add a real zero anywhere on the real axis (this will be the PD zero).
- Move the zero until the root locus passes through the intersection point of the constraint lines
- Move the closed-loop poles along the root locus at the point of intersection: through this action we actually set the closed-loop system gain
- The resulting controller is given in the Control and Estimation Tools Manager window.

SE 8.2 Magnetic levitation systems are often used in high speed trains. On a single axis, the problem is similar - in principle - with the one where a metal ball levitates at a (vertical) position depended on the magnetic force generated by the electromagnet (Fig. 8.6).

If u is the voltage control signal, and y the ball position measured through a photo emitter-detector pair, the transfer function of the linearized system is, [6]:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{3148}{s^2 - 4551}. \tag{8.6}$$

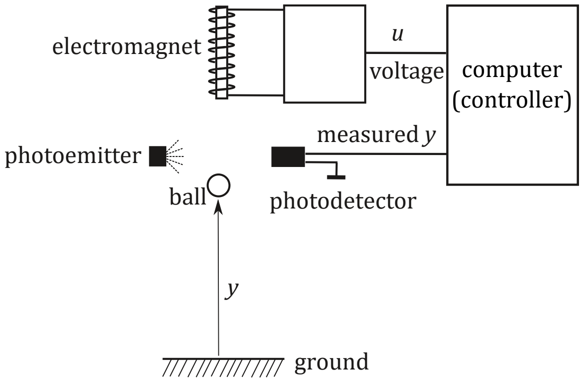


Figure 8.6: *Single axis magnetic levitation system*

(a) Design an ideal PD controller with the transfer function:

$$C(s) = K(s + z_d) \tag{8.7}$$

that ensures a settling time $t_s = 0.01$ seconds and a damping factor $\zeta = 0.95$ for the step response of the closed-loop system (Figure 8.7). Calculate K and z_d.

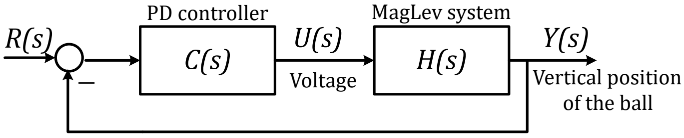


Figure 8.7: *Block diagram of the position control system*

(b) Compute the steady-state error for a unit step input.

Solution:

(a) The closed-loop poles $r_{1,2}$ that meet the performance specifications are computed as follows:

$$t_s = \frac{4}{\zeta\omega_n} \quad \Rightarrow \quad \omega_n = \frac{4}{\zeta t_s} = \frac{4}{0.95 \cdot 0.01} = 421.053$$

$$r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}j = -400 \pm 131.47j.$$

The process transfer function can be written as:

$$H(s) = \frac{3148}{s^2 - 4551} = \frac{3148}{(s + 67.46)(s - 67.46)}.$$

By adding the controller, the open-loop transfer function of the control system becomes:

$$C(s)H(s) = \frac{3148 \cdot K(s + z_d)}{(s + 67.46)(s - 67.46)}$$

The controller zero, $-z_d$, can be determined from the phase angle condition:

$$\angle C(s)H(s)|_{s=r_1} = -180^\circ$$

which ensures that the poles $r_{1,2}$ are on the root locus of the compensated system. The angle of $C(s)H(s)$ measured for $s = r_1$ is:

$$\left(\angle(s + z_d) - \angle(s + 67.46) - \angle(s - 67.46)\right)|_{s=r_1} = -180^\circ$$

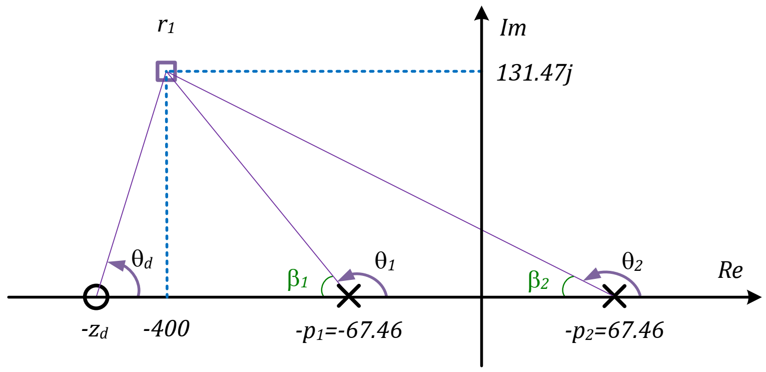


Figure 8.8: *Step respone of the closed-loop system*

From Figure 8.8 we have:

$$\theta_d - \theta_1 - \theta_2 = \theta_d - (180^\circ - \beta_1) - (180^\circ - \beta_2) = -180^\circ$$

and

$$\theta_d = 180^\circ - \beta_1 - \beta_2 = 180^\circ - \arctan\frac{131.47}{400-67.46} - \arctan\frac{131.47}{400+67.46} = 142.72^\circ$$

Because the angle to the controller zero is obtuse $90^\circ < \theta_d < 180^\circ$, the location of $-z_d$ is actually on the right of -400, or $-z_d > -400$ and the value z_d can be computed from:

$$\tan(180^\circ - \theta_d) = \frac{131.47}{400 - z_d} \quad \Rightarrow \quad z_d = 227.29$$

The PD controller is now: $C(s) = K(s + 227.29)$. The controller gain is obtained from the magnitude condition:

$$|C(s)H(s)|_{s=r_1} = \left|\frac{3148 \cdot K(s+227.29)}{(s+67.46)(s-67.46)}\right|_{s=-400+131.47j} = 1$$

$$K = \left|\frac{(s+67.46)(s-67.46)}{3148(s+227.29)}\right|_{s=-400+131.47j} = 0.254.$$

The step response of the closed-loop system is shown in Figure 8.9. Notice that the real overshoot is larger than the one imposed through design. A damping factor $\zeta = 0.95$ would give an overshoot $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \cdot 100 = 0.007\%$ for a second-order system with no zeros. In our case, the closed-loop system has also a zero $-z_d$ introduced by the PD controller which has the effect of increasing the overshoot of the step response.

(+) Do the design in MATLAB using *rltool*. Try to further lower the overshoot by moving the closed-loop poles.

(b) The steady-state error for a unit step input is computed from Figure 8.7:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s(R(s) - Y(s)) = \lim_{s\to 0} sR(s)\frac{1}{1 + C(s)H(s)}$$

By replacing the transfer functions of the controller and the process we obtain:

$$e_{ss} = \lim_{s\to 0} s\frac{1}{s}\frac{1}{1 + 0.254(s+227.29)\frac{3148}{s^2-4551}} = -0.02$$

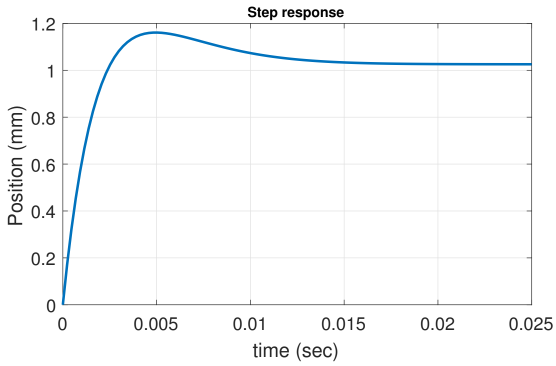


Figure 8.9: *Step respone of the closed-loop system*

SE 8.3 Consider a four-way-hydraulic-valve that commands a linear actuator (a double-acting single rod). Such systems can be used to control a robot joint, when high forces/torques are required. The valve is controlled electronically, while the position of the rod is measured with a potentiometer. The transfer function of the linearized dynamics is [14]:

$$G(s) = \frac{1}{T_c s + 1} \cdot \frac{2AK_q/K_c}{Ms^2 + (D + 2A^2/K_c)s + K_l}, \tag{8.8}$$

with the parameters:

- $K_q = 0.359\ \ m^2/s$ (flow gain of the control valve),
- $K_c = 1.70 \cdot 10^{-11}\ \ m^3/Pa/s$ (pressure-flow coefficient),
- $Am = 550 \cdot 10^{-6}\ \ m^2$ (area of the main spool),
- $M = 4\ \ kg$ (mass of the load),
- $D = 1\ \ N \cdot s/m$ (damping of the load),
- $K_l = 1\ \ N/m$ (Load spring stiffness),
- $A = 1.1 \cdot 10^{-3}\ \ m^2$ (area of the actuator piston),
- $T_c = A_m/K_q$ (valve time constant).

Design a PID controller such that the step response of the closed loop system has an overshoot less than 5% and is as fast as possible.

Solution:

For standard low-order process models, analytical optimization-based methods for tuning PID controllers have been developed in the literature, which give excellent results if the model is accurate enough. We will use here the Modulus Optimum (BO) method described in [3], which is very often used in practice because it usually gives good results and it is easy to apply. The method imposes a frequency response as close as possible to $0dB$, for low frequencies.

First we rewrite the process transfer function (8.8) in order to highlight the time constants and the process gain:

$$G(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)(T_3 s + 1)}, \quad \text{with} \quad T_1 > T_2 > T_3 \tag{8.9}$$

and

$$\begin{aligned} K &= (2A \cdot K_q/K_c)/K_l = 4.6459 \cdot 10^7, \\ T_1 &= 1.4235 \cdot 10^5, \\ T_2 &= T_c = 0.001532, \\ T_3 &= 2.8099 \cdot 10^{-5}. \end{aligned}$$

The ideal PID controller in parallel form has the transfer function:

$$G_c(s) = K_p + K_i \frac{1}{s} + K_d s. \tag{8.10}$$

For a third order process with 3 real negative poles and no zeros, the relations that give the controller parameters with the BO method are [3]:

$$\begin{aligned} K_p &= \frac{T_1 + T_2}{2T_3 K}, \\ K_i &= \frac{K_p}{T_1 + T_2}, \\ K_d &= K_p \frac{T_1 T_2}{T_1 + T_2} \end{aligned}$$

In the end, the calculated PID parameter values are:

$$K_p = 54.52, \quad K_i = 0.00038, \quad K_d = 0.084$$

The step response of the closed-loop system is shown in Figure 8.10.

