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SPECIAL MATHEMATICS. PROBLEMS

UNIVERSITATEA TEHNICĂ DIN CLUJ-NAPOCA

U.T. PRESS

CLUJ-NAPOCA

2014

ISBN 978-973-662-988-4

Editura U.T.PRESS
Str.Observatorului nr. 34
C.P.42, O.P. 2, 400775 Cluj-Napoca
Tel.:0264-401.999 / Fax: 0264-430.408
e-mail: utpress@biblio.utcluj.ro
www.utcluj.ro/editura

Director: Prof. dr. Daniela Manea
Consilier editorial: ing. Călin D. Câmpean

Multiplicarea executată la editura U.T.PRESS.

ISBN978-973-662-988-4

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Preface

The aim of this book is to cover the analytical program for the course of Special Mathematics of different sections or faculties from Technical University of Cluj-Napoca. It is however, mainly addressed for the students of Electrical Engineering Faculty, which follow this course in the second semester of the first year. The book has four chapters and each chapters ends with a section of proposed problems. The first three chapters are dedicated to differential equations and in the fourth chapter we introduce the reader into the basics of complex analysis. One of the main features of this book is, in the opinion of the authors, the multitude of detailed solved problems which come to help the students. We used as sources for selecting some proposed and solved problems the following references: [1], [2], [3].

Cluj-Napoca, 2014
The Authors

Chapter 1

Differential equations effectively integrable

In this chapter we present the most relevant types of differential equations of order one and some basic and elementary techniques techniques to solve them. We end this chapter with a section regarding some higher order differential equations.

1.1 Differential equations with separable variables

A differential equation of the form

$$y' = f(x)g(y) \tag{1.1.1}$$

where $f \in \mathrm{C}(I), g \in \mathrm{C}(J), I, J \subseteq \mathbb{R}$ are intervals is called *equation with separable variables*. For $y \in J_1 \subseteq J$, J_1 interval with $g(y) \neq 0$ the equation 1.1.1 is equivalent to

$$\frac{dy}{g(y)} = f(x)dx$$

and the solution follows by integration

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

If $y_0 \in J$ and $g(y_0) = 0$ then the equation (1.1.1) admits the singular solution $y(x) = y_0$, for all $x \in I$.

Example 1.1.1. Integrate $\quad xy(1 + x^2)y' = 1 + y^2.$

Proof. We have $xy(1+x^2)\frac{dy}{dx} = 1+y^2$, hence $\frac{ydy}{1+y^2} = \frac{dx}{x(1+x^2)}$. We integrate to obtain $\int \frac{ydy}{1+y^2} = \int \frac{dx}{x(1+x^2)}$. Equivalently $\int \frac{ydy}{1+y^2} = \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx$, hence

$$\frac{1}{2}\ln(y^2+1) = \ln \mid x \mid - \frac{1}{2}\ln(1+x^2) + C.$$

It follows that $\ln(y^2+1) = \ln x^2 - \ln(1+x^2) + C^2$. For $C^2 = \ln k, k \geq 1$ we obtain the solution

$$y^2 + 1 = \frac{kx^2}{1+x^2}.$$

□

Example 1.1.2. Integrate $y' = (x+y+1)^2$.

Proof. Let $x+y+1 = z,\ z = z(x)$. Then $y = z - x - 1$ and $y' = z' - 1$ hence the equation becomes $z' = 1 + z^2$, that is $\frac{dz}{1+z^2} = dx$. Integrating the previous relation it follows that

$$\int \frac{dz}{1+z^2} = \int dx$$

or

$$\arctan z = x + C$$

so $z = \tan(x+C)$ hence the solution

$$y(x) = \tan(x+C) - x - 1.$$

□

1.2 Homogenous equations

A differential equation of the form

$$y' = f(\frac{y}{x}) \tag{1.2.1}$$

where $f \in \mathrm{C}(I)$, with $I \subseteq \mathbb{R}$ an interval and $f(u) \neq u$ for any $u \in I$, is called *homogenous equation.*

Let $z : I \to \mathbb{R}$ be the function defined by $z(x) = \frac{y(x)}{x}$, $x \in I$. Then $y = xz$ and $y' = xz' + z$ hence the equation becomes $xz' = f(z) - z$, i.e.

$$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

which is an equation with separable variables.

Example 1.2.1. Integrate $2x^3y' = y(3x^2y + y^2), x \in (0, \infty)$.

Proof. We have $y' = \frac{3x^2+y^3}{2x^3}$, that is $y' = \frac{3}{2}\frac{y}{x} + \frac{1}{2}(\frac{y}{x})^3$. The substitution $\frac{y}{x} = z, z = z(x)$ leads to

$$xz' + z = \frac{3}{2}z + \frac{1}{2}z^3.$$

Equivalently, to $xz' = \frac{1}{2}z + \frac{1}{2}z^3$. The new equation becomes $\frac{2dz}{z(1+z^2)} = \frac{dx}{x}$ and by integration we get

$$2\int\left(\frac{1}{z} - \frac{z}{1+z^2}\right)dz = \int\frac{dx}{x}$$

hence

$$\ln z^2 - \ln(1 + z^2) = \ln x + \ln C, C \geq 0.$$

Finally by replacing $z = \frac{y}{x}$ in $\frac{z^2}{1+z^2} = Cx$, we get $y^2 = \frac{Cx^3}{1-Cx}, x \in (0, \infty)$. □

Example 1.2.2. Prove that the differential equation

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$$

$f \in \mathrm{C}(I), a_k, b_k, c_k \in \mathbb{R}, k \in \{1, 2\}$ becomes an homogenous equation if the system of equations

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

admits a unique solution (x_0, y_0).

Proof. Consider the change of variables $\begin{cases} x = t + x_0 \\ y = u + y_0 \end{cases}$, and let $u = u(t)$ be the unknown function. Then

$$a_1x + b_1y + c_1 = a_1(t + x_0) + b_1(u + y_0) + c_1 = a_1t + b_1u,$$

$$a_2x + b_2y + c_2 = a_2(t + x_0) + b_2(u + y_0) + c_2 = a_2t + b_2u$$

and $y' = \frac{dy}{dx} = \frac{du}{dt}$. Therefor we obtain

$$\frac{du}{dt} = f\left(\frac{a_1t + b_1u}{a_2 + b_2u}\right) = f\left(\frac{a_1 + b_1\frac{u}{t}}{a_2 + b_2\frac{u}{t}}\right) = g\left(\frac{u}{t}\right),$$

which is an homogenous equation. □

1.3 Linear equations of order one

A differential equation of the form

$$y' + f(x)y = g(x) \tag{1.3.1}$$

where $f, g \in \mathrm{C}(I), I \subseteq \mathbb{R}$ is an interval is called a *linear differential equation of order one.*

Let $F(x) = \int_{x_0}^{x} f(t)dt, x_0 \in I$ be an antiderivative of f. Multiplying the equation (1.3.1) by $e^{F(x)}$ we get

$$y(x) = e^{-F(x)}\left(\int_{x_0}^{x} g(t)e^{F(t)}dt\right), C \in \mathbb{R}.$$

The function $h : I \to \mathbb{R}$ is called the integrating factor of the above equation.

Example 1.3.1. Integrate $y' + 2y = e^{-x}, x \in \mathbb{R}$.

Proof. We compute $F(x) = e^{2x}, x \in \mathbb{R}$, so we multiply with e^{2x} and the equation becomes $y'(x)e^{2x} + 2e^{2x}y(x) = e^x$ or $(y(x)e^{2x})' = e^x$, hence by integration we obtain

$$y(x) = e^{-2x}(x + C), C \in \mathbb{R}.$$

□

Example 1.3.2. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that there exists $\lim\limits_{x\to\infty} f(x) = l, l \in \mathbb{R}$ and $a > 0$. Prove that any solution of the equation

$$y' + ay = f(x)$$

admits an horizontal asymptote at $+\infty$.

Proof. Let y be a solution of the equation. Multiplying with e^{ax} the equation becomes $(y(x)e^{ax})' = f(x)e^{ax}, x \in [0,\infty)$. Integrating on $[0,x], x > 0$ it follows

$$y(x)e^{ax} = \int_0^x f(t)dt + C, C \in \mathbb{R}$$

hence

$$y(x) = \frac{\int_0^x f(t)e^{at}dt + C}{e^{ax}}, x > 0.$$

Then

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} \left(\frac{\int_0^x f(t)e^{at}dt}{e^{ax}} + \frac{C}{e^{ax}} \right) = \lim_{x\to\infty} \frac{\left(\int_0^x f(te^{at}dt)\right)'}{ae^{ax}} =$$

$$\lim_{x\to\infty} \frac{f(x)e^{ax}}{ae^{ax}} = \frac{l}{a}.$$

Thus $y = \frac{l}{a}$ is the horizontal asymptote of f at ∞. □

Example 1.3.3. Find all continuous function $y : \mathbb{R} \to \mathbb{R}$ satisfying the following integral equation

$$\int_0^x (x-s)y(s)ds = \int_0^x y(s)ds + \sin x, x \in \mathbb{R}.$$

Proof. Since y is a continuous function, the functions from the left and the right hand are differentiable. First we put the equation in the form

$$x\int_0^x y(s)ds - \int_0^x sy(s)ds = \int_0^x y(s)ds + \sin x$$

and by differentiation with respect to x we get

$$xy(x) + \int_0^x y(s)ds - xy(x) = y(x) + \cos x \qquad (\star)$$

A second differentiation leads to $y(x) = y'(x) - \sin x$, a linear equation of order one, which can be written in the form $(y(x)e^{-x})' = e^{-x}\sin x$. By integration we get

$$y(x) = Ce^x - \frac{\sin x + \cos x}{2}, x \in \mathbb{R}.$$

Take $x = 0$ in relation $(\star)$ to obtain $y(0) = -1$ so $y(0) = C - \frac{1}{2} = -1$, hence $C = -\frac{1}{2}$. The solution is

$$y(x) = -\frac{e^x + \sin x + \cos x}{2}, x \in \mathbb{R}.$$

□

1.4 Bernoulli's equations

A differential equation of the form

$$y' + f(x)y = g(x)y^\alpha, \alpha \in \mathbb{R} \setminus \{0, 1\}$$

where $f, g \in \mathrm{C}(I), I \subseteq \mathbb{R}$ interval, is called *Bernoulli's equation.* For $\alpha > 0$ the equation admits the solution $y(x) = 0, x \in I$. On an interval $I_1 \subseteq I$ where $y(x) \neq 0, x \in I_1$ the substitution $z(x) = y^{1-\alpha}(x), x \in I$ leads to

$$z' + (1-\alpha)f(x)z = (1-\alpha)g(x)$$

which is a linear equation.

Example 1.4.1. Integrate the equation $xy^2y' = x^2 + y^3$.

Proof. We can divide with y^2 to obtain the rigorous form of this Bernoulli equation, or directly we make the substitution $z = y^3$ and notice that $z' = 3y^2y'$. By replacing these in the equation we obtain

$$\frac{x}{3}z' = x^2 + z,$$

next we multiply by $\frac{3}{x}$ to obtain the linear equation $z' - \frac{3}{x}z = 3x$. We multiply it by x^{-3} and we have $(x^{-3}z)' = 3x^{-2}$, hence $z = 3x^3 \int x^{-2}dx$. Equivalently $z = 3x^3(C - x^{-1})$. Finally we obtain $y^3 = Cx^3 - 3x^2$. □

Example 1.4.2. Integrate the equation $y' + xy = xy^2, x \in \mathbb{R}$.