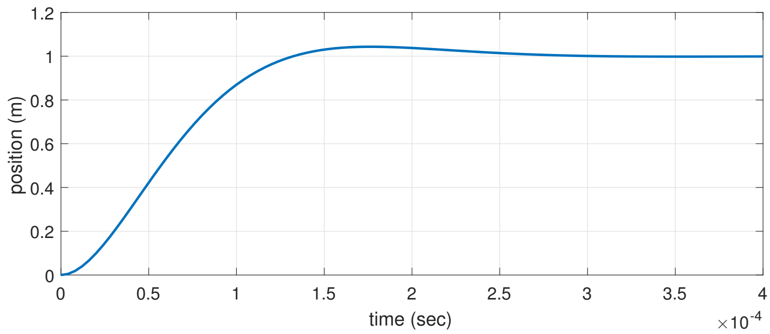


Figure 8.10: *Step response of the closed-loop system*

The design method used here guarantees an overshoot of the step response of the closed-loop system of about 4% and a settling time $t_s = 6/\omega_n$, where $\omega_n = 0.7/T_3$. Indeed, the overshoot of the step response from Figure 8.10 is 4.32% and the settling time is $t_s = 2.4$ sec.

SE 8.4 Automatic mean arterial pressure regulation is crucial in ensuring acceptable quality of donated organs for heart-beating brain death subjects. Clinical trials were recently performed for the design of controllers based on pharmacological models, with experiments on animals.

Consider the dynamic model of nitroglycerin pharmacology identified from experiments on anesthetized pigs [28]:

$$G(s) = \frac{Y_{map}(s)}{I_{nitro}(s)} = \frac{K}{(T_1 s + 1)(T_2 s + 1)} e^{-s\tau}, \tag{8.11}$$

where the input I_{nitro} represents nitroglycerine infusion rate, while the output Y_{map} represents mean arterial pressure.

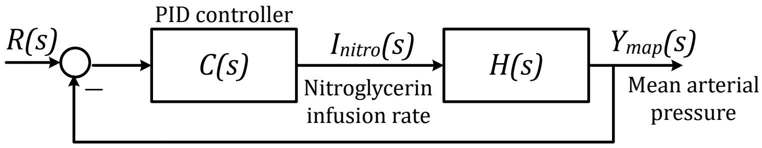


Figure 8.11: *Closed-loop control system for mean arterial pressure*

This low-order model represents an over-simplification of the actual dynamics, but it is usually good enough for controller design due to the robustness of the control feedback loop.

Design a PID controller:

$$C(s) = k_p + k_i \frac{1}{s} + k_d s \tag{8.12}$$

for the plant $G(s)$, with the parameters [28]:

$$T_1 = 30 \text{ sec}, \quad T_2 = 70 \text{ sec}, \quad \tau = 40 \text{ sec}, \quad K = 1.06.$$

The controller must ensure the following performance specifications for the step response of the closed-loop system:

- the overshoot less than 20%
- the settling time less than 500 seconds.

Solution:

There are wide range of techniques for designing PID controllers reported in the literature. For our current design problem, we will limit ourselves to a classical method proposed by Haalman [3] and a more recent method proposed by Skogestad [27]. Both methods are model based.

(a) Haalman's method was derived by imposing the desired open-loop transfer function (an integrator with delay element), considering also that the process poles are compensated by the controllers zeros (a detailed discussion can be found in [3] - pp. 190). The PID controller can be calculated based on the following relations:

$$k_p = \frac{2(T_1 + T_2)}{3K\tau},$$
$$k_i = \frac{2}{3K\tau},$$
$$k_d = \frac{2T_1T_2}{3K\tau}.$$

Thus, for our process $G(s)$, the controller parameters are:

$$k_p = 1.572, \quad k_i = 0.0157, \quad k_d = 33.019$$

The step response of the closed loop system is illustrated in Figure 8.12. The input is a step signal $r(t) = 10, \ t \geq 0$.

(b) Now, we will design the PID controller with Skogestad's method. The method is derived by imposing the desired closed-loop transfer function (a first order element with delay) and approximating the delay using Taylor approximations (for details,

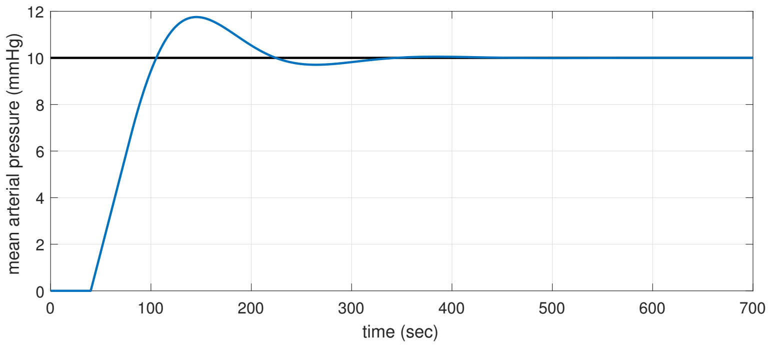


Figure 8.12: *Step respone of the closed loop system - PID designed using Haalman method*

see [27]). The PID controller can be calculated based on the following relations:

$$\begin{aligned} k_i &= \frac{T_1}{K(c+\tau)\alpha}, \\ k_p &= k_i(\alpha + T_2), \\ k_d &= k_i(\alpha T_2), \\ \alpha &= min\{T_1, 4(c+\tau)\}. \end{aligned}$$

The design parameter c can be chosen in order ensure a good trade-off between a fast response and stability (robustness). Usually $c = \tau$ gives good results.
For the PID controller $C(s)$ we obtain the parameters:

$$k_i = 0.0118, \quad k_p = 1.179, \quad k_d = 24.764.$$

The closed-loop response for a step input $r(t) = 10,\ t \geq 0$, is illustrated in Figure 8.13. The results seem better than those obtained with the Haalman method and we can improve them by further tuning c (for larger values we can decrease the overshoot at the price of increasing the settling time).

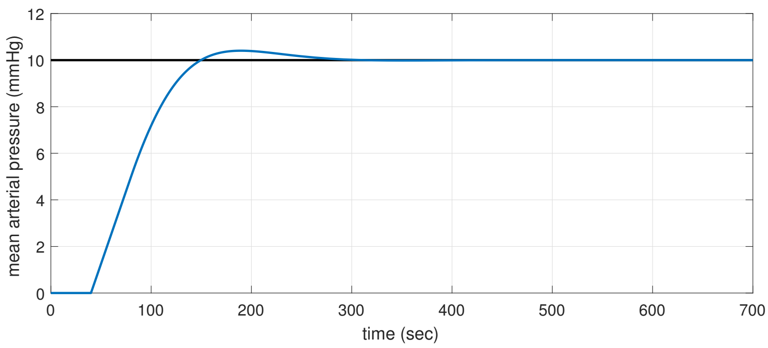


Figure 8.13: *Step respone of the closed loop system - PID designed using Skogestad method*

The performance specifications are fulfilled in both cases: the overshoot is less than 20% and the settling time is smaller than 500 seconds for the step response of the closed-loop system.

8.2 Proposed exercises

PE 8.1 For a system having the transfer function $G(s) = \frac{1}{s^2}$ determine a lead compensator with the transfer function $G_c(s) = \frac{k(s+z)}{s+p}$, with $|z| < |p|$, as shown in Figure 8.14, so that the dominant closed-loop poles are located at $r_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$.

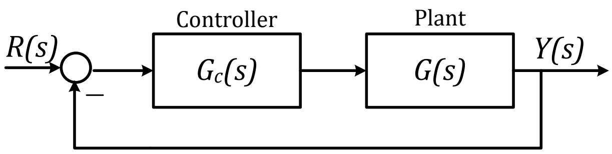


Figure 8.14: *Closed-loop control system*

PE 8.2 Consider a unity negative feedback control system with the transfer function of the plant $G(s) = \frac{1}{s(s+1)}$.

(a) Design an ideal PD compensator with the transfer function $G_{PD}(s) = K_P + K_D s$ so that the closed-loop system has a settling time $t_s = 4$ sec and a damping factor $\zeta = 0.5$.

(b) Add another compensator $G_c(s) = \frac{s+z}{s+p}$ with $|z| > |p|$ (see Figure 8.15) so that the velocity error constant is $K_{vcomp} = 20$ and the dominant closed-loop poles are located in approximately the same position as in case (a).

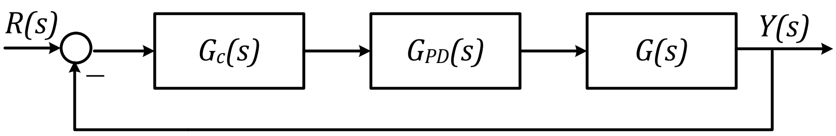


Figure 8.15: *Closed-loop control system*

PE 8.3 For a closed-loop control system (Figure 8.16) with the plant transfer function $G(s) = \frac{1}{s+1}$ determine a compensator with the transfer function $G_c(s) = \frac{k(s+a)}{s}$ so that the overshoot of the step response is about $M_p = 10\%$ and the settling time is about $t_s = 1$ *sec*.

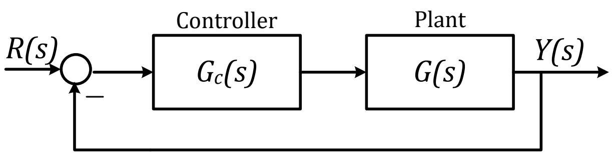


Figure 8.16: *Closed-loop control system*

(a) Determine the controller analytically and check the result with *rltool*. Analyze the step response of the closed-loop system and check the overshoot and the settling time obtained with this controller.

(b) Move the closed-loop poles on the root locus (this will change the controller gain) and try to obtain the desired specifications.

PE 8.4 Consider the control system shown in Figure 8.17. The specifications for a unit step response are:

- the settling time is $t_s = \dfrac{2}{3}$ sec
- the peak time is about $t_p = \dfrac{\pi}{2\sqrt{3}}$ sec.

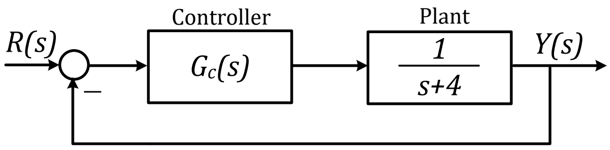


Figure 8.17: *Closed-loop control system*

(a) Can the criteria be satisfied with a proportional P-controller? Justify your answer.
(b) Design a PI controller to meet the requirements.

PE 8.5 Consider a closed-loop control system from Figure 8.18, where $G(s) = \dfrac{1}{(s+1)(s+4)}$ is a plant to be controlled and $G_c(s) = \dfrac{k(s+z)}{s+p}$ is a compensator with the parameters k, z and p.

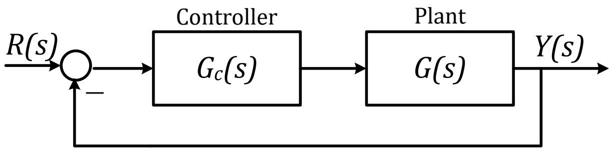


Figure 8.18: *Closed-loop control system*

(a) Determine the pole of the compensator, $-p$, so that the steady-state error for a unit step input is zero.
(b) Determine the gain k and the zero, $-z$, of the compensator so that the dominant closed-loop poles are located at $-1 \pm \sqrt{3}j$.

PE 8.6 Consider a unity feedback control system where the transfer function of the plant is $G(s) = \dfrac{1}{s^2}$. It is required that for the dominant complex poles of the closed-loop system the settling time is about $t_s = 4$ seconds and the peak time is about $t_p = \dfrac{\pi}{\sqrt{3}}$.

(a) Can you fulfill the requirement with a proportional (P) controller? Why?
(b) Design a controller with the transfer function $G_c(s) = \dfrac{k(s+z)}{s+p}$ with $0 < z < p$, so that the requirement is fulfilled.

PE 8.7 Consider a closed-loop control system as shown in Figure 8.19 where the plant has a transfer function $G(s) = \dfrac{5}{(0.5s+1)(s+1)(10s+1)}$.

(a) Compute the static position error constant and the steady-state error of the uncompensated system (when $G_c(s) = 1$) for a unit step input. Plot the step response of the closed-loop system and check the steady-state error.
(b) Determine the dominant poles of the uncompensated closed-loop system.

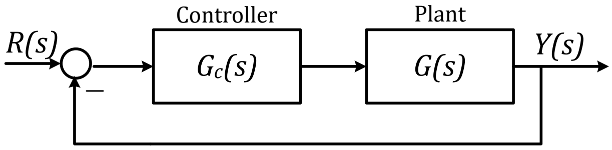


Figure 8.19: *Closed-loop control system*

(c) It is desired to increase the static position error constant 10 times, without changing to much the location of the dominant closed-loop poles. Design a lag compensator $G_c(s) = \dfrac{s+z}{s+p}$, with $0 < p < z$, to meet this specification.

(d) Compare the step response of the closed-loop system for $G_c(s) = 1$ and for $G_c(s)$ designed at point **(c)** and comment the result.

PE 8.8 For a unity feedback closed-loop control system with a plant $G(s) = \dfrac{5}{s^2 + 2s + 2}$:

(a) Determine the static position error constant K_p and the closed-loop poles, when the compensator transfer function is $G_c(s) = 1$.

(b) Design a lag compensator $G_c(s)$ that will increase the position error constant 30 times, while keeping the closed-loop complex poles in approximately the same location. Compare the step response of the closed-loop system with, and without the compensator.

(c) Compute the poles of the closed-loop system with, and without the compensator and compare their values. Determine also the real zero of the compensated closed-loop system and compare it with the real pole.

PE 8.9 Consider a closed-loop unity feedback control system with a plant having the transfer function $G(s) = \dfrac{1}{s(s+3)}$:

(a) Design a proportional-derivative (PD) controller $G_{PD}(s)$, such that the dominant closed-loop poles are located at $r_{1,2} = -3 \pm 3j$.

(b) For the closed-loop system determine the static velocity error constant, K_v.

(c) Add a lag compensator $G_c(s)$ (see Figure 8.20) in order to obtain a static velocity error constant $K_{vcomp} = 10K_v$.

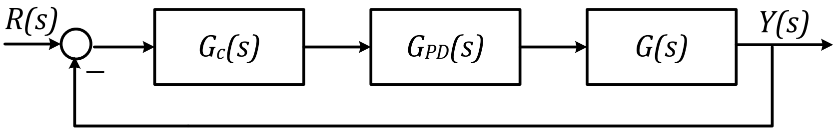


Figure 8.20: *Closed-loop control system*

PE 8.10 DC motors are extensively used in control applications, including robotic manipulators and mobile robots, machine tool industry, electric vehicles, etc. Transfer function models of DC motors can be determined as presented in SE 2.2 and [9, 14].

If the input signal is the voltage applied to the motor, U and the output is the rotational speed ω, the transfer function of a DC motor is written as:

$$G(s) = \frac{\omega(s)}{U(s)} = \frac{k}{(Ls + R)(Js + b) + kk_e}. \tag{8.13}$$

where:

- k_e is the electromotive force constant, $k_e = 5 \cdot 10^{-2}\ V/rad/sec$
- k - the motor torque constant, $k = 5 \cdot 10^{-2}\ Nm/A$
- R - the armature resistance, $R = 3\ \Omega$
- L - the armature inductance, $L = 0.5H$
- J - the rotor inertia, $J = 9 \cdot 10^{-3}\ kgm^2$
- b - coefficient of viscous friction, $b = 2 \cdot 10^{-2}\ Nms$

(a) Plot the open-loop unit step response of $G(s)$ and determine the steady-state error.

(b) Consider a unity-feedback control system as shown in Figure 8.21 with an ideal PID-controller and the plant $G(s)$. The transfer function of an ideal PID-controller is:

$$G_{PID} = K_P + \frac{K_I}{s} + K_D s$$

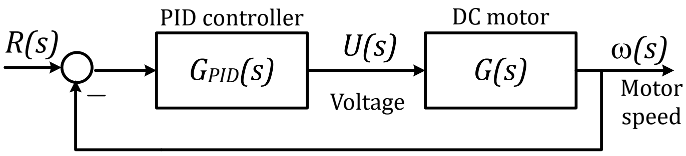


Figure 8.21: *Closed-loop control system*

Determine the effect of the PID parameters (K_P, K_I and K_D) on the overshoot, rise time, settling time and steady-state error.

Fill in the Table 8.1 using the results of the following simulations:

(i) Consider first a proportional (P) controller and determine the influence of K_P on the step response of the closed-loop system. Set the parameters K_I and K_D to zero and simulate the step response of the system for K_P given in Table 8.1.

(ii) Analyze the influence of the derivative term K_D term by taking a proportional-derivative (PD) controller. Set $K_I = 0$, K_P is constant and K_D takes the values from Table 8.1.

(iii) Consider a proportional-integral (PI) controller and determine the effect of K_I. Set $K_D = 0$, K_P is constant and K_I takes values according to Table 8.1.

Controller		*Overshoot*	*Rise time*	*Settling time*	*Steady-state error*
P	$K_P = 1$				
	$K_P = 5$				
	$K_P = 50$				
PD	$K_P = 50,\ K_D = 0.5$				
	$K_P = 50,\ K_D = 1$				
	$K_P = 50,\ K_D = 3$				
PI	$K_P = 1,\ K_I = 1$				
	$K_P = 1,\ K_I = 3$				
	$K_P = 1,\ K_I = 5$				

Table 8.1: *Closed-loop step response. Effect of PID controller parameters*

PE 8.11 Consider a plant described by a transfer function:

$$G(s) = \frac{1}{s^3 + 10s^2 + 20s}$$

The design specifications for a closed-loop control system are:

- The closed-loop system is stable
- Zero steady-state error for a step input

(a) Plot the unit step response for the open-loop system.

(b) Tune a PID controller using the Ziegler-Nichols ultimate sensitivity method:

(i) Consider the closed-loop system with the plant $G(s)$ and a proportional controller with the transfer function $G_{PID}(s) = K_P$. Determine the value of K_P so that the closed-loop system is critically stable ($K_0 = K_P$).

(ii) Determine the period of oscillations T_0 from the equivalent transfer function of the closed-loop system.

(iii) Simulate the closed-loop system step response for $G_{PID} = K_0$ and compare the period of oscillations with the one obtained at point (ii).

(iv) Set the controller parameters K_P, T_i and T_d according to Ziegler-Nichols table. If the transfer function of the controller is in the form:

$$G_{PID}(s) = K_P \left(1 + \frac{1}{T_i s} + T_d s\right), \quad (8.14)$$

then the parameters are computed as:

$$K_P = 0.6K_0, \quad T_i = 0.5T_0, \quad T_d = 0.125T_0$$

Simulate the closed-loop system and verify the closed-loop system specifications. *Note that the Simulink PID block has various controller forms that may be different than (8.14). If using Simulink, check the controller form before setting the parameters.*

(c) Modify the controller parameters to obtain a smaller overshoot.