Proof. We use the substitution $z(x) = y^{-1}(x)$, then $z'(x) = -y^{-2}(x)y'(x)$. We multiply the given equation by y^{-2} and we obtain

$$y^{-2}y' + xy^{-1} = x.$$

From the above substitutions the linear equation in z is $-z' + xz = x$, that is

$$z' - xz = -x.$$

Since the antiderivative is $F(x) = -\frac{x^2}{2}$, we multiply this equation by $e^{-\frac{x^2}{2}}$ to obtain

$$e^{-\frac{x^2}{2}}z' - xe^{-\frac{x^2}{2}}z = -xe^{-\frac{x^2}{2}}.$$

Equivalently we have $(e^{-\frac{x^2}{2}}z)' = xe^{-\frac{x^2}{2}}$, thus $e^{-\frac{x^2}{2}}z = \int xe^{-\frac{x^2}{2}}dx$. Integrating we have

$$z = e^{\frac{x^2}{2}}(e^{-\frac{x^2}{2}} - C).$$

We recall that $z = y^{-1}$ to conclude that

$$y^{-1}(x) = e^{\frac{x^2}{2}}(e^{-\frac{x^2}{2}} - C).$$

□

1.5 Riccati's equations

A differential equation of the form

$$y' = f(x)y^2 + g(x)y + h(x)$$

where $f, g, h \in \mathrm{C}(I), I \subseteq \mathbb{R}$ interval, is called *Riccati's equation.* Generally Riccati's equations cannot be effectively integrated. But if y_0 is a particular solution of it, then the substitution $y = y_0 + \frac{1}{z}$ gives the linear differential equation

$$z' - (2f(x)y_0(x) + g(x))z = f(x).$$

Example 1.5.1. Integrate the equation $y' = y^2 - \frac{1}{x}y - \frac{1}{x^2}$, if it admits the particular solution $y_0(x) = -\frac{1}{x}$.

Proof. The substitution $y = -\frac{1}{x} + \frac{1}{z}$ leads to the equation

$$(-\frac{1}{x} + \frac{1}{z})' = (-\frac{1}{x} + \frac{1}{z})^2 - \frac{1}{x}(-\frac{1}{x} + \frac{1}{z}) - \frac{1}{x^2},$$

equivalent to

$$\frac{1}{x^2} - \frac{z'}{z^2} = \frac{1}{x^2} - \frac{2}{xz} + \frac{1}{z^2} + \frac{1}{x^2} - \frac{1}{xz} - \frac{1}{x^2}.$$

We obtain the linear equation in z:

$$-\frac{z'}{z^2} = -\frac{3}{xz} + \frac{1}{z^2},$$

next we multiply by xz^2 to obtain $-xz' = -3z + x$ or $z' - \frac{3}{x}z = -1$. We multiply this equation by x^{-3} to obtain $z'x^{-3} - 3x^{-4} = -x^{-3}$, that is $(zx^{-3})' = -x^{-3}$. Integrating we have $z = -x^3 \int x^{-3}dx$, hence $z(x) = -x^3(-\frac{x^{-2}}{2} + C)$, with the final solution

$$y = -\frac{1}{x} + \frac{1}{-x^3(-\frac{x^{-2}}{2} + C)}, \quad C \in \mathbb{R}.$$

□

Example 1.5.2. Integrate $(1+x^3)y' - y^2 - x^2y - 2x = 0, x > -1$, if the equation admits a particular solution of the form $y_0(x) = ax^n, a \in \mathbb{R}, n \in \mathbb{N}$.

Proof. Replacing y_0 in the equation we get

$$(a + x^3)nax^{n-1} - a^2x^{2n} - ax^{n+2} - 2x = 0$$

or

$$-a^2x^{2n} + (na - a)x^{n+2} + na^2x^{n-1} - 2x = 0, x > -1.$$

It follows $n = 2$ and $-a^2x^4 + ax^4 + 2ax - 2x = 0$ for all $x > -1$, hence $a = 1$. The particular solution is $y_0(x) = x^2$. The substitution is $y = x^2 + \frac{1}{z}$ and leads to

$$z'(1 + x^3) + 3x^2z = -1 \quad \texttt{or} \quad (z(1 + x^3))' = -1.$$

So $z(1 + x^3) = -x + C$ hence $z = \frac{-x+C}{1+x^3}$. Finally

$$y = x^2 + \frac{1 + x^2}{-x + C} = \frac{Cx^2 + 1}{C - x}, \quad C \in \mathbb{R}.$$

□

1.6 Exact differential equations. Integrant factor

Let $D \subseteq \mathbb{R}$ be a rectangle and $P, Q \in \mathrm{C}^1(D)$. A differential equation of the form

$$P(x, y)dx + Q(x, y)dy = 0 \tag{1.6.1}$$

where $\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y)$ for all $(x, y) \in D$ is called an *exact differential equation.* Under the previous conditions there exists a function $F \in \mathrm{C}^2(D)$ given by the relation

$$F(x, y) = \int_{x_0}^{x} P(t, y)dt + \int_{y_0}^{y} Q(x_0, t)dt, (x_0, y_0) \in D,$$

such that $dF(x,y) = P(x,y)dx + Q(x,y)dy, (x,y) \in D$. Since the exact differential equation is equivalent to $dF(x,y) = 0$ the solutions are implicitly defined by

$$F(x,y) = C, C \in \mathbb{R}.$$

The function F is called an antiderivative (primitive) of the differential form $Pdx + Qdy$.

If an equation of the form 1.6.1 is not an exact equation then a function $\mu \in \mathrm{C}^1(D)$ with the property that the equation

$$\mu(x,y)P(x,y)dx + \mu(x,y)Q(x,y)dy = 0 \tag{1.6.2}$$

is an exact differential equation is called *integrant factor*. Denoting $P_1(x,y) = \mu(x,y)P(x,y),\ Q_1(x,y) = \mu(x,y)Q(x,y),\ (x,y) \in D$ the equation 1.6.2 is an exact equation if

$$\frac{\partial P_1(x,y)}{\partial y} = \frac{\partial Q_1(x,y)}{\partial x}, \quad (x,y) \in D \tag{1.6.3}$$

The equation 1.6.3 is equivalent to the equation of integrant factor

$$Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu. \tag{1.6.4}$$

In practice, usually we are looking for integrant factors of the form $\mu = \mu(x), \mu = \mu(y)$. If the equation

$$Q\mu'(x) = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu(x)$$

depends only on x. then there exists $\mu = \mu(x)$. If the equation

$$-P\mu'(y) = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu(y)$$

depends only on y, then there exists $\mu = \mu(y)$.

Example 1.6.1. Integrate the equation $e^{-y}dx - (2y + xe^{-y})dy = 0, x, y \in \mathbb{R}$.

Proof. It is easy to check that $\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{x} = -e^{-y}$ where

$$P(x,y) = e^{-y}, Q(x,y) = -2y - xe^{-y}.$$

We apply the integral formula for F to obtain

$$F(x,y) = \int_{x_0}^{x} e^{-y}dt + \int_{y_0}^{y}(-2t - x_0e^{-t})dt = e^{-y}t|_{x_0}^{x} - 2\left[\frac{t^2}{2}\right]_{y_0}^{y} - x_0\left[-e^{-y}\right]_{y_0}^{y} =$$

$$e^{-y}(x - x_0) - (y^2 - y_0^2) + x_0(e^{-y} - e^{-y_0}) = xe^{-y} - x_0e^{-y} - y^2 + y_0^2 + x_0e^{-y} - x_0e^{-y_0} =$$

$$xe^{-y} - y^2 + y_0^2 - x_0e^{-y_0}.$$

Since x_0, y_0 are constants the solution of the equation is $xe^{-y} - y^2 = C$. □

Example 1.6.2. Find the integrant factor $\mu = \mu(y)$, depending on y, for the equation

$$(2xy^2 - 3y^3)dx + (7 - 3xy^2)dy = 0$$

Proof. We have

$$\frac{\partial P(x,y)}{\partial y} = 4xy - 9y^2, \quad \frac{\partial Q(x,y)}{x} = -3y^2.$$

Since μ depends on y, $\mu = \mu(y)$ we apply the above formula to obtain

$$-(2xy^2 - 3y^3)\mu'(y) = (4xy - 9y^2 + 3y^2)\mu(y)$$

We divide by y to obtain $-y(2x - 3y)\mu'(y) = 2(2x - 3y)\mu(y)$. Next we divide by $2x - 3y$ to obtain an equation with separable variables $-y\frac{d\mu}{dy} = 2\mu$ hence $\frac{d\mu}{\mu} = -2\frac{dy}{y}$. Integrating we have $\ln \mid \mu \mid= -2\ln \mid y \mid +C$, thus $\mu(y) = Cy^{-2}$. □

Example 1.6.3. Find the integrating factor $\mu = \mu(x + y^2)$, depending on $x + y^2$ for the equation $(3y^2 - x)dx + (2y^3 - 6xy)dy = 0$.

Proof. We have $\frac{\partial P(x,y)}{\partial y} = 6y$, $\frac{\partial Q(x,y)}{\partial x} = -6y$. For shortness we denote by t the expression $x + y^2$, hence $t = t(x, y) = x + y^2$. We apply formula 1.6.3 to obtain

$$Q\mu'(t)\frac{\partial t}{\partial x} - P\mu'(t)\frac{\partial t}{\partial y} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu(t).$$

Since $\frac{\partial t}{\partial x} = 2x$ and $\frac{\partial t}{\partial y} = 1$ we obtain

$$\mu'(t)[(2y^3 - 6xy)1 - (3y^2 - x)2y] = [6y - (-6y)]\mu(t),$$

equivalently $\mu'(t)(2y^3 - 6xy - 6y^3 + 2xy) = 12y\mu(t)$. Divide by y and we get $\mu'(t)(-4y^2 - 4x) = 12\mu(t)$. Divide also by -4 and we obtain the equation

$$(x + y^2)\mu'(t) = -3\mu(t)$$

that is $t\frac{d\mu}{dt} = -3\mu$ which is a separable variable equation, hence

$$\frac{d\mu}{\mu} = -3\frac{dt}{t}.$$

We integrate and obtain $\ln \mid \mu \mid = -3 \ln \mid t \mid + \ln \mid C \mid$, thus $\mu(t) = Ct^{-3}$. We conclude that an integrant factor can be chosen $\mu(x, y) = (x + y^2)^{-3}$. □

1.7 Equations of Clairaut and Lagrange

A differential equation of the form

$$y = xy' + g(y'))$$

where $g \in \mathrm{C}(I), I \subseteq \mathbb{R}$ interval, is called *Clairaut's equation*. We make the substitution $y' = p$ to obtain the equation $y = xp + g(p)$. By differentiating we get $pdx = pdx + dxp + g'(p)dp$ hence $[x + g'(p)]dp = 0$. We obtain the *singular solution* (given by parametric equations)

$$x = -g'(p), \quad y = -pg'(p) + g(p)$$

and from $dp = 0$ we obtain the general solution $(p = C)$

$$y = Cx + g(C).$$

A differential equation of the form

$$y = xf(y') + g(y'))$$

where $f, g \in C(I), I \subseteq \mathbb{R}$ interval, with $f(y') \neq y'$, is called *Lagrange's equation.* The same substitution $y' = p$ leads us to $y = xf(p) + g(p)$ hence, by differentiating we get $dy = dxf(p) + xf'(p)dp + g'(p)dp$. We obtain the linear equation

$$(p - f(p))dx = (xf'(p) - g'(p))dp,$$

with the unknown $x = x(p)$, which has a general solution $x = h(p, C)$. WE obtain the general parametric solution of the Lagrange's equation

$$x = h(p, C), \quad y = h(p, C)f(p) + g(p).$$

Example 1.7.1. Integrate the equation $y = xy' + \sqrt{1 + y'^2}$.