PE 8.12 Consider a system with the transfer function $G(s) = \frac{1}{s(s-5)}$ for which we want to obtain a settling time less than 1 second, an overshoot as small as possible and zero steady-state error for a step reference input.

(a) Analyze the open-loop system stability. Represent graphically the system's response to a unit step input.

(b) Consider the closed-loop control system from Figure 8.22. Try to solve the problem

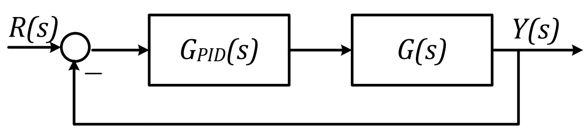


Figure 8.22

with a proportional controller (P), $G_{PID} = K_P$. Calculate the closed-loop transfer function and check if the design specifications can be met with a P controller.

(c) Consider a PI controller with the transfer function:

$$G_{PID} = K_P + \frac{K_I}{s}$$

Can you achieve the desired specifications with this controller? Calculate the closed-loop transfer function and justify your answer.

(d) Try to solve the problem using a PD controller with the transfer function:

$$G_{PD} = K_P + K_D s \tag{8.15}$$

- Calculate the closed-loop transfer function and determine the constraints on the system's parameters such that the closed-loop system is stable.
- Simulate the closed-loop system and find a set of suitable parameter values for the controller such that the design requirements are met. A convenient method to design the controller may be using *rltool*: draw the root locus for $G(s)$, then add and move the zero of the compensator and the location of the closed-loop poles until the system response meets the requirements. Write the controller in the form (8.15).

(e) Consider a PID controller with the transfer function

$$G_{PID} = K_P + \frac{K_I}{s} + K_D s \tag{8.16}$$

Simulate the closed-loop system step response for different values of the controller parameters. Use *rltool* to design the controller: draw the root locus for $G(s)$, add one pole at the origin and two zeros for the compensator. Change the zeros and the location of the closed-loop poles until the closed-loop step response meets the requirements. Write the controller in the form (8.16).

PE 8.13 Consider the linearized model of a hydro-turbine and penstock system ([13] - pp. 211):

$$H(s) = \frac{P(s)}{U(s)} = \frac{1 - sT_w}{\left(1 + \frac{sT_w}{2}\right)(1 + sT_q)},$$

where P is the power generated by the turbine, U is the control signal and the time constants are $T_w = 2$ sec and $T_q = 0.5$ sec.

Design a PID controller that ensures a settling time less than 5 seconds, an overshoot less than 20% and zero steady-state error for a step input.

9 State Feedback

Topics: State feedback controller, Pole placement, Stabilization, Tracking

9.1 Solved exercises

SE 9.1 Robotic arms are used in a wide range of applications, from manufacturing and automation industry, to surgical robotics. Consider a simple 2 DOF (Degrees Of Freedom) robot with two revolute joints as in Figure 9.1. The robot has a base fixed to the table and the gripper holds a cylindrical weight (load).

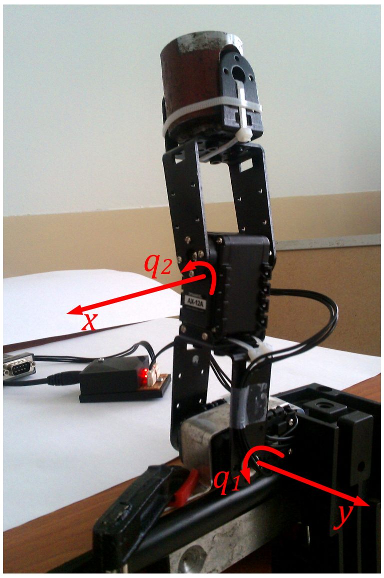


Figure 9.1: *2DOF Robot Arm, [23]*

The goal is to design a controller that controls the joints angles (q_1, q_2) such that the robot can follow accurately different trajectories in space.

Consider that the state vector $\mathbf{x}(t)$ is composed of the angular positions and angular velocities, the input vector $\mathbf{u}(t)$ represents the voltage signals that control the motor for each joint and the output vector $\mathbf{y}(t)$ is given by the joint angles:

$$\mathbf{x}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}, \quad \mathbf{y}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

The linearized model of the robot arm in the operation point $[0 \quad 0 \quad 0 \quad 0]^T$ corresponding to the upward position of the robot is [23]:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (9.1)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (9.2)$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.24 & 0 & -7.82 & 0 \\ 0 & 7.68 & 0 & -6.77 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 32.58 & 0 \\ 0 & 42.33 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(a) Analyze the stability of the open-loop system.

(b) *Stabilization problem.* Design a state-feedback control law

$$\mathbf{u}(t) = -\mathbf{K} \cdot \mathbf{x}(t), \quad (9.3)$$

that will return the system states to zero from non-zero initial conditions, for the imposed closed-loop poles: -3,-33, -43,-3. The block diagram of the state-feedback control system is presented in Figure 9.2.

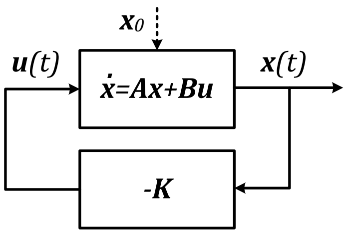


Figure 9.2: *State-feedback control structure*

Test in simulation the closed-loop response for non-zero initial conditions.

(c) *Tracking problem.* Consider that the robot joint angles q_1 and q_2 have to follow some constant reference inputs $r_1 = 0.5$ and $r_2 = 1$, respectively. If the closed-loop poles of the process are the same as in case **(b)**, design the state-feedback control system that will make the steady-state error equal to zero using the following strategies:

(i) Add a pre-compensation gain N to scale the reference input as presented in Figure 9.3. The control law is:

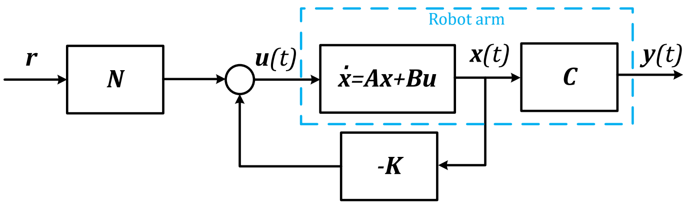


Figure 9.3: *State-feedback with pre-compensation gain*

$$\mathbf{u}(t) = -\mathbf{K} \cdot \mathbf{x}(t) + \mathbf{N} \cdot \mathbf{r}(t), \quad (9.4)$$

where $\mathbf{r} = \begin{bmatrix} r_1(t) & r_2(t) \end{bmatrix}^T$. Compute the gain matrices $\mathbf{K}$ and $\mathbf{N}$, then simulate the closed-loop system to verify the steady-state error.

(ii) Add an integrator in the control loop as shown in Figure 9.4. The control law is

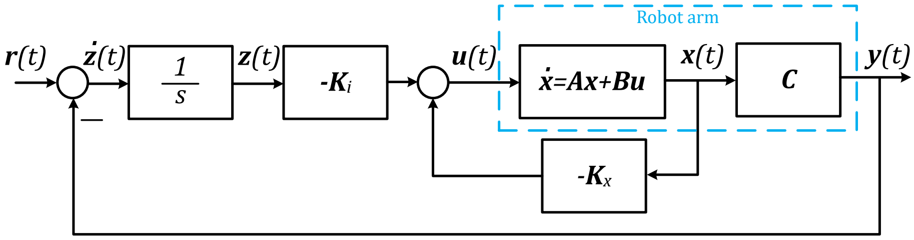


Figure 9.4: *State-feedback with integral control structure*

given by:

$$\mathbf{u}(t) = -\mathbf{K}_x \cdot \mathbf{x}(t) - \mathbf{K_i} \cdot \mathbf{z}(t), \tag{9.5}$$

where

$$\dot{\mathbf{z}}(t) = \mathbf{e}(t) = \mathbf{r}(t) - \mathbf{y}(t). \tag{9.6}$$

Compute the gains $\mathbf{K}_x$ and $\mathbf{K}_i$, then simulate the closed-loop response for a step reference input vector $\mathbf{r} = [0.5 \;\; 1]^T, \;\; t \geq 0$ and check the steady-state error.

Solution:

(a) The open-loop process poles are the eigenvalues of the system matrix $\mathbf{A}$, i.e. the solutions of the equation:

$$\det(s\mathbf{I}_4 - \mathbf{A}) = 0 \quad \Rightarrow \quad p_1 = 1.14, \quad p_2 = -8.96, \quad p_2 = 0.98, \quad p_4 = -7.75.$$

Because two poles are positive, the open-loop system is unstable.

(b) *Stabilization problem.* In order to design the state-feedback controller we need to check first the controllability of this system. The rank of the controllability matrix:

$$\mathbf{P}_c = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B}\end{bmatrix}$$

can be calculated in MATLAB as:

$$rank(P_c)$$

and it is equal to 4, which is the system order (the number of state variables). Therefore the system is controllable.

Through state-feedback - that is a control structure like in Figure 9.2 - we can impose the closed-loop poles:

$$p_{i1} = -3, \quad p_{i2} = -33, \quad p_{i3} = -43, \quad p_{i4} = -3$$

for which the closed-loop system is stable and achieves some desired dynamic behavior (for example: no oscillations, fast response).

We can calculate the gain $\mathbf{K}$ using the *place* function from MATLAB:

$$K = place(A, B, [p_{i1} \quad p_{i2} \quad p_{i3} \quad p_{i4}]),$$

which leads to the values:

$$K = \begin{bmatrix} 4.27 & 0 & 1.17 & 0 \\ 0 & 2.52 & 0 & 0.69 \end{bmatrix}. \tag{9.7}$$

The response of the closed-loop system with the initial conditions $[\pi/2 \quad 1.87 \quad 0 \quad 0]^T$ is shown in Figure 9.5.

(c) *Tracking problem.*

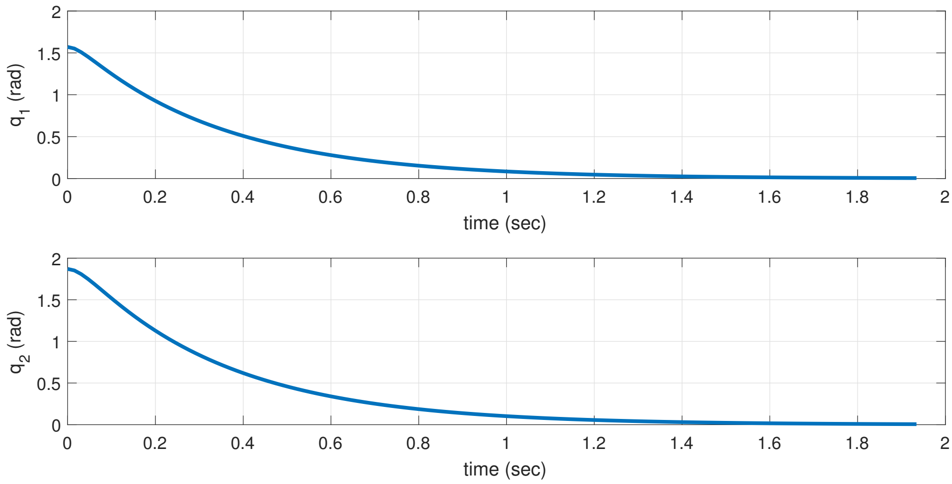


Figure 9.5: *Closed-loop system response - stabilization problem - non-zero initial conditions*

(i) The feedback gain matrix $\mathbf{K}$ is computed in the same manner as in case (b) and for the same closed-loop poles it is given by (9.7).
The gain matrix $\mathbf{N}$ is computed so that the output vector tracks the reference input $\mathbf{r}$ at steady-state ($\mathbf{r}_{ss}$) with zero steady-state error ($\mathbf{e}_{ss} = \mathbf{r}_{ss} - \mathbf{y}_{ss} = 0$).
With the feedback control law (9.4), the closed-loop state equation is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(\mathbf{N}\mathbf{r} - \mathbf{K}\mathbf{x}(t)) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{N}\mathbf{r}(t)$$

At steady-state $\dot{\mathbf{x}} = 0$ and the steady-state values of the states and outputs are denoted by $\mathbf{x}_{ss}$ and $\mathbf{y}_{ss}$, respectively. Then:

$$0 = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}_{ss} + \mathbf{B}\mathbf{N}\mathbf{r}_{ss} \quad \Rightarrow \quad \mathbf{x}_{ss} = -(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{N}\mathbf{r}_{ss}$$

If the output at steady-state is equal to $\mathbf{r}$ we obtain:

$$\mathbf{y}_{ss} = \mathbf{C}\mathbf{x}_{ss} = -\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{N}\mathbf{r}_{ss} = \mathbf{r}_{ss}$$

and then, the gain matrix $\mathbf{N}$ is given by:

$$\mathbf{N} = -\left(\mathbf{C}\left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)^{-1}\mathbf{B}\right)^{-1} \tag{9.8}$$

For the system matrices given in the problem statement and the matrix $\mathbf{K}$ from (9.7) we obtain:

$$\mathbf{N} = \begin{bmatrix} 3.96 & 0 \\ 0 & 2.34 \end{bmatrix}$$

Notice from the simulation results presented in Figure 9.6 that the robot joint angles reach the steady-state values equal to the reference inputs $\mathbf{r} = [0.5 \;\; 1]^T$, $t \geq 0$ in less than 2 seconds, which makes the steady-state error equal to zero.

(ii) When we add the integrator component, the control structure becomes the one from Figure 9.4. Now we have to design both gains: $\mathbf{K}_x$ and $\mathbf{K}_i$. Our approach here will be to design both gains at once by extending the system (9.1) with the error dynamics given by (9.6):

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \dot{\mathbf{z}}(t) &= \mathbf{e}(t) = \mathbf{r}(t) - \mathbf{y}(t) = \mathbf{r}(t) - \mathbf{C}\mathbf{x}(t) \end{cases} \tag{9.9}$$

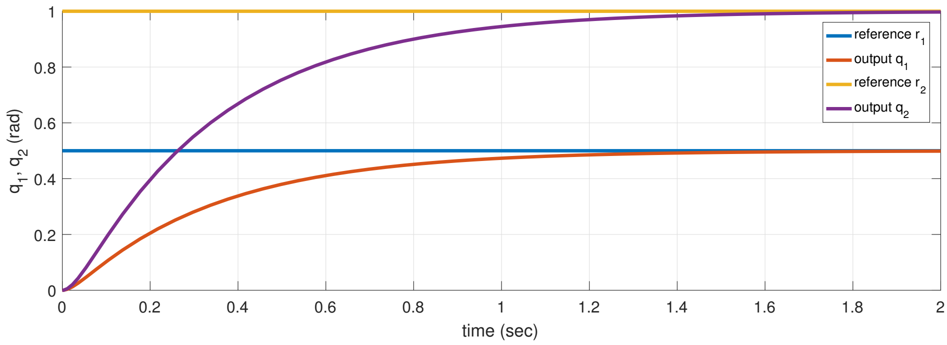


Figure 9.6: *Closed-loop step response*

or:

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}}_{\dot{\mathbf{x}}_e} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0}_{4\times 2} \\ -\mathbf{C} & \mathbf{0}_{2\times 2} \end{bmatrix}}_{\mathbf{A}_e} \underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix}}_{\mathbf{x}_e} + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0}_{2\times 2} \end{bmatrix}}_{\mathbf{B}_e} \mathbf{u}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_2 \end{bmatrix} \mathbf{r}(t) \tag{9.10}$$

In equation (9.10), $\mathbf{x}_e(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix}$ is the extended state vector. Two new states are introduced, included in the vector $\mathbf{z}(t)$ that describe the error dynamics, i.e. the variation of the error between the reference inputs and the system outputs. The matrices $\mathbf{0}_{n\times m}$ are matrices of zeros and $\mathbf{I}_m$ is an identity matrix.
The control law is written as:

$$\mathbf{u}(t) = -\mathbf{K}_x \cdot \mathbf{x}(t) - \mathbf{K_i} \cdot \mathbf{z}(t) = -\begin{bmatrix} \mathbf{K}_x & \mathbf{K_i} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} = -\mathbf{K_e}\mathbf{x}_e(t) \tag{9.11}$$

We can now design the gain $\mathbf{K}_e$ in a similar manner as at point (b), by imposing the eigenvalues of the closed-loop system matrix $\mathbf{A}_e - \mathbf{B}_e\mathbf{K}_e$.
First, we have to verify whether or not the extended system is controllable. The controllability matrix is:

$$\mathbf{P}_c = \begin{bmatrix} \mathbf{B}_e & \mathbf{A}_e\mathbf{B}_e & \mathbf{A}_e^2\mathbf{B}_e & \cdots & \mathbf{A}_e^5\mathbf{B}_e \end{bmatrix}$$

with the rank calculated in MATLAB:

$$rank(P_c) = 6$$

which is also the system order. The extended system is thus controllable.
The next step is to obtain a set of poles to be imposed as the eigenvalues of the closed-loop system matrix. We will keep the same values for the closed-loop process poles as at point (b) and add two new values corresponding to the newly introduced states. The closed-loop poles for the extended system will be placed at:

$$p_{i1} = -3, \quad p_{i2} = -33, \quad p_{i3} = -43, \quad p_{i4} = -3, \quad p_{i5} = -6, \quad p_{i6} = -8$$

The matrix $\mathbf{K}_e$, computed using the function *place*:

$$Ke = place(Ae, Be, [p_{i1}\ p_{i2}\ p_{i3}\ p_{i4}\ p_{i5}\ p_{i6}])$$

is

$$\mathbf{K}_e = \begin{bmatrix} 11.73 & 0.92 & 1.09 & 0.03 & -23.04 & -2.50 \\ 1.04 & 10.22 & 0.03 & 1.08 & -2.83 & -19.60 \end{bmatrix}.$$

We separate the gain matrices $\mathbf{K}_x$ and $\mathbf{K}_i$ of appropriate sizes from $\mathbf{K}_e$ and obtain:

$$\mathbf{K}_x = \begin{bmatrix} 11.73 & 0.92 & 1.09 & 0.03 \\ 1.04 & 10.22 & 0.03 & 1.08 \end{bmatrix}, \quad \mathbf{K}_i = \begin{bmatrix} -23.04 & -2.50 \\ -2.83 & -19.60 \end{bmatrix}$$

The simulation results of the control structure from Figure 9.4 with the reference inputs $\mathbf{r} = [0.5 \;\; 1]^T, \;\; t \geq 0$ is shown in Figure 9.7. As in the previous case, the steady-state error for each output is zero.