Proof. Let $y' = p$, so $y = xp + \sqrt{1 + p^2}$. Differentiate to obtain

$$dy = dxp + xdp + \frac{2p}{2\sqrt{1 + p^2}}dp$$

hence

$$pdx = pdx + xdp + \frac{p}{\sqrt{1 + p^2}}dp$$

First we have $dp = 0$ that is $p = C$ and the general solution $y = xC + \sqrt{1 + C^2}$. Secondly we obtain the singular solution

$$x = -\frac{p}{\sqrt{1 + p^2}}, \quad y = -\frac{p^2}{\sqrt{1 + p^2}} + \sqrt{1 + p^2}.$$

□

Example 1.7.2. Integrate the equation $y = x(1 + y') + y'^2$.

Proof. Let $y' = p$ then $y = x(1 + p) + p^2$ and we differentiate to obtain $dy = dx(1 + p) + xdp + 2pdp$, hence $pdx = dx + pdx + xdp + 2pdp$ that is

$$-dx = (x + 2p)dp$$

which is a linear equation with the unknown $x = x(p)$.We write this equation with derivative $-\frac{dx}{dp} = x + 2p$, that is $-x' - x = 2p$, hence $x' + x = -2p$. We multiply this equation by e^p to obtain $e^p x' + e^p x = -2pe^p$, that is $(e^p x)' = -2pe^p$. Equivalently we get

$$x = -2e^{-p} \int pe^p dp = -2e^{-p}(pe^p - e^p + C) = -2p + 2 - 2Ce^{-p}.$$

We replace x in the first formula of y to obtain

$$y = (1 + p)(-2p + 2 - 2Ce^{-p}) + p^2.$$

So, the parametric equations are

$$x = -2p + 2 - 2Ce^{-p}, \quad y = (1 + p)(-2p + 2 - 2Ce^{-p}) + p^2.$$

□

1.8 Higher order differential equations

We present some classes of differential equations of order n, with n a positive integer greater than one, which can be reduced to differential equations of order strictly less than n.

1. Equations of the form $F(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}) = 0$ with the unknown $y \in \mathrm{C}^n(I), y = y(x), I \subseteq \mathbb{R}$ interval. The substitution $z = y^{(k)}$ leads to the equation of order $n - k$

$$F(x, z, z', \ldots, z^{(n-k)}) = 0.$$

Example 1.8.1. Integrate $y'' + 2y' = e^{-2x}, x \in \mathbb{R}$.

Proof. The substitution $z = y'$ leads to the equation $z' + 2z = e^{-2x}$. Multiplying by e^{2x} we get $(ze^{2x})' = 1$ so $z(x) = e^{-2x}(x + C_1)$ and

$$y(x) = \int e^{-2x}(x + C_1)dx = -\frac{1}{2}e^{-2x}(x + C_1 + \frac{1}{2}) + C_2, C_1, C_2 \in \mathbb{R}.$$

□

2. Equations of the form $F(y, y', y'', \ldots, y^{(n)}) = 0$. Let $y' = p(y)$ where p becomes the new unknown of the equation. We have

$$y' = p;$$

$$y'' = p'(y)y' = pp';$$
$$y''' = p''(y)(y')^2 + p'(y)y'' = p^2p'' + p(p')^2;$$
$$\ldots$$

so we get a differential equation of order $(n-1)$ with the unknown p.

Example 1.8.2. Integrate $2yy'' = (y')^2 + 1$.

Proof. The substitution $y' = p, p = p(y)$ leads to $2ypp' = p^2 + 1$ which can be written as $\frac{2pdp}{p^2+1} = \frac{dy}{y}$. It follows $\int \frac{2pdp}{p^2+1} = \int \frac{dy}{y}$, hence $\ln(p^2+1) = \ln \mid Cy \mid$ so $p = \pm\sqrt{Cy-1}$. To obtain the solution we have to integrate the equation $y' = \pm\sqrt{Cy-1}$, which is equivalent to $\frac{dy}{\sqrt{Cy-1}} = \pm dx$. We get $4(Cy-1) = C^2(x+C_1), C, C_1 \in \mathbb{R}$. □

3. Equations of the form $F(x, \frac{y'}{y}, \frac{y''}{y}, \ldots, \frac{y^{(n)}}{y}) = 0$**.** Remark that these equations are homogenous with respect to $y, y', \ldots, y^{(n)}$. The substitution

$$z = \frac{y'}{y}, z = z(x)$$

leads to a differential equation of order $n-1$. Indeed

$$\frac{y''}{y} = z^2 + z'; \quad \frac{y'''}{y} = z^3 + 3zz' + z'', \ldots$$

Example 1.8.3. Integrate the equation $x^2yy'' = (y - xy')^2$.

Proof. Divide by y^2 to obtain $x^2\frac{y''}{y} = (1 - x\frac{y'}{y})^2$. Take the substitution $\frac{y'}{y} = z$, hence $\frac{y''}{y} = z' + z^2$ and replace it to obtain the equation $x^2(z' + z^2) = (1 - xz)^2$, that is

$$x^2z' + x^2z^2 = 1 + x^2z^2 - 2xz.$$

We have a linear equation $x^2z' + 2xz = 1$ which is equivalent to $(x^2z)' = 1$, hence $x^2z = x + C$. Since $z = \frac{y'}{y}$ we obtain the separable variable equation

$$\frac{y'}{y} = \frac{x+C}{x^2}.$$

Equivalently we get $\frac{dy}{y} = (\frac{C}{x^2} + \frac{1}{x})dx$. Integrating we obtain $\ln \mid y \mid= -C\frac{1}{x} + \ln \mid x \mid + C_1$ hence the solution

$$y = xe^{\frac{C}{x}}C_1.$$

□

1.9 Problems

Problem 1.9.1. Integrate the following differential equations of order one:

(1) $xydx + (x+1)dy = 0$;

(2) $\sqrt{y^2+1}dx = xydy$;

(3) $2x^2yy' + y^2 = 2$;

(4) $(x-y)dx + (x+y)dy = 0$;

(5) $xy' - y = x\tan\frac{y}{x}$;

(6) $x - y - 1 + (y - x - 2)y' = 0$;

(7) $xy' - 2y = 2x^4$;

(8) $y' + y\tan x = \frac{1}{\cos x}$;

(9) $y' + 2y = y^2e^x$;

(10) $xy' - y^2 + (2x+1)y = x^2 + 2x$, if admits the particular solution $y_0 = x$;

(11) $y' = y^2 - 2ye^x + e^{2x} + e^x$, if admits the particular solution $y_0 = e^x$;

(12) $y' + y^2 - 2y\sin x + \sin^2 x = \cos x$, if admits the particular solution $y_0 = \sin x$;

(13) $y = xy' - \ln y'$;

(14) $y = xy' + y'^2$;

(15) $y = 2xy' + \sin y'$;

(16) $y = 2xy' + \ln y'$;

Problem 1.9.2. Integrate the exact differential equations:

(1) $(2 - 9xy^2)xdx + (4y^2 - 6x^3)ydy = 0$;

(2) $\frac{y}{x}dx + (y^3 + \ln x)dy = 0$;

(3) $\frac{3x^2+y^2}{y^2}dx - \frac{2x^3+5y}{y^3}dy = 0$;

(4) $ydx + xdy = 0$;

Problem 1.9.3. Find the integrant factor μ for the equations:

(1) $(x + \sin x + \sin y)dx + cosydy = 0$ depending on x, $\quad \mu = \mu(x)$;

(2) $(x - y)dx + (y + x^2)dy = 0$ depending on $x^2 + y^2$, $\quad \mu = \mu(x^2 + y^2)$;

Problem 1.9.4. Integrate the following equations of order higher than one:

(1) $x^2y'' = y'^2$;

(2) $xy'' = y' + x^2$;

(3) $y'^2 + 2yy'' = 0$;

(4) $y^3y'' = 1$;

(5) $yy'' - y'^2 = y^2y'$;

(6) $x^2yy'' = (y - xy')^2$;

(7) $y' + 2xyy'' = 0$;

(8) $xyy'' - xy'^2 = yy'$.

Solutions: **1.9.1** (1). $y = C(x+1)e^{-x}$; (2). $\ln \mid x \mid = C + \sqrt{y^2+1}$; (3). $y^2 - 2 = Ce^{\frac{1}{x}}$; (4). $\ln(x^2+y^2) = C - 2\arctan\frac{y}{x}$; (5). Divide by x to obtain $y' - \frac{y}{x} = \tan\frac{y}{x}$ and then take the substitution $\frac{y}{x} = z$ to obtain a variable separable equation with the final solution $\sin\frac{y}{x} = Cx$; (6). Make the substitution $z = y - x, z = z(x)$ and obtain the final solution $(y - x + 2)^2 + 2x = C$; (7). $y = Cx^2 + x^4$; (8). $y = \sin x + C\cos x$; (9). $y = 0$ and $y(e^x + Ce^{2x}) = 1$; (10). Take the substitution $y = z + x$ which leads to the Bernoulli equation in z,$xz' - z^2 + z = 0$ and the final solution $y = \frac{1}{Cx+1} + x$; (11). take the substitution $y = z + e^x$ and get the final solution $y = \frac{1}{C-x} + e^x$; (12). $y = \frac{1}{C+x} + \sin x$; (13). $y = Cx - \ln C$ and $y = 1 + \ln x$; (14). $y = Cx + C^2$ and $x^2 + 4y = 0$; (15). $x = \frac{C - p\sin p - \cos p}{p^2}$, $y = \frac{2C - p\sin p - 2\cos p}{p}$; (16). $x = \frac{C-p}{p^2}$, $y = \frac{2(C-p)p}{p} + \ln p$; **1.9.2** (1). $x^2 - 3x^3y^2 + y^4 = C$; (2). $4y\ln x + y^4 = C$; (3). $x + \frac{x^3}{y^2} + \frac{5}{y} = C$; (4). $xy = C$; **1.9.3** (1). $\mu = e^x$; (2). $\mu = (x^2+y^2)^{\frac{3}{2}}$; **1.9.4** (1). The substitution is $y' = z$ and the final solution

$$C_1x - C_1^2y = \ln \mid C_1x + 1 \mid + C - 2;$$

(2). $y = \frac{x^3}{3} + C_1\frac{x^2}{2} + C_2$; (3). $y' = p,\ y'' = pp'$ and the final solution is $y = C$ and $y^3 = C_1(x + C_2)^2$; (4). $C_1y^2 - 1 = (C_1x + C_2)^2$; (5). The equation in $p = p(y)$ is $p' - \frac{1}{y}p = y$ with the solution $y = C_1$ and $\frac{y}{y+C_2} = C_3e^{Cx}$; (6). Divide by y^2 to obtain the equation $x^2\frac{y''}{y} = (1 - x\frac{y'}{y})$. Use the substitution $\frac{y'}{y} = z$ to obtain the equation $z' = -\frac{2z}{x} + \frac{1}{x^2}$ The final solution is $y = xe^{\frac{C_1}{x}}C_2$; (7). Divide by y^2; (8). $y = C_2e^{C_1x^2}$,

Chapter 2

Linear differential equations of order higher than one

2.1 Linear differential equations with constant coefficients

To integrate a *homogenous linear differential equation with constant (real) coefficients*

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y^{'} + a_n y = 0$$

we attache the (polynomial) *characteristic equation*

$$a_0 r^n + a_1 r^{n-1} + \ldots + a_{n-1} + a_n = 0$$

and we find all its roots $r_1, \ldots, r_n \in \mathbb{C}$. A root r from this set may be a multiple real root of order k with $k \geq 1$ or it may be a multiple complex root and its conjugate $r = \alpha + i\beta,\ \overline{r} = \alpha - i\beta,\ \ \beta \neq 0$ of order k with $k \geq 1$. The general solution of the above differential equation is the sum of all terms (associated to the roots) of form

$$(C_1 + C_2 x + \ldots + C_k x^{k-1}) e^{rx}$$

if $r \in R$ is a multiple root of order k and of form

$$P_{k-1}(x) e^{\alpha x} \cos \beta x + Q_{k-1}(x) e^{\alpha x} \sin \beta x$$

if $\alpha + \beta i$ and $\alpha - \beta i$ are roots of order k. Here P_{k-1}, Q_{k-1} are polynomials of degree $k-1$ with coefficients arbitrary constants.