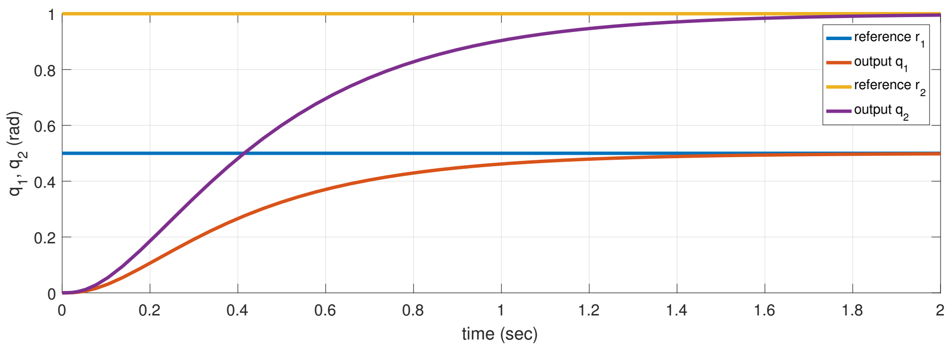


Figure 9.7: *Closed-loop step response with integral control component*

Compare the strategies (i) and (ii) when the process model is slightly changed (for example, alter the elements of the matrix $\mathbf{A}$ by 10%). This would simulate the situation when the process parameters are uncertain meaning that a discrepancy is possible between the process model and the real process. Simulate the closed-loop system for the same values of $\mathbf{N}$ and $\mathbf{K}$ (in case (i)) and $\mathbf{K}_x$ and $\mathbf{K}_i$ (in case (ii)). Comment on the results.

Simulate the closed-loop system when the reference inputs are: $r_1 = 0.5$ and r_2 given in Figure 9.8. This would simulate the situation when the joint angle q_1 has to reach a constant value of 0.5 rad, while the angle q_2 has to reach successively the positions +1 rad and −1 rad, in less than 2 seconds.

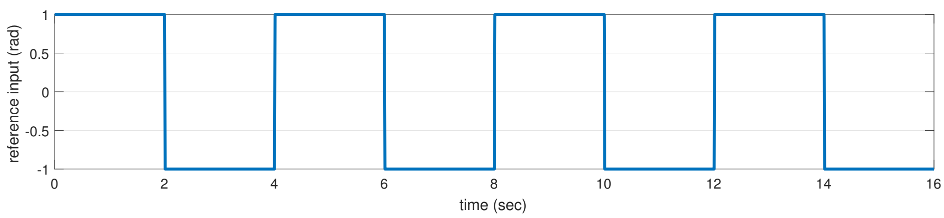


Figure 9.8: *Square wave reference input for joint angle q_2*

SE 9.2 pH (potential of hydrogen) neutralization processes are used in wastewater treatement, electrochemistry and precipitation plants [29]. The process is represented schematically in Figure 9.9. The control input is the acid stream flow rate (u), while the output is the measured pH level (y). The control problem is challenging, due to nonlinearities and sensibility to small perturbations. We will limits our discussion here to the dynamic behavior of the process near a nominal operating point.

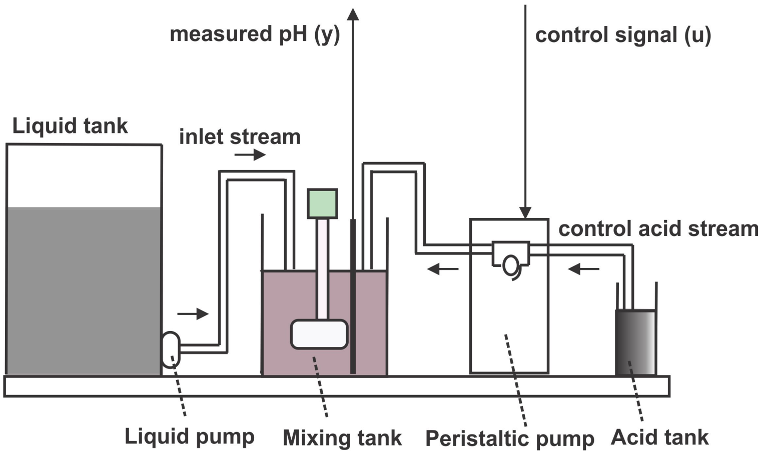


Figure 9.9: *pH neutralization process (adapted from [29])*

The linearized model in the nominal operation point is [29]:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{9.12}$$

with

$$A = \begin{bmatrix} -0.525 & -0.01265 & -0.000078 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = 10^{-4} \cdot \begin{bmatrix} 0 & -0.958 & -0.01197 \end{bmatrix}, \; D = \begin{bmatrix} 0 \end{bmatrix}$$

(a) Analyze the system stability and controllability.

(b) Design a state-feedback control law:

$$\mathbf{u}(t) = -\mathbf{K} \cdot \mathbf{x}(t) + \mathbf{N} \cdot r(t), \tag{9.13}$$

where r is the prescribed pH level (reference), that meets the following control requirements for a step reference $r(t) = 2$, $(t > 0)$:

- no overshoot,
- a settling time less than 4 minutes,
- a zero steady state error.

Test the result in simulation.

Solution:

(a) In order to determine whether the system is stable or not, we compute the system poles or the eigenvalues of matrix $\mathbf{A}$ from:

$$\det(s\mathbf{I} - \mathbf{A}) = 0 \quad \Rightarrow \quad p_1 = -0.5, \quad p_2 = -0.0128, \quad p_3 = -0.0122 \tag{9.14}$$

The system is stable because all poles are negative.

When designing a state-feedback controller, we have to check controllability. The controllability matrix is calculated as:

$$\mathbf{P}_c = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}1 & -0.525 & 0.2630\\ 0 & 1 & -0.525\\ 0 & 0 & 1\end{bmatrix}.$$

The rank of $\mathbf{P}_c$ is equal to the order of the system, that is 3. This means that the system is controllable.

(b) We shall design a state-feedback controller that makes the steady-state error equal to zero for a constant reference input. Notice first that the open-loop system is stable and the system poles are real and negative, as resulted from (9.14). We shall analyze the open-loop step response to gain some insight on the process.

The transfer function representation of the system can be determined as:

$$H(s) = \mathbf{C}(s\mathbf{I}_3 - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = -10^{-4}\frac{0.9580s + 0.01197}{s^3 + 0.5250s^2 + 0.01265s + 0.000078}. \quad (9.15)$$

The step response of the open-loop system when the input is a step $r(t) = 2,\ t \geq 0$ is shown in Figure 9.10.

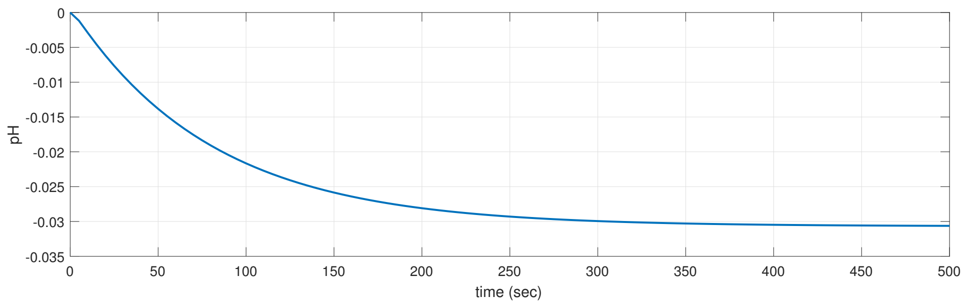


Figure 9.10: *Open-loop step response*

For the steady-state requirements we'll analyze the steady-state error and the settling time of the open-loop system.

The DC gain of $H(s)$ is:

$$\lim_{s\to 0} H(s) = -10^{-4} \cdot \frac{0.01197}{0.000078} = -0.0153.$$

So the steady-state error for a step input is zero (ideally) if we add an open-loop gain $K_0 = -\dfrac{1}{0.0153}$ to scale the reference input, as shown in Figure 9.11.

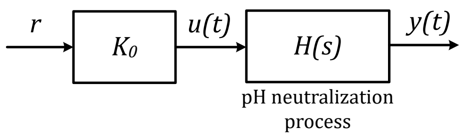


Figure 9.11: *Open-loop system with the gain K_0*

The step response of the system $K_0 \cdot H(s)$ for an input $r(t) = 2,\ \ t \geq 0$ is shown in Figure 9.12. The settling time is approximately 400 sec or 6.6 minutes.

One of the design requirements is a shorter settling time. This can be accomplished is we design a closed-loop control system with the closed-loop poles, imposed in the

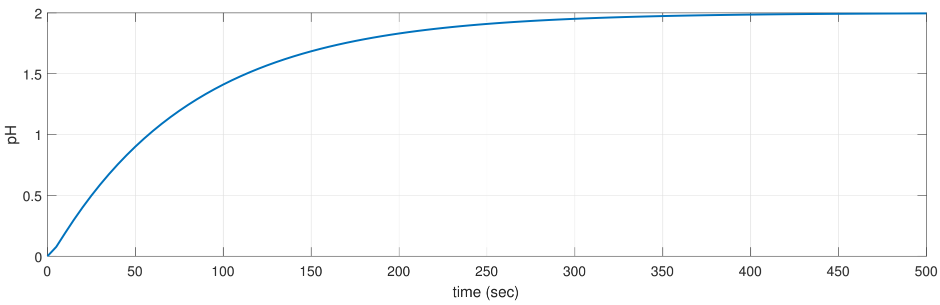


Figure 9.12: *Step response of the open-loop system with K_0*

process of designing a state-feedback controller, are faster than the open-loop poles of the process. A pole is "faster" if it has a larger absolute value - for real values this is easy to see as the poles are the negative inverse of time constants $p_i = -1/T_i$. In our specific case, if the open-loop poles are given by (9.14) then the closed-loop poles should be to the left in respect with the real axis (see Figure 9.13).

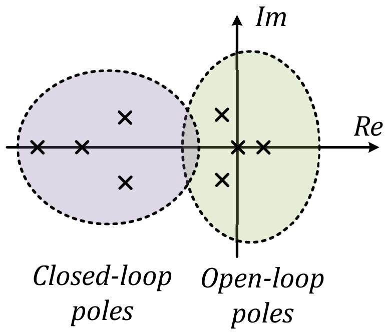


Figure 9.13: *Location of open-loop poles and closed-loop poles: a generic example*

We shall design a state feedback controller that stabilizes the system, using the pole placement method, and then compute a pre-compensation gain N to scale the reference input so that the steady-state error is zero, according to the block diagram shown in Figure 9.14.

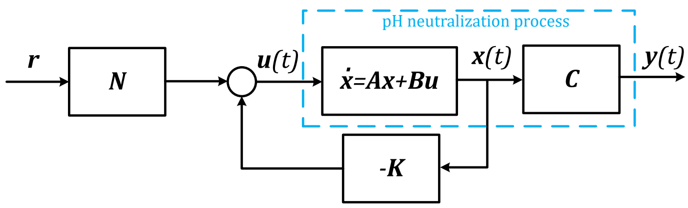


Figure 9.14: *Closed-loop control system with pre-compensation gain*

Let's impose the closed-poles four times "faster" than the open-loop poles:

$$p_{i1} = -2, \quad p_{i2} = -0.0512, \quad p_{i3} = -0.0487.$$

This means the characteristic polynomial of the closed-loop system is:

$$\mu_i(s) = (s - p_{i1})(s - p_{i2})(s - p_{i3}) = s^3 + 2.1s^2 + 0.2024s + 0.005 \qquad (9.16)$$

On the other hand, the closed-loop system given by the state equation (9.12) and the control law (9.13) with $N = 0$, has the characteristic polynomial:

$$\mu_{cl}(s) = det(s\mathbf{I}_3 - (\mathbf{A} - \mathbf{BK}))$$
$$\mu_{cl}(s) = s^3 + (k_1 + 0.525)s^2 + (k_2 + 0.0127)s + (k_3 + 0.000078), \qquad (9.17)$$

where $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$.
By equating term by term (9.16) and (9.17), we obtain a system of three equations with three unknowns, which has the unique solution:

$$k_1 = 1.5750 \quad k_2 = 0.1898 \quad k_3 = 0.0049.$$

The response of the closed loop system when $N = -1$ is shown in Figure 9.15. Although now, the settling time is within limits, what stands out is that we have a big overshoot, although all poles are real.

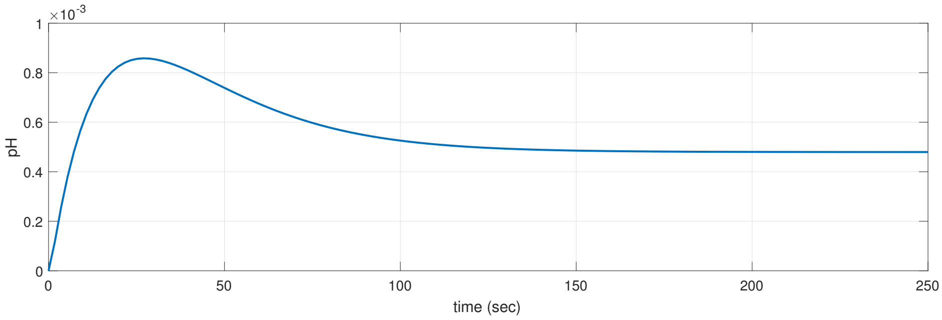


Figure 9.15: *Closed loop step response with state feedback - when the open loop zero is ignored*

This is because, so far, we ignored the process zero - see 9.15, which persists even in closed-loop (open-loop zeros are the same as the closed-loop zeros). A quick fix is to impose a closed-loop pole with the same value as the zero: -0.0125. This way, the closed-loop pole cancels the zero. Consequently, we will try the imposed poles:

$$p_{i1} = -2, \quad p_{i2} = -0.0512, \quad p_{i3} = -0.0125,$$

that give the characteristic polynomial:

$$\mu_i(s) = s^3 + 2.0637s^2 + 0.128s + 0.0013. \qquad (9.18)$$

The new feedback gains are:

$$k_1 = 1.5387 \quad k_2 = 0.1154 \quad k_3 = 0.0012.$$

The response of the closed loop system when $N = -1$ is shown in Figure 9.16. Now the response has no overshoot.
Finally, we need to eliminate the steady-state error. This can be achieved by properly designing the pre-compensation gain N. It can be computed as presented in SE 9.1:

$$\mathbf{N} = -\left(\mathbf{C}\left(\mathbf{A} - \mathbf{BK}\right)^{-1}\mathbf{B}\right)^{-1} = -1069.3$$

The response of the closed-loop system is illustrated in Figure 9.17. We can see that now all the design constraints are met.

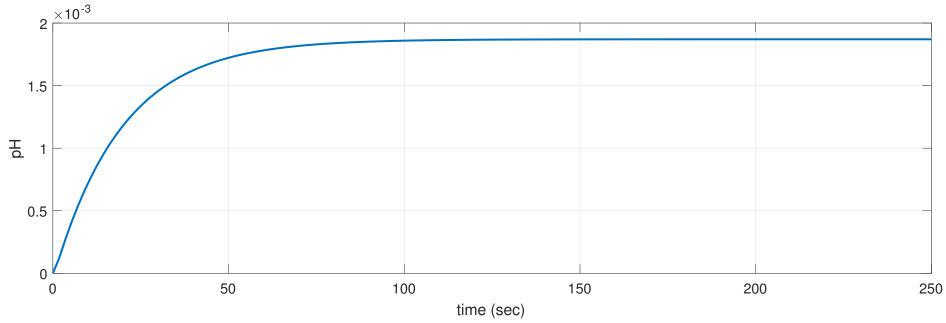


Figure 9.16: *Closed loop step response with state feedback - when the open loop zero is compensated*

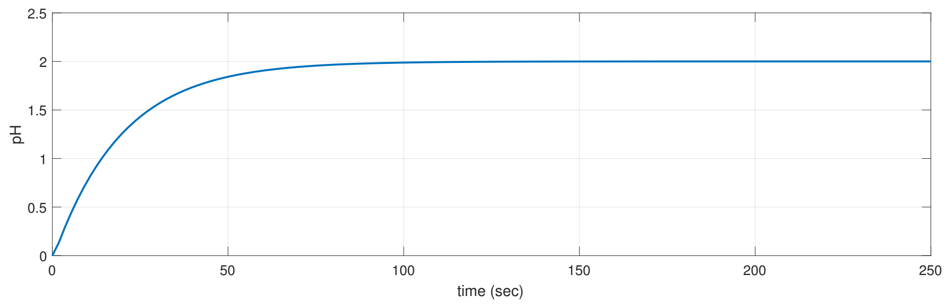


Figure 9.17: *Closed loop step response with state feedback and pre-compensation: final design*

9.2 Proposed exercises

PE 9.1 Consider the process model:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) + u(t)\\ \dot{x}_2(t) &= -2x_2(t) - u(t)\\ y(t) &= x_1(t)\end{aligned}$$

(a) Analyze the internal stability of this system.

(b) Determine the transfer function of the system and analyze the external stability.

(c) Determine whether the system is controllable.

PE 9.2 Consider a system with the state equations:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= 4x_1(t) - u(t)\end{aligned}$$

(a) Analyze the internal stability of this system.

(b) Show that the system is controllable.

(c) Stabilize the system using a state-feedback control law $u(t) = -\mathbf{K}\mathbf{x}(t)$ so that the closed-loop poles are located at $r_{1,2} = -1 \pm j$.

PE 9.3 Consider the process model

$$\begin{aligned}\dot{x}_1(t) &= 0.5x_1(t) - 2x_2(t) + u(t)\\ \dot{x}_2(t) &= -0.5x_1(t) - 0.3x_2(t)\end{aligned}$$

(a) Analyze the internal stability of the system.

(b) Show that the system is controllable.

(c) Stabilize the system using a state feedback controller

$$u(t) = -\mathbf{K} \cdot \mathbf{x}(t), \quad \text{where} \quad \mathbf{K} = [k_1 \quad k_2] \quad \text{and} \quad \mathbf{x} = [x_1 \quad x_2]^T$$

Choose a set of poles such that the closed-loop system is stable, compute the feedback gain matrix **K** *using pole placement and simulate the closed-loop system with a zero setpoint.*

PE 9.4 Consider the process model

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

with

$$A = \begin{bmatrix} 1 & -4 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}.$$

(a) Determine the internal stability of the system.

(b) Determine the transfer function of the system and the external stability. Interpret the results in respect with point **(a)**.

(c) Design a state feedback controller

$$u(t) = -\mathbf{K} \cdot \mathbf{x}(t),$$

so that the closed-loop system is stable and the closed-loop poles are real. Simulate the closed-loop system with a zero setpoint and non-zero initial conditions.

PE 9.5 Consider the process with the state equations:

$$\begin{aligned} \dot{x}_1(t) &= -5x_1(t) - 20x_2(t) + u(t) \\ \dot{x}_2(t) &= 8x_1(t) - 3x_2(t) + x_3(t) \\ \dot{x}_3(t) &= x_1(t) - 2x_2(t) + 4x_3(t) \end{aligned}$$

(a) Show the the system is unstable and controllable.

(b) Design a state-feedback controller such that the closed-loop poles of the system are:

$$p_1 = -20 - 4j, \; p_2 = -20 + 4j, \; p_3 = -10.$$

Check the result with the MATLAB function *place*.

(c) Consider

$$y(t) = x_1(t)$$

as the output of the system. Add a gain such that

$$u(t) = -\mathbf{K}\mathbf{x}(t) + Nr$$

where r is a step reference signal. Design the gain N such that the steady-state error is zero. Simulate the closed-loop system and check the result.