Example 2.1.1. Integrate the equation $y^{(5)} - 2y^{(4)} - 16y' + 32y = 0$.

Solution. The characteristic equation is $r^5 - 2r^4 - 16r + 32 = 0$ hence, by decomposing we have

$$r^4(r-2) - 16(r-2) = 0$$

$$(r-2)(r-2)(r+2)(r^2+4) = 0.$$

The roots are $r_1 = r_2 = 2$ multiple of order 2; $r_3 = -2$ simple of order one; $r_4 = 2i, r_5 = -2i$ simple of order one. We apply the above regulae to obtain the general solution

$$y = (C_1 + C_2x)e^{2x} + C_3e^{-2x} + C_4\cos 2x + C_5\sin 2x.$$

To integrate a *non-homogenous linear differential equation with constant (real) coefficients*

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny = f(x), f(x) \neq 0$$

where $f(x)$ is sums and products of functions like: $b_0+b_1x+\ldots+b_mx^m$, $e^{\alpha x}, \cos\beta x, \sin\beta x$ we solve first the homogenous equation and find its general solution y_o and then we search for a particular solution y_p with *the method of nondeterminate coefficients.*The general solution will be

$$y = y_o + y_p.$$

If $f(x)$ is of the form

$$e^{\alpha x}(P(x)\cos\beta x + Q(x)\sin\beta x)$$

we search a particular solution of the form

$$y_p = x^s e^{\alpha x}(R_m(x)\cos\beta x + T_m(x)\sin\beta x),$$

where s is equal to zero if $\alpha + \beta i$ is not a root of the characteristic equation and is the order of multiplicity of the root $\alpha + \beta i$ otherwise. R_m, T_m are polynomials of degrees greater or equal to the degrees of P and Q. For finding the coefficients of R_m, T_m we replace y_p in the non-homogenous equation and we identify the same terms.

Example 2.1.2. Integrate the equation $y''' - 6y'' + 9y' = xe^{3x} + e^{3x}\cos 2x$.

Solution. The characteristic equation is $r^3 - 6r^2 + 9r = 0$ and has $r_1 = r_2 = 3$ a root of order 2 and $r_3 = 0$ a simple root. So, the general solution of the homogenous equation is

$$y_o = (C_1 + C_2x)e^{3x} + C_3.$$

The second member of the equation has two terms of different forms: for the first xe^{3x}, we have $\gamma = \alpha + \beta i = 3$ and for the second $e^{3x}\cos 2x$, we have $\alpha + \beta i = 3 + 2i$. These numbers are different hence we search two separate particular solutions of the equations

$$y''' - 6y'' + 9y' = xe^{3x}, \quad (1)$$
$$y''' - 6y'' + 9y' = e^{3x}\cos 2x \quad (2).$$

The number $\gamma = 3$ is a root of order 2 so a particular solution for (1) is of the form $y_{p_1} = x^2(ax + b)e^{3x}$. We replace it in (1) to obtain $a = \frac{1}{18}, b = -\frac{1}{18}$. Moreover $\alpha + \beta i = 3 + 2i$ is not a root for the characteristic equation hence we search for a particular solution of the equation (2) of the form $y_{p_2} = e^{3x}(c\cos 2x + d\sin 2x)$. We replace it in (2) to obtain $c = -\frac{3}{52}, d = -\frac{1}{26}$. The general solution is

$$y = y_o + y_{p_1} + y_{p_2} = (C_1 + C_2x)e^{3x} + C_3 + x^2(\frac{1}{18}x - \frac{1}{18})e^{3x} + e^{3x}(-\frac{3}{52}\cos 2x - \frac{1}{26}\sin 2x).$$

Example 2.1.3. Integrate the equation $y'' + 4y = \sin x \sin 2x$.

Proof. The characteristic equation is $r^2 + 4 = 0$ with the complex conjugate roos of order one $r_1 = 2i, r_2 = -2i$. So, the solution of the homogenous equation is $y_o = C_1\cos 2x + C_2\sin 2x$.

For the non-homogenous part $f(x) = \sin x \sin 2x$ we use the formula $\sin a \sin b = \frac{1}{2}(\cos(a - b) - \cos(a + b))$ to obtain $f(x) = \frac{1}{2}\cos x - \frac{1}{2}\cos 3x$. The first term $\frac{1}{2}\cos x$ give the number $\gamma = 1$ which is not a root of the characteristic equation and the second term $-\frac{1}{2}\cos 3x$ gives a different number $\delta = 3$ which again is not a root of the characteristic equation. Hence we will search for two separate particular solutions of the equations

$$y'' + 4y = \frac{1}{2}\cos 2x, \quad (1)$$
$$y'' + 4y = -\frac{1}{2}\cos 3x \quad (2).$$

A particular solution is of the form $y_{p_1} = a\cos x$ with $y'_{p_1} = -a\sin x$ and $y''_{p_2} = -a\cos x$. We replace these terms in (1) to obtain

$$-a\cos x + 4a\cos x = \frac{1}{2}\cos 2x,$$

thus $3a = \frac{1}{2}$ which gives $a = \frac{1}{6}$. The other particular solution for (2) is of the form $y_{p_2} = b\cos 3x$ and if we replace it we obtain $b = \frac{1}{10}$. The general solution is

$$y = C_1\cos 2x + C_2\sin 2x + \frac{1}{6}\cos x + \frac{1}{10}\cos 3x.$$

2.2 Euler's equations

The *equation of Euler* is

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \ldots + a_{n-1}xy' + a_ny = f(x).$$

To solve this equation we reduce it to a linear equation with constant coefficients by using the substitution $x = e^t$ for $x > 0$ (or $x = -e^t$ for $x < 0$). The new linear equation has the attached characteristic equation of the form

$$a_0r(r-1)(r-2)\ldots(r-n+1) + \ldots + a_{n-2}r(r-1) + a_{n-1}r + a_n = 0.$$

To write the linear equation we change each term $x^ky^{(k)}$ with a product of k factors decreasing by unity; $r(r-1)\ldots(r-k+1)$. Alternatively, in practice we prefer to change in the given equation: x by e^t; $xy'(x)$ by $y'(t)$; $x^2y''(x)$ by $y''(t)-y'(t)$; $x^3y'''(x)$ by $y'''(t) - 3y''(t) + 2y'(t) + 2$ and so on.

Example 2.2.1. Integrate the equation $x^3y''' - x^2y'' + 2xy' - 2y = x^3$

Solution.. The characteristic equation is

$$r(r-1)(r-2) - r(r-1) + 2r - 2 = 0$$

hence by decomposition we obtain $(r-1)(r^2 - 3r + 2) = 0$ with the multiple root $r_1 = r_2 = 1$ of order 2 and the simple root $r_3 = 2$. The general solution of the homogenous linear equation is

$$y_o = (C_1 + C_2t)e^t + C_3e^{2t}.$$

To solve the given Euler's equation we multiply all factors in the characteristic equation an we obtain $r^3 - 4r^2 + 5r - 2 = 0$. Now, immediately we obtain from this the linear non-homogenous equation with constant coefficients (obtained with the substitution $x = e^t > 0$):

$$y''' - 4y'' + 5y' - 2y = e^{3t}.$$

Since 3 is not a root of the characteristic equation we search for a particular solution of the form $y_p = ae^{3t}$. Replacing it we obtain

$$27ae^{3t} - 36ae^{3t} + 15ae^{3t} - 2ae^{3t} = e^{3t}$$

hence $4ae^{3t} = e^{3t}$. We obtain $a = \frac{1}{4}$ and the general solution is

$$y = y_o + y_p = (C_1 + C_2 t)e^t + C_3 e^{2t} + \frac{1}{4}e^{3t} = (C_1 + C_2 \ln x) + C_3 x^2 + \frac{1}{4}x^3, \quad x > 0.$$

For $x < 0$ the formula is similar, so the general solution will be

$$y = (C_1 + C_2 \ln |x|) + C_3 x^2 + \frac{1}{4}x^3.$$

Example 2.2.2. Integrate the equation $x^2 y'' - xy' - 3y = 8x^3$.

Solution. The characteristic equation is $r(r-1) - r - 3 = 0$ that is $r^2 - 2r - 3 = 0$ which has two roots $r_1 = 3$ and $r_2 = -1$. The linear equation is

$$y'' - 2y' - 3y = 8e^{3t}$$

with the homogenous solution $y_o = C_1 e^{3t} + C_2 e^{-t}$. Since $f(t) = 8e^{3t}$ and 3 is a root of the characteristic equation we search for a particular solution of the form $y_p = Ate^{3t}$. We have $y_p' = Ae^{3t}(3t+1)$ and $y_p'' = Ae^{3t}(9t+6)$. We replace these expressions to obtain

$$Ae^{3t}(9t+6) - 2Ae^{3t}(3t+1) - 3Ae^{3t}t = 8e^{3t}.$$

It follows

$$Ae^{3t}(9t + 6 - 6t - 2 - 3t) = 8e^{3t},$$

hence $4A = 8$ and $A = 2$. We obtain

$$y = y_o + y_p = C_1 e^{3t} + C_2 e^{-t} + 2te^{3t} = C_1 e^{3\ln x} + C_2 e^{-\ln x} + 2\ln x \cdot e^{3\ln x}.$$

The general solution is

$$y = C_1 x^3 + \frac{C_2}{x} + 2x^3 \ln |x|.$$

2.3 Problems

Problem 2.3.1. Integrate the following differential equations:

(1) $y'' + y' - 2y = 0$;

(2) $y'' + 4y' + 3y = 0;$

(3) $y'' - 2y' = 0;$

(4) $y''' - 8y = 0;$

(5) $y^{(v)} - 6y^{(iv)} + 9y''' = 0;$

(6) $y'' + y' - 2y = 3xe^x;$

(7) $y'' - y = 2e^x - x^2;$

(8) $y'' - 3y' + 2y = \sin x;$

(9) $y''' - y'' + y' - y = x^2 + x$

(10) $y^{(iv)} + y'' = 7x - 3\cos x.$

Problem 2.3.2. Integrate the following Euler's equations:

(1) $x^2y'' - 4xy' + 6y = 0;$

(2) $x^2y'' - xy' + y = 8x^3;$

(3) $x^2y'' - 3xy' + 4y = x + 2;$

(4) $x^2y'' - 2y = \sin\ln x;$

(5) $x^2y''' = 2y';$

Solutions: **2.3.1** (1). $y = C_1e^x + C_2e^{-2x}$; (2). $y = C_1e^{-x} + C_2e^{-3x}$; (3). $y = C_1 + C_2e^{2x}$; (4). $y = C_1e^{2x} + e^{-x}(C_2\cos(x\sqrt{3}) + C_4\sin(x\sqrt{3}))$; (5). $y = C_1 + C_2x + C_3x^2 + e^{3x}(C_4 + C_5x)$; (6). $y = C_1e^x + C_2e^{-2x} + (\frac{x^2}{2} - \frac{x}{3})e^x$; (7). $y = C_1e^x + C_2e^{-x} + xe^x + x^2 + 2$; (8). $y = C_1e^x + C_2e^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x$; (9). $y = C_1e^x + C_2\cos x + C_3\sin x - x^2 + 3x - 1.$

2.3.2 (1). $y = C_1x^2 + C_2x^3$; (2). $y = x(C_1 + C_2\ln|x|) + C_3\ln^2|x|$; (3). $y = C_1x^2 + C_2x^2\ln x + x + \frac{1}{2}$; (4). $y = C_1x^2 + C_2x^{-1} + \frac{1}{10}\cos\ln x - \frac{3}{10}\sin\ln x$; (5). $y = C_1 + C_2\ln|x| + C_3x^3.$

Chapter 3

Systems of differential equations and partial differential equations

3.1 Linear systems with constant coefficients

A *linear system with constant coefficients* is a system of the form

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \ldots a_{1n}y_n + f_1(x) \\ y_2' = a_{21}y_1 + a_{22}y_2 + \ldots a_{2n}y_n + f_2(x) \\ \ldots \\ y_n' = a_{n1}y_1 + a_{1n2}y_2 + \ldots a_{nn}y_n + f_n(x) \end{cases}$$

where $y_1 = y_1(x), \ldots y_n = y_n(x)$ are the unknown differentiable functions and $a_{ij} \in \mathbb{R}, f_i \in \mathrm{C}^1(\mathbb{R}), i, j \in \{1, \ldots, n\}$. To solve the above system the shortest method is the *elimination method* which allow us to reduce the above system to a linear differential equation of order n with constant coefficients. We will describe this method in the following two relevant examples. In practice when we have systems with three unknowns we use the notations for the unknown functions: $y = y(x), z = z(x), w = w(x), x \in \mathbb{R}$ or $x = x(t), y = y(t), z = z(t), t \in \mathbb{R}$.

Example 3.1.1. Integrate the following system

$$\begin{cases} y' = y - z + w \\ z' = y + z - w \\ w' = 2y - z \end{cases} .$$

Solution. We will transform by elimination, the above system into a linear differential equation of order 3 in y. We write the first equation $y' - y = -z + w$ and we differentiate to obtain

$$y'' - y' = -z' + w' \quad (1).$$

We replace y', z' and w' from the first, the second and the third equation of the system to obtain

$$y'' - (y - z + w) = -(y + z - w) + 2y - z.$$

All the terms with y are kept in the left part hence

$$y'' - 2y = -3z + 2w \quad (2).$$

Again we differentiate and make the same replacements to obtain $y''' - 2y' = -3z' + 2w'$ hence $y''' - 2y + 2z - 2w = -3y - 3z + 3w + 4y - 2z$. We have a third equation with y in the left part

$$y''' - 3y = -7z + 5w \quad (3).$$

We write the equivalent system formed by the equations (1),(2),(3) (viewed as a system with 3 equations and two unknowns z, w) and we compute the rank of the attached matrix

$$\begin{cases} -z + w = y' - y \\ -3z + 2w = y'' - 2y \\ -7z + 5w = y''' - 3y \end{cases} .$$

The attached matrix is

$$A = \begin{pmatrix} -1 & 1 \\ -3 & 2 \\ -7 & 5 \end{pmatrix}$$

which has rank 2. For the above system to be compatible the extended attached matrix must have rank two as well, hence its determinant of order 3 must be zero.

We obtain the equation

$$\begin{vmatrix} -1 & 1 & y' - y \\ -3 & 2 & y'' - 2y \\ -7 & 5 & y''' - 3y \end{vmatrix} = 0$$

that is

$$-2y''' + 6y - 15y' + 15y - 7y'' + 14y + 14y - 14y' - 14y + 5y'' - 10y + 3y'' - 9y = 0.$$

We have $y'' - 2y'' - y' + 2y = 0$ with the characteristic equation $r^3 - 2r^2 - r + 2 = 0$ and decomposing $(r-2)(r^2-1) = 0$ we obtain the simple roots $r_1 = 1, r_2 = 2, r_3 = -1$. The solution is $y = C_1e^x + C_2e^{2x} + C_3e^{-x}$. For finding z we multiply equation (1) by -2 and add equation (2) to get $2z - 3z = -z = -2y' + 2y + y'' - 2y$ hence $z = 2y' - y''$ We obtain

$$z = 2(C_1e^x + 2C_2e^{2x} - C_3e^{-x}) - (C_1e^x + 4C_2e^{2x} + C_3e^{-x})$$
$$= C_1e^x - 3C_3e^{-x}.$$

Finally from equation (1) we have

$$w = y' - y + z = C_1e^x + 2C_2e^{2x} - C_3e^{-x} - C_1e^x - C_2e^{2x} - C_3e^{-x} + C_1e^x - 3C_3e^{-x}$$
$$= C_1e^x + C_2e^{2x} - 5C_3e^{-x}.$$

Example 3.1.2. Integrate the system

$$\begin{cases} y' = -z + w \\ z' = w \\ w' = -y + w \end{cases}.$$

Solution. The first equation remains

$$y' = -z + w \quad (1).$$

Differentiating we obtain $y'' = -z' + w'$ hence $y'' = -w - y + w = -y$. The second equation is now

$$y'' + y = 0 \quad (2).$$

Differentiating this equation once more we get $y'''+y' = 0$ and by replacing $y' = -z+w$ (which is the first equation of the system) we obtain

$$y''' = z - w \quad (3).$$

Equations (1), (2) and (3) give us the system $\begin{cases} -z + w = y' \\ 0 = y'' - 2y \\ z - w = y''' \end{cases}$, with the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

This matrix has rank 1 and the extended matrix is

$$\overline{A} = \begin{pmatrix} -1 & 1 & y' \\ 0 & 0 & y'' + y \\ 1 & -1 & y''' \end{pmatrix}.$$

We want that this matrix $\overline{A}$ to have the same rank 1 thus we require that all minors of order 2 to be 0. We obtain

$$\begin{vmatrix} -1 & y' \\ 0 & y'' + y \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & y' \\ 1 & y''' \end{vmatrix} = 0,$$

which give us the resolvent equations $\begin{cases} y'' + y = 0 \\ y''' + y' = 0 \end{cases}$, with the characteristic equations $\begin{cases} r^2 + 1 = 0 \\ r^3 + r = 0 \end{cases}$. We search for the *common solutions*! Since the first equation has roots $r_{1,2} = \pm i$ and the second equation has roots $r_{1,2} = \pm i, r_3 = 0$ we obtain that

$$y = C_1 \cos x + C_2 \sin x.$$

For finding w we use the third equation of the given system which tells us $w'-w=-y$ hence $w'-w=-C_1\cos x-C_2\sin x$. This is a linear non-homogenous equation of order one with the homogenous solution $w_o=C_3e^x$. Since i is not a root for the characteristic equation $(r-1=0)$ we search for a particular solution of the form $w_p=A\cos x+B\sin x$. We replace it to obtain

$$-A\sin x+B\cos x-A\cos x-B\sin x=-C_1\cos x-C_2\sin x,$$

hence $-A+B=-C_1$ and $-A-B=-C_2$. We add these equation to get $-2A=-C_1-C_2$ and we get $A=\frac{C_1+C_2}{2}, B=\frac{C_2-C_1}{2}$. The solution w is now

$$w=w_o+w_p=C_3e^x+\frac{C_1+C_2}{2}\cos x+\frac{C_2-C_1}{2}\sin x.$$

For finding z we use the second equation of the given system $z'=w$, that is $z=\int(C_3e^x+\frac{C_1+C_2}{2}\cos x+\frac{C_2-C_1}{2}\sin x)dx$ hence

$$w=C_3e^x+\frac{C_1+C_2}{2}\sin x-\frac{C_2-C_1}{2}\cos x.$$

Example 3.1.3. Integrate the system

$$\begin{cases} x'=x+y+e^t \\ y'=x+y-e^t \end{cases}, \quad x=x(t), y=y(t), t\in\mathbb{R}.$$

Solution. In the first equation we keep all terms in x on left side: $x'-x=y+e^t$. We differentiate and replace x' and y':

$$x''-x'=y'+e^t,$$

$$x''-x-y-e^t=x+y-e^t+e^t.$$

We get $x''-2x=2y+e^t$. Form the first equation we have

$$y=x'-x-e^t \quad (1)$$

which we replace in the previous equation ,hence

$$x''-2x=2x'-2x-2e^t+e^t$$

We obtain the linear non-homogenous equation of order 2

$$x'' - 2x' = -e^t.$$

The characteristic equation is $r^2 - 2r = 0$ with the roots $r_1 = 0, r_2 = 2$ hence $x_o = C_1 + C_2 e^2 t$. Since 1 is not a root of the characteristic equation we search for a particular solution of the form $x_p = Ae^t$. We obtain $Ae^t - 2Ae^t = -e^t$ hence $A = 1$ and the solution is

$$x = x_o + x_p = C_1 + C_2 e^{2t} + e^t.$$

For finding y we use (1) and we get

$$y = (C_1 + C_2 e^{2t} + e^t)' - (C_1 + C_2 e^{2t} + e^t) - e^t = -C_1 + C_2 e^{2t} - e^t.$$

3.2 Symmetric Systems

A *symmetric system of order n* (where n is a non-negative integer) is a system of the form

$$\frac{dy_1}{f_1(y_1, \dots y_{n+1})} = \dots = \frac{dy_n}{f_n(y_1, y_2, \dots, y_{n+1})} = \frac{dy_{n+1}}{f_{n+1}(y_1, \dots, y_{n+1})},$$

where $y_1, \dots, y_{n+1}$ are the unknown functions. For solving such a system we search for n independent *prime integrals*. A prime integral F is a constant map on any solution of the system, that is a map of the form

$$F : \mathbb{R}^{n+1} \to \mathbb{R}, \quad F(y_1, \dots, y_{n+1}) = C.$$

In this book we concentrate on systems with 3 unknowns denoted

$$\frac{dx}{f(x,y,z)} = \frac{dy}{g(x,y,z)} = \frac{dz}{h(x,y,z)}.$$

For this system we must find 3 independent prime integrals. We will solve also systems with 4 unknowns denoted

$$\frac{dx}{f(x,y,z,u)} = \frac{dy}{g(x,y,z,u)} = \frac{dz}{h(x,y,z,u)} = \frac{du}{j(x,y,z,u)}.$$

For this kind of system we must find 3 independent prime integrals. To determine prime integrals we have the following methods:

-if two rapports depend only on 2 unknowns, the equality of these 2 rapports is a differential equation which in general can be integrated;

-if from a prime integral we can express an unknown as a function of the rest of the unknowns we can reach sometimes to the above situation;

-we apply *integrable combinations* of the form

$$\frac{dy_1}{f_1} = \ldots = \frac{dy_{n+1}}{f_{n+1}} = \frac{g_1 dy_1 + \ldots g_{n+1} dy_{n+1}}{g_1 f_1 + \ldots + g_{n+1} f_{n+1})},$$

where we choose convenable functions (usually constant functions) $g_1, \ldots g_{n+1}$ such that

$$g_1 dy_1 + \ldots g_{n+1} dy_{n+1} = dG$$

$$g_1 f_1 + \ldots g_{n+1} f_{n+1} = f \circ G.$$

Next, we apply the above methods to this new system, and if necessary again the above methods until we find n independent prime integrals. .

Example 3.2.1. Integrate the symmetric system

$$\frac{dx}{z^2 - y^2} = \frac{dy}{z} = \frac{dz}{-y}.$$

Solution. For the first prime integral we take the equality of the last two rapports $\frac{dy}{z} = \frac{dz}{-y}$ which become $-ydy = zdz$. We integrate $\int -ydy = \int zdz$ to obtain $-\frac{y^2}{2} = \frac{z^2}{2} + C_1$, hence the first prime integral

$$y^2 + z^2 = C_1.$$

Next we do some integrable combinations by amplifying the second rapport with z, the third rapport with z and adding the obtained terms:

$$\frac{dx}{z^2 - y^2} = \frac{dy}{z} = \frac{dz}{-y} = \frac{zdy + ydz}{z^2 - y^2}.$$

We separate the equality of the first and the last rapport and using the well-known formula $d(yz) = dyz + ydz$ we obtain

$$\frac{dx}{z^2 - y^2} = \frac{d(yz)}{z^2 - y^2},$$

hence $dx = d(yz)$ which is integrable with the solution $x - yz = C_2$. The solution of this system is

$$\begin{cases} y^2 + z^2 = C_1 \\ x - yz = C_2 \end{cases}.$$

Example 3.2.2. Integrate the system

$$\begin{cases} y' = y(y+z) \\ z' = z(y+z) \end{cases}.$$

Solution. This is a symmetric system since if we use $y' = \frac{dy}{dx}$, $z' = \frac{dz}{dx}$ we have

$$\frac{dy}{y(y+z)} = \frac{dz}{z(y+z)} = dx.$$

We apply the first method for the first two rapports

$$\frac{dy}{y(y+z)} = \frac{dz}{z(y+z)}.$$

Simplifying with $y + z$ we get the integrable equation $\frac{dy}{y} = \frac{dz}{z}$, which leads to

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

with the prime integral

$$\frac{y}{z} = C_1.$$

Next we apply the second method by replacing $y = zC_1$ and obtaining the equation

$$\frac{dz}{z^2(C_1+1)} = dx.$$

Integrating $\int \frac{dz}{z^2(C_1+1)} = \int dx$ we obtain the solution

$$-\frac{1}{z(C_1+1)} = x - C_2$$

hence the second prime integral (by replacing $\frac{y}{z} = C_1$) is

$$-\frac{1}{z(\frac{y}{z}+1)} = x - C_2,$$

that is

$$x + \frac{1}{y+z} = C_2.$$

These two prime integrals are independent.

3.3 Partial differential equations

A *homogenous partial differential equation of order one* is an equation of the form

$$f_1 \frac{\partial u}{\partial x_1} + \ldots + f_n \frac{\partial u}{\partial x_n} = 0$$

where $f_1 = f_1(x_1, \ldots, x_n), \ldots, f_n = f_n(x_1, \ldots, x_n)$ are functions depending on n variables $x_1, \ldots, x_n$ and $u \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R})$ is the unknown function. To solve this equation we consider the symmetric system

$$\frac{dx_1}{f_1} = \ldots = \frac{dx_n}{f_n}$$

and we find $n-1$ prime integrals $F_1, \ldots, F_{n-1}$. Now, for any function G on $n-1$ variables the composed function

$$u = G(F_1, \ldots, F_{n-1})$$

is the general solution of the above partial differential equation.

A *Cauchy problem* for the equation

$$f_1 \frac{\partial u}{\partial x_1} + \ldots + f_n \frac{\partial u}{\partial x_n} = 0,$$

is the problem of finding that solution for this equation which for a fixed value of some variable, for example $x_i = a \in \mathbb{R}, i \in \{1, \ldots, n\}$, we can reduce it (the solution) to a given function

$$y(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

We also say that we look for the integral surface which contains a given curve.

Example 3.3.1. Find the solution of the Cauchy's problem for the following equation

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + xy \frac{\partial u}{\partial z} = 0$$

$$u(x, y, 0) = x^2 + y^2.$$

Solution. The symmetric system is $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy}$. We take the firs equality and we integrate $\int \frac{dx}{x} = \int \frac{dy}{y}$ to obtain $\ln|x| = \ln|y| + \ln|C_1|$. The first prime integral is

$$\frac{x}{y} = C_1.$$

Next we amplify the first rapport with y, the second with x and we add to obtain

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy} = \frac{ydx + xdy}{2xy}.$$

We separate the last equality to get $\frac{dz}{xy} = \frac{d(xy)}{2xy}$, hence we can integrate $\int 2dz = \int d(xy)$ to obtain $xy - 2z = C_2$. The general solution is

$$u(x, y, z) = G(\frac{x}{y}, xy - 2z).$$

For the Cauchy problem we write the system formed by the prime integrals and the initial conditions

$$\begin{cases} \frac{x}{y} = C_1 \\ xy - 2z = C_2 \\ z = 0 \\ u = x^2 + y^2 \end{cases}.$$

So, if we solve the system of the first 3 equations we get $x = yC_1$ and $xy = C_2$. We multiply these 2 relations to obtain $x^2y = yC_1C_2$ hence $x^2 = C_1C_2$. Similarly we obtain (by dividing the two relations) $y^2 = \frac{C_2}{C_1}$, hence if we replace in the last equation of the above system we get $u = C_1C_2 + \frac{C_2}{C_1}$. We use the prime integrals to get the solution of Cauchy's problem

$$u(x, y, z) = \frac{x}{y}(xy - 2z) + \frac{xy - 2z}{\frac{x}{y}}.$$

A *quasi-linear partial differential equation of order one* is an equation of the form

$$f_1\frac{\partial u}{\partial x_1} + \ldots + f_n\frac{\partial u}{\partial x_n} = g$$

where $f_1 = f_1(x_1, \ldots, x_n, u), \ldots, f_n = f_n(x_1, \ldots, x_n, u), g = g(x_1, \ldots, x_n, u)$ are functions depending on $n + 1$ variables $x_1, \ldots, x_n, u$ and $u \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R})$ is the unknown function. To solve this equation we consider the symmetric system

$$\frac{dx_1}{f_1} = \ldots = \frac{dx_n}{f_n} = \frac{du}{g}$$

and we find n prime integrals $F_1, \dots, F_n$. Notice that in this case the prime integrals are depending on $n+1$ variables

$$F_1 = F_1(x_1, \dots, x_n, u), \dots, F_n = F_n(x_1, \dots, x_n, u).$$

Now, for any function G on n variables the relation

$$G(F_1, \dots, F_n) = 0$$

is the *implicit* solution of the above partial differential equation. As in the case of homogenous equation we can solve similarly Cauchy's problems.

Example 3.3.2. Find the general solution of the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -xy$$

and the surface passing through the curve $y = x^2,\ u = x^3$.

Solution. The symmetric system is

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-xy}.$$

The equality of the first rapports gives us the prime integral $\frac{dx}{xu} = \frac{dy}{yu}$, that is

$$\frac{x}{y} = C_1 \quad (1.)$$

Next we amplify the first rapport with y, the second with x and we add to obtain

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-xy} = \frac{ydx + xdy}{2xyu}.$$

The equality of the last rapports is $\frac{du}{-xy} = \frac{ydx+xdy}{2xyu}$, thus

$$\frac{du}{-xy} = \frac{d(xy)}{2xyu}.$$

We integrate $\int 2udu = -\int d(xy)$ to obtain

$$u^2 + xy = C_2 \quad (2).$$

The general solution is

$$G(\frac{x}{y}, u^2 + xy) = 0.$$

To solve the Cauchy problem, or equivalently to find the surface passing through the given curve we take x as a parameter:

$$x = x, y = x^2, u = x^3$$

and we replace these relations in the prime integrals (1) and (2) to obtain

$$\frac{1}{x} = C_1, \quad x^6 + x^3 = C_2.$$

We eliminate x in these equations to get

$$\frac{1}{C_1^6} + \frac{1}{C_1^3} = C_2.$$

Using again the prime integrals (1) and (2) we obtain the final solution

$$\left(\frac{y}{x}\right)^6 + \left(\frac{y}{x}\right)^3 = u^2 + xy.$$

Example 3.3.3. Find the solution of the following Cauchy problem

$$x\frac{\partial u}{\partial x} + (xu + y)\frac{\partial u}{\partial y} = u$$

satisfying the initial conditions $x + y = 2u,\ xu = 1$.

Solution. The symmetric system is

$$\frac{dx}{x} = \frac{dy}{xu + y} = \frac{du}{u}.$$

The equality of the first and the third rapport $\frac{dx}{x} = \frac{du}{u}$ gives by integration

$$\frac{x}{u} = C_1.$$

From this prime integral we express x as $x = C_1 u$ and we replace it in the symmetric system

$$\frac{C_1 du}{C_1 u} = \frac{dy}{C_1 u^2 + y} = \frac{du}{u}.$$

Next we have $\frac{dy}{du} = \frac{C_1u^2+y}{u}$ hence we have the equation

$$\frac{dy}{du} = C_1u + \frac{y}{u} \quad (1).$$

We denote $\frac{y}{u} = t$ so $dy = tdu + udt$. It follows that

$$\frac{dy}{du} = t + u\frac{dt}{du} \quad (2).$$

From (1) and (2) we have $t + u\frac{dt}{du} = C_1u + t$ so $u\frac{dt}{du} = C_1u$. We get $dt = C_1du$, that is $t = C_1u + C_2$. Moreover since $\frac{y}{u} = t$ and $\frac{x}{u} = C_1$ we obtain the second prime integral

$$\frac{y}{u} - x = C_2.$$

The general solution is

$$G(\frac{x}{u}, \frac{y}{u} - x) = 0.$$

For the Cauchy problem we take, in the initial conditions, u as parameter, that is

$$x = \frac{1}{u},\ y = 2u - \frac{1}{u},\ u = u.$$

We replace these in the prime integrals for obtaining

$$\frac{1}{u^2} = C_1 \qquad (3)$$

$$\frac{2u - \frac{1}{u}}{u} - \frac{1}{u} = C_2 \quad (4).$$

We eliminate u, by using $u = \frac{1}{\sqrt{C_1}}$, obtained from (3) and replacing in (4)

$$\frac{\frac{2}{\sqrt{C_1}} - \sqrt{C_1}}{\frac{1}{\sqrt{C_1}}} - \sqrt{C_1} = C_2.$$

We have that $2 - C_1 - \sqrt{C_1} = C_2$ and replacing the above prime integrals we find the final solution

$$2 - \frac{x}{u} - \sqrt{\frac{x}{u}} = \frac{y}{u} - x.$$

3.4 Problems

Problem 3.4.1. Integrate the following linear system with constant coefficients

$$(1).\quad \begin{cases} y' = 2y + z \\ z' = 3y + 4z \end{cases};$$

$$(2).\quad \begin{cases} y' = y - z \\ z' = z - 4y \end{cases};$$

$$(3).\quad \begin{cases} y' = y - z + w \\ z' = y + z - w \\ w' = 2y - z \end{cases};$$

$$(4).\quad \begin{cases} y' = y - z - w \\ z' = y + z \\ w' = 3y + w \end{cases};$$

$$(5).\quad \begin{cases} y' = 3y - z + w \\ z' = y + z + w \\ w' = 4y - z + 4w \end{cases};$$

$$(6).\quad \begin{cases} y' = y - z + w \\ z' = y + z - w \\ w' = 2w - z \end{cases};$$

$$(7).\quad \begin{cases} x' = y + 2e^t \\ y' = x + t^2 \end{cases};$$

(8). $\begin{cases} x' = y - 5\cos t \\ y' = 2x + y \end{cases}$;

(9). $\begin{cases} x' = 3x + 2y + 4e^{5t} \\ y' = x + 2y \end{cases}$;

(10). $\begin{cases} x' = 2x - 4y + 4e^{-2t} \\ y' = 2x - 2y \end{cases}$;

(11). $\begin{cases} x' = 2x + y - 2z - t \\ y' = 1 - x \\ z' = x + y - z - t \end{cases}$;

(12) $\begin{cases} x' = 4x + y - e^{2t} \\ y' = y - 2x \end{cases}$.

Problem 3.4.2. Integrate the symmetric systems

(1). $\frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{y+x}$;

(2). $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{yx}$;

(3). $\frac{dx}{z^2-y^2} = \frac{dy}{z} = \frac{dz}{-y}$;

(4). $\frac{dx}{x(y+z)} = \frac{dy}{z(z-y)} = \frac{dz}{y(y-z)}$.

Problem 3.4.3. Find the general solution for the following partial differential equations

(1). $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$;

(2). $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$;

(3). $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = x - y;$

(4). $2x\frac{\partial u}{\partial x} + (y-x)\frac{\partial u}{\partial y} - x^2 = 0;$

(5). $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + (z+u)\frac{\partial u}{\partial z} = xy.$

Problem 3.4.4. Find the general solution and the solution for the Cauchy problem of the following equations

(1). $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0, \quad u(x,1) = 2x;$

(2). $2\sqrt{x}\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0, \quad u(1,y) = y^2;$

(3). $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 2\frac{\partial u}{\partial z} = 0, \quad u(1,y,z) = yz;$

(4). $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + xy\frac{\partial u}{\partial z} = 0, \quad u(x,y,0) = x^2 + y^2;$

(5). $x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = x^2 + y^2, \quad y = 1,\ u = x^2;$

(6). $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u - x^2 - y^2, \quad y = -2, u = x - x^2;$

(7). $u\frac{\partial u}{\partial x} + (u^2 - x^2)\frac{\partial u}{\partial y} + x = 0, \quad y = x^2, \quad u = 2x;$

(8). $x\frac{\partial u}{\partial x} + (xu + y)\frac{\partial u}{\partial y} = u, \quad x + y = 2u,\ xu = 1.$

Solutions: **3.4.1** (1). $y = C_1e^x + C_2e^{5x},\ z = -C_1e^x + 3C_2e^{5x}$; (2). $y = C_1e^{-x} + C_2e^{3x},\ z = 2C_1e^{-x} - 2C_2e^{3x}$; (3). $y = C_1e^x + C_2e^{2x} + C_3e^{-x},\ z = C_1e^x - 3C_3e^{-x},\ w = C_1e^x + C_2e^{2x} - 5C_3e^{-x}$; (4).$y = e^x(2C_2\sin 2x + 2C_3\cos 2x),\ z = e^x(C_1 - C_2\cos 2x + C_3\sin 2x),\ w = e^x(-C_1 - 3C_2\cos 2x + 3C_3\sin 2x)$; (5).$y = C_1e^x + C_2e^{2x} + C_3e^{5x},\ z = C_1e^x - 2C_2e^{2x} + C_3e^{5x},\ w = -C_1e^x - 3C_2e^{2x} + 3C_3e^{5x}$; (6). $y = (C_1 + C_2x)e^x + C_3e^{2x},\ z = (C_1 - 2C_2 + C_2x)e^x,\ w = (C_1 - C_2 + C_2x)e^x + C_3e^{2x}$; (7). $x = C_1e^t + C_2e^{-t} + te^t - t^2 - 2,\ y = C_1e^t - C_2e^{-t} + (t-1)e^t - 2t$; (8).$x = C_1e^{2t} + C_2e^{-t} + te^t - 2\sin t - \cos t,\ y = 2C_1e^{2t} - C_2e^{-t} + \sin t - 3\cos t$; (9). $x = C_1e^t + 2C_2e^{4t} + 3e^{5t},\ y = -C_1e^t + C_2e^{4t} + e^{5t}$; (10). $x = C_1(\cos 2t - \sin 2t) + C_2(\cos 2t + \sin 2t),\ y = C_1\cos 2t + C_2\sin 2t + e^{-2t}$; (11). $x = C_1e^t + C_2\cos t + C_3\sin t,\ y = -C_1e^t - C_2\sin t - C_3\cos t + 1,\ z = C_2\cos t +$

$C_3 \sin t - \frac{t-1}{2}$; (12). $x = C_1e^{2t} + C_2e^{3t} + (t+1)e^{2t},\ y = -2C_1e^{2t} - C_2e^{3t} - 2te^{2t}$. **3.4.2** (1). $\frac{y-x}{z-x} = C_1,\ (x+y+z)(z-x)^2 = C_2$; (2). $y^2-x^2 = C_1,\ z^2-x^2 = C_2$; (3). $y^2+z^2 = C_1,\ yz - x = C_2$; (4). $y^2 + z^2 = C_1,\ x(y-z) = C_2$. **3.4.3** (1). $u(x,y) = G(x^2+y^2)$; (2). $u(x,y,z) = G(\frac{y}{x}, \frac{z}{x})$; (3). $G(x^2 - y^2, x - y + u) = 0$; (4). $G(x^2 - 4u, \frac{(x+y)^2}{x}) = 0$; (5). $G(\frac{x}{y}, xy - 2u, \frac{z+u-xy}{x}) = 0$. **3.4.4** (1). $u(x,y) = 2xy$; (2). $u(x,y) = y^2e^{2\sqrt{x}-2}$; (3). $u(x,y,z) = (1-x+y)(2-2x+z)$; (4). $u(x,y,z) = (xy-2z)(\frac{x}{y}+\frac{y}{x})$; (5). $2x^2(y+1) = y^2 + 4u - 1$; (6). $x - 2y = x^2 + y^2 + u$; (7). $x^2 + u^2 = 5(xu - y)$; (8). $xu = (xu - y - x + 2u)^2$.

Chapter 4

Complex Analysis

4.1 Complex numbers. Basic results

Recall that the set of complex numbers is denoted $\mathbb{C}$. A complex number $z \in \mathbb{C}$ has an *algebraic form* $z = x + iy$ where $x = \mathrm{Re}z \in \mathbb{R}$ is the real part of z and $y = \mathrm{Im}z \in \mathbb{R}$ is the imaginary part of z. The symbol i is the imaginary unity and has the fundamental property that $i^2 = -1$. The *conjugate* of a complex number z is denoted $\overline{z} = x - iy$.

In some exercises is important to remember the following formulas

$$\mathrm{Re}z = x = \frac{z + \overline{z}}{2};$$

$$\mathrm{Im}z = y = \frac{z - \overline{z}}{2i}.$$

We recall some basic properties of complex numbers in the next proposition. The proof is proposed as a problem in the last section.

Proposition 4.1.1. *Let $z_1, z_2 \in \mathbb{C}$. The next statements holds*

a) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$;

b) $\overline{z_1 z_2} = \overline{z_1 z_2}$;

c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

We associate to z its geometrical image $M(x, y)$ in the plane xOy. The *module* of a complex number z is the non-negative real number $|z| = \sqrt{x^2 + y^2}$ and represents the distance form the origin O to M. A complex number has also a *trigonometrical form*

$$z = |z|(\cos\theta + i\sin\theta)$$

where $\theta = \arg z$ is the angle between Ox and OM in a positive sense and is called the *principal argument* of z. The set of all arguments is denoted

$$\text{Arg}z = \{\arg z + 2k\pi \mid k \in \mathbb{Z}\}.$$

To obtain the principal argument we apply $\arg z = \arctan\frac{y}{x} + k\pi$ with $k = 0$ if M is in the first quadrant, $k = 1$ if M is in the second or third quadrant and $k = 2$ if M is in the fourth quadrant. To recall the operations of complex numbers we solve the following problem.

Problem 4.1.2. Compute the following expressions:

a) $(2 + i) - 5(1 + i) + 2 + 3i$;

b) $(-1 - 4i)(3 + i)$;

c) $\frac{i+1}{5+2i}$;

Solution.

a) $(2+i)-5(1+i)+2+3i = 2+i-5-5i+2+3i = 2-5+2+i-5i+3i = -1-i$;

b) $(-1 - 4i)(3 + i) = -3 - i - 12i - 4i^2 = -3 - 4(-1) - i - 12i = 1 - 13i$;

c) To compute fractions of complex numbers (to divide complex numbers) we amplify the fraction with the conjugate of the denominator

$$\frac{i+1}{5+2i} = \frac{(i+1)(5-2i)}{(5+2i)(5-2i)} = \frac{5i - 2i^2 + 5 - 2i}{5^2 - 4i^2} = \frac{7+3i}{29}.$$

For multiplication it is sometimes more convenient to use the trigonometrical form. So let $z_1 = |z|(\cos\theta_1 + i\sin\theta_1),\ z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$ and let n be a nonnegative integer. Then

$$z_1 z_2 = |z_1| \cdot |z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)];$$
$$z^n = |z|^n[\cos(n\theta) + i\sin(n\theta)].$$

The roots of order n are given using the trigonometrical form:

$$z_k = \sqrt[n]{|z|}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

are the n distinct roots of z where $k = 0, \ldots, n-1$.

We identify sets of complex numbers with sets of points from the complex plane. For example the *circle* with center z_0 and radius $r > 0$ is

$$C(z_0; r) = \{z \in \mathbb{C};\ |\ z - z_0| = r\}.$$

The *disc* with center z_0 and radius $r > 0$ is

$$D(z_0; r) = \{z \in \mathbb{C}; |z - z_0| < r\},$$

and the *circular crown* with center z_0 and radiuses r, R is

$$C(z_0; r; R) = \{z \in \mathbb{C}; r < |z - z_0| < R\}.$$

Let $(z_n)_{n\geq 0}$ be a sequence of complex numbers where $z_n = x_n + iy_n$ or

$$z_n = \rho_n(\cos\theta_n + i\sin\theta_n),\ \rho_n \geq 0, \theta_n \in [0, 2\pi), n \geq 0.$$

As in the real case we say that the sequence is *convergent* if there is $z \in \mathbb{C}$ such that $\lim\limits_{n\to\infty} z_n = z$. We have that

$$\lim_{n\to\infty} z_n = z = x + iy$$

if and only if $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y$. Similarly we can prove that

$$\lim_{n\to\infty} z_n = \rho(\cos\theta + i\sin\theta), \theta \in [0, 2\pi)$$

if and only if $\lim\limits_{n\to\infty} |z_n| = |z| = \rho$ and if $\rho \neq 0,\ \lim\limits_{n\to\infty} \arg(z_n) = \arg z = \theta$. Using this last formula if we denote with e^z the limit $\lim\limits_{n\to\infty}\left(1 + \frac{z}{n}\right)^n$ we obtain Euler's formula

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y).$$

4.2 Complex functions of a complex variable

A complex function of a complex variable has the form

$$f : D \to \mathbb{C}, \quad f(z) = u(x,y) + iv(x,y)$$

where $u = \mathrm{Re}f, v = \mathrm{Im}f$ and $z = x + iy, \quad x, y \in \mathbb{R}, D \subseteq \mathbb{C}$. Let $z_0 = x_0 + iy_0$ be a cluster point of D. The derivative of f at z_0 is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

if the previous limit exists. If $f'(z_0)$ is finite we say that f is differentiable at z_0. A function which is differentiable at every point of a domain D is called holomorphic on D.

Theorem 4.2.1. *If f is differentiable at z_0 then the following relations are satisfied*

$$(C-R)\begin{cases} \frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \\ \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0) \end{cases}$$

(C-R) are called the Cauchy-Riemann equations. The converse is also true in appropriate conditions.

Theorem 4.2.2. *If u and v are function of class C^1 in a neighborhood of z_0 and the conditions (C-R) are satisfied then f is differentiable at z_0. Moreover*

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).$$

In then next lines we recall some properties of holomorphic functions. We denote by $\mathcal{H}(D)$ the set of all functions $f : D \to C$ which are holomorphic on D. Let $f \in \mathcal{H}(D), f = u + iv, u, v : D \to \mathbb{R}$.

Proposition 4.2.3. *If u or v are constant functions on D then f is a constant function on D.*

Proposition 4.2.4. *The real and the imaginary part of the holomorphic function f are harmonic functions on D, i.e.*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

on D.

Proposition 4.2.5. *If the real part of the holomorphic function f is known then f is determined up to a constant. The imaginary part v can be determined from*

$$v(x, y) = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(t, y)dt + \int_{y_0}^{y} \frac{\partial u}{\partial x}(x_0, t) + k, \quad k \in \mathbb{R},$$

where $(x_0, y_0) \in D$ and D is a simple connected domain.

Some complex functions. We define the exponential, sinus and cosinus by the folllowing power series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots,$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + - \dots,$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots,$$

for any $z \in \mathbb{C}$.

Euler's relation is satisfied

$$e^{iz} = \cos z + i \sin z, \quad z \in \mathbb{C}.$$

For $z \in \mathbb{C} \setminus \{0\}$ the complex logarithm is defined by

$$\mathrm{Log} z = \ln |z| + i(\mathrm{arg} z + 2k\pi), k \in \mathbb{Z}.$$

For $k = 0$ we get $\ln z = \ln |z| + i\mathrm{arg} z$ is called the principal branch of $\mathrm{Log} z$. The power function is defined by

$$z^\alpha = e^{\alpha \mathrm{Log} z}, \; z \in \mathbb{C} \setminus \{0\}, \alpha \in \mathbb{C}.$$

From Euler's formula we get

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}.$$

The hyperbolic functions are

$$\mathrm{ch} z = \frac{e^{z} + e^{-z}}{2}, \quad \mathrm{sh} z = \frac{e^{z} - e^{-z}}{2}, \quad z \in \mathbb{C}.$$

The following relations hold

$$\sin^2 z + \cos^2 z = 1; \quad \mathrm{ch}^2 z - \mathrm{sh}^2 z = 1, z \in \mathbb{C}.$$

We continue this section with some important solved problems.

Problem 4.2.6. Find the real part and the imaginary part of the following numbers:

a) $\sin(2 - i)$;

b) i^i;

Solution.

a)

$$\sin(2-i) = \frac{e^{i(2-i)} - e^{-i(2-i)}}{2i} = \frac{e^{1+2i} - e^{-1-2i}}{2i}$$
$$= \frac{e(\cos 2 + i\sin 2) - e^{-1}(\cos 2 + i\sin 2)}{2i} = \frac{(e - \frac{1}{e})\cos 2 + i(e + \frac{1}{e})\sin 2}{2i}$$
$$= \frac{i(e - \frac{1}{e})\cos 2 - (e + \frac{1}{e})\sin 2}{-2} = \frac{e^2+1}{2e}\sin 2 + i\frac{1-e^2}{2e}\cos 2.$$

b) $i^i = e^{i\mathrm{Log} i}$, where

$$\mathrm{Log} i = \{\ln|i| + i(\arg i + 2k\pi) \mid k \in \mathbb{Z}\}.$$

Since $|i| = \sqrt{1^2+0^2} = 1$ and $\arg i = \arctan\frac{1}{0} = \arctan\infty = \frac{\pi}{2}$ we obtain that

$$\mathrm{Log} i = \{i(\frac{\pi}{2} + 2k\pi) \mid k \in \mathbb{Z}\},$$

thus

$$i^i = \{e^{i(\frac{\pi}{2} + 2k\pi)} \mid k \in \mathbb{Z}\}.$$

Problem 4.2.7. Find the holomorphic functions $f : \mathbb{C} \to \mathbb{C},\ f(z) = u(x,y) + iv(x,y), \quad x, y \in \mathbb{R}$ if:

a) $u(x,y) = x^2 - y^2 - x$;

b) $v(x,y) = \frac{y}{x^2+y^2},\ f(1) = 0$;

Solutions.

a) *Solution 1.* From the (C-R) equations we get $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, hence the relations

$$\frac{\partial v}{\partial y} = 2x - 1; \tag{4.2.1}$$

$$\frac{\partial v}{\partial x} = 2y; \tag{4.2.2}$$

From (4.2.1) and (4.2.2) it follows

$$dv = 2ydx + (2x-1)dy = P(x,y)dx + Q(x,y)dy,$$

thus

$$v(x,y) = \int_0^x P(t,y)dt + \int_0^y Q(0,t)dt + k = \int_0^x 2ydt + \int_0^y (-1)dt$$
$$= 2xy - y + k, \quad k \in \mathbb{R}.$$

Now the expression of f is

$$f(z) = u(x,y) + iv(x,y) = x^2 - y^2 - x + i(2xy - y) + ik$$
$$= (x^2 - y^2 + 2ixy) - (x + iy) + ik = z^2 - z + ik.$$

Solution 2. Integrate the above relation (4.2.1) to obtain $v(x,y) = (2x-1)y + \phi(x)$. Now we apply (4.2.2) and it follows that $2y + \phi'(x) = 2y$, thus $\phi'(x) = 0$ and $\phi(x) = k, k \in \mathbb{R}$. It follows analogously $f(z) = z^2 - z + ik, k \in \mathbb{R}$.

b) We have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x^2 + y^2 - 2y^2}{(x^2+y^2)^2}; \tag{4.2.3}$$

$$\frac{\partial y}{\partial y} = -\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}; \tag{4.2.4}$$

From (4.2.4) we have that $u(x,y) = \int \frac{2xy}{(x^2+y^2)}dy = -\frac{x}{x^2+y^2} + \phi(y)$. Replacing in (4.2.3) we get $\frac{\partial u}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2}$ hence

$$-\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \phi'(y) = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

We obtain $\phi'(y) = 0$, that is $\phi(y) = k, k \in \mathbb{R}$. We get $u(x,y) = -\frac{x}{x^2+y^2} + k$.

$$f(z) = -\frac{x}{x^2+y^2} + k + i\frac{y}{x^2+y^2} = \frac{-x+iy}{x^2+y^2} + k$$

$$= -\frac{x-iy}{(x+iy)(x-iy)} + k$$

$$= -\frac{1}{x+iy} + k = \frac{-1}{z} + k.$$

The condition $f(1) = 0$ leads to $k = 1$, so

$$f(z) = 1 - \frac{1}{z}, \quad z \in \mathbb{C} \setminus \{0\}.$$

Problem 4.2.8. Find all holomorphic functions $f(z) = u(x,y) + iv(x,y), z = x + iy, x, y \in \mathbb{R}$ if $u(x,y) = \phi(x^2-y^2), \phi \in C^2(\mathbb{R})$.

Solution. The function u satisfies Laplace equation $\Delta u = 0$, that is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. We obtain

$$\frac{\partial u}{\partial x} = \phi'(x^2-y^2)2x;$$

$$\frac{\partial^2 u}{\partial x^2} = 2\phi'(x^2-y^2) + 4x^2\phi''(x^2-y^2);$$

$$\frac{\partial u}{\partial y} = \phi'(x^2-y^2)(-2y);$$

$$\frac{\partial^2 u}{\partial y^2} = -2\phi'(x^2-y^2) + 4y^2\phi''(x^2-y^2).$$

Since $\Delta u = 0$ it follows that for any $(x,y) \in \mathbb{R}^2$ we have $4(x^2+y^2)\phi''(x^2-y^2) = 0$ hence $\phi''(x^2-y^2) = 0$. Let $x^2-y^2 = t, t \in \mathbb{R}$. The relation $\phi''(t) = 0$ implies

$$\phi(t) = C_1 t + C_2,\ C_1, C_2 \in \mathbb{R},$$

so $u(x,y) = C_1(x^2 - y^2) + C_2$. (C-R) equations lead to

$$\frac{\partial v}{\partial x} = 2C_1 y$$

$$\frac{\partial v}{\partial y} = 2C_1 x.$$

The first relation leads to $v(x,y) = \int 2C_1 y dx = 2C_1 xy + \phi(y)$. Replacing in the second relation we obtain $2C_1 + \phi'(y) = 2C_1 x$ hence $\phi'(y) = 0$ and $\phi(y) = k, k \in \mathbb{R}$ so $v(x,y) = 2C_1 xy + k$. Finally

$$f(z) = C_1(x^2 - y^2) + C_2 + i(2C_1 xy + k)$$

$$= C_1(x^2 - y^2 + 2ixy) + C_2 + ik$$

$$= C_1 z^2 + C_2 + ik.$$

Let $\lambda = C_2 + ik, \lambda \in \mathbb{C}$ then $f(z) = Cz^2 + \lambda, C \in \mathbb{R}$.

Problem 4.2.9. Find $z \in \mathbb{C}$ such that $\sin z = \frac{4i}{3}$,

Solution. The equation is equivalent to $\frac{e^{iz} - e^{-iz}}{2i} = \frac{4i}{3}$, that is $e^{iz} - e^{-iz} = -\frac{8}{3}$. We obtain the equation $e^{2iz} + \frac{8}{3}e^{iz} - 1$. Let $e^{iz} = w$. Then $3w^2 + 8w - 3 = 0$ which is an equation with $\Delta = 100$ and $w_1 = \frac{1}{3}; w_2 = -3$.

In the first case we obtain $e^{iz} = \frac{1}{3}$ with

$$iz = \text{Log}\frac{1}{3} = \ln|\frac{1}{3}| + i(\arg\frac{1}{3} + 2k\pi) = -\ln 3 + 2k\pi i.$$

In the second case we obtain $e^{iz} = -3$ with the solution

$$iz = \text{Log}(-3) = \ln|-3| + i(\arg(-3) + 2k\pi) = \ln 3 + (2k+1)\pi i.$$

The solution is

$$z \in \{2k\pi + i\ln 3 | k \in \mathbb{Z}\} \cup \{(2k+1)\pi - i\ln 3 | k \in \mathbb{Z}\}.$$

Problem 4.2.10. Prove the following relations:

a) $\text{Re}\sin z = \sin x \text{ch} y$;

b) $|\sin z| = \sqrt{\text{ch}^2 y - \cos^2 x}$.

Solution.

a)

$$\begin{aligned}\sin z &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix}e^{-iy} - e^{-ix}e^{y}}{2i} \\ &= \frac{(\cos x + i\sin x)e^{-y} - (\cos x - i\sin x)e^{y}}{2i} \\ &= \frac{\cos xe^{-y} + i\sin xe^{-y} - \cos xe^{y} + i\sin xe