PE 9.6 Consider the linearized model of the levitation system from SE 8.2, given by the transfer function [6]:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{3148}{s^2 - 4551}.$$

where the input $u(t)$ is the voltage control signal and the output $y(t)$ is the ball vertical position.

(a) Determine a state-space model for this system. Choose the system states as the ball position $x_1(t) = y(t)$ and the ball velocity $x_2(t) = \dot{y}(t)$.

(b) Design a state-feedback controller to stabilize the system for the closed-loop poles: $p_{i1} = -390$ and $p_{i2} = -410$. Simulate the closed-loop system from non-zero initial conditions and a zero reference input.

(c) Design a pre-compensator gain N so that the steady-state error for a unit step input is zero. Analyze the simulated step response of the closed-loop system, determine the overshoot and the settling time.

PE 9.7 Consider the stabilization problem for a rotary inverted pendulum - Figure 9.18. The

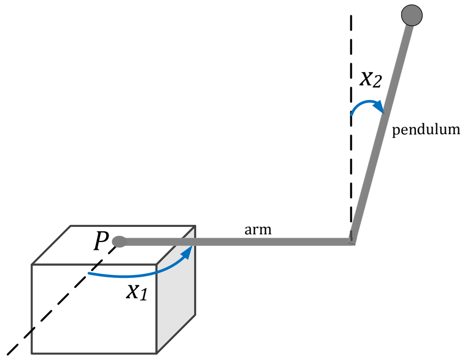


Figure 9.18: *Rotary Inverted Pendulum*

arm of the pendulum rotates around the pivot point P, actuated by a DC motor. The input signal is the voltage $u(t)$ that drives the motor and the states are chosen as the angular positions of the arm and the pendulum $x_1(t)$ and $x_2(t)$, respectively, as well as their velocities $x_3(t) = \dot{x}_1(t)$, $x_4(t) = \dot{x}_2(t)$.

The linearized model at the upward position (unstable equilibrium) is [17]:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 39.32 & -14.52 & 0 \\ 0 & 81.78 & -13.98 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 25.54 \\ 24.59 \end{bmatrix}.$$

(a) Analyze the system stability.

(b) Design a state-feedback controller the stabilizes the system, for the closed loop poles: $p_{i1} = -41$, $p_{i2} = -6$, $p_{i3,4} = -2 \pm 2i$. Check the results in simulation for non-zero initial conditions.

A Laplace and Z-transform

A.1 Table of Laplace and Z-transforms

	Time domain $f(t)$	Laplace transform $F(s) = \mathscr{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt$	Z-transform $F(z) = \mathcal{Z}[f(t)\vert_{t=kT}] = \sum_{k=0}^{\infty} f(kT)z^{-k}$
1	$\delta(t)$	1	1
2	1	$\frac{1}{s}$	$\frac{z}{z-1}$
3	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
4	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
5	$\sin at$	$\frac{a}{s^2+a^2}$	$\frac{z \sin aT}{z^2 - 2z\cos aT + 1}$
6	$\cos at$	$\frac{s}{s^2+a^2}$	$\frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$
7	$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$	$\frac{ze^{-aT}\sin bT}{z^2 - 2ze^{-aT}\cos bt + e^{-2aT}}$
8	$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$	$\frac{z(z-e^{-aT}\cos bT)}{z^2 - 2ze^{-aT}\cos bt + e^{-2aT}}$
9	te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
10	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{z(1-e^{-aT})}{a(z-1)(z-e^{-aT})}$
11	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$\frac{z(e^{-aT}-e^{-bT})}{(b-a)(z-e^{-aT})(z-e^{-bT})}$

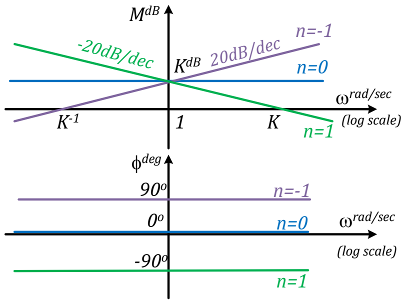
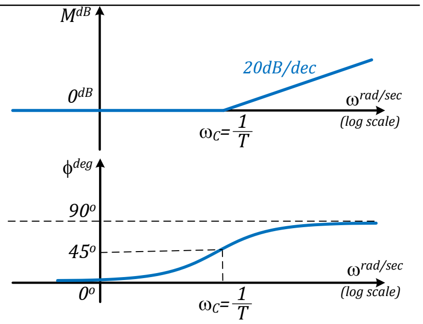
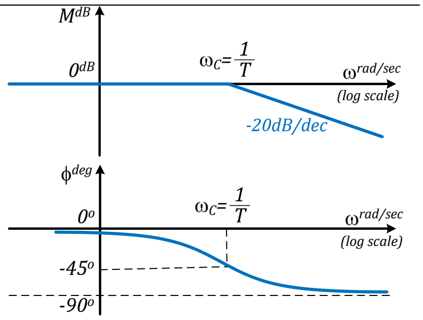
A.2 Properties of the Laplace transform

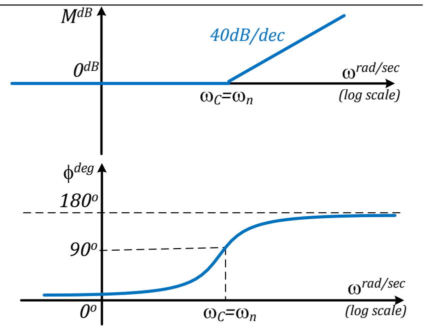
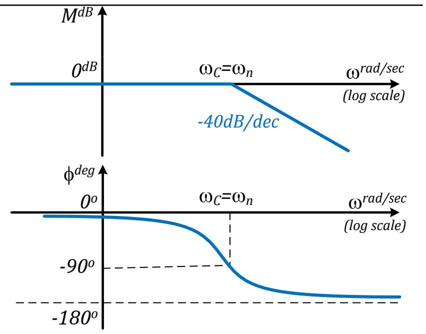
	Property	*Time domain* $f(t)$	*s-domain* $F(s) = \mathscr{L}[f(t)]$
1	Linearity	$af_1(t) + bf_2(t)$	$aF_1(s) + bF_2(s)$
2	First derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0)$
3	Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0) - f'(0)$
4	n-th derivative	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
5	Time integration	$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
6	Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
7	Time shift	$f(t-T)$	$e^{-sT}F(s)$
8	Initial value theorem	$f(0^+) = \lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$
9	Final value theorem	$f(\infty) = \lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$, if all poles of $sF(s)$ are in the left half-plane

A.3 Properties of the Z-transform

	Property	*Discrete time domain* $f(kT)$	*z-domain* $F(z) = \mathcal{Z}[f(kT)]$
1	Linearity	$af_1(kT) + bf_2(kT)$	$aF_1(z) + bF_2(z)$
2	Shift right by T	$f((k-1)T)$	$z^{-1}F(z)$
3	Shift right by nT	$f((k-n)T)$	$z^{-n}F(z)$
4	Shift left by T	$f((k+1)T)$	$zF(z) - zf(0)$
5	Shift left by nT	$f((k+n)T)$	$z^nF(z) - \sum_{i=0}^{n-1} f(iT)z^{k-i}$
6	First difference	$f(kT) - f((k-1)T)$	$(1-z^{-1})F(z)$
7	Initial value theorem	$f(0) = \lim_{k\to 0} f(kT)$	$\lim_{z\to\infty} F(z)$
8	Final value theorem	$f(\infty) = \lim_{k\to\infty} f(kT)$	$\lim_{z\to 1}(z-1)F(z)$ if the poles of $(z-1)F(z)$ are inside the unit circle

B Rules for sketching Bode plots

Factor	Magnitude, M^{dB}	Phase, ϕ^{deg}	Sketch
$\frac{K}{s^n}$	▪ straight line ▪ $M^{dB}\|_{\omega=1} = K^{dB}$ ▪ $\omega\|_{M^{dB}=0} = K^{1/n}$ ▪ slope: $-20n$ dB/dec	▪ $\phi = -90° \cdot n$	
$T \cdot s + 1$	▪ Low frequency asymptote at 0dB ▪ High frequency asymptote: slope = 20dB/dec ▪ Corner frequency $\omega_c = \frac{1}{T}$	▪ arctangent ▪ $\phi \in (0,\ 90°)$ ▪ inflection $(\omega_c,\ 45°)$	
$\frac{1}{T \cdot s + 1}$	▪ Low frequency asymptote at 0dB ▪ High frequency asymptote: slope = -20dB/dec ▪ Corner frequency $\omega_c = \frac{1}{T}$	▪ arctangent ▪ $\phi \in (0,\ -90°)$ ▪ inflection $(\omega_c,\ -45°)$	

Factor	**Magnitude**, M^{dB}	**Phase**, ϕ^{deg}	**Sketch**
$\frac{1}{\omega_n^2}s^2+\frac{2\zeta}{\omega_n}s+1$	▪ Low frequency asymptote at 0dB ▪ High frequency asymptote: slope $=$ 40dB/dec ▪ Corner frequency $\omega_c = \omega_n$	▪ arctangent ▪ $\phi \in (0,\ 180°)$ ▪ inflection $(\omega_c,\ 90°)$	
$\dfrac{1}{\frac{1}{\omega_n^2}s^2+\frac{2\zeta}{\omega_n}s+1}$	▪ Low frequency asymptote at 0dB ▪ High frequency asymptote: slope $=$ -40dB/dec ▪ Corner frequency $\omega_c = \omega_n$	▪ arctangent ▪ $\phi \in (0,\ -180°)$ ▪ inflection $(\omega_c,\ -90°)$	

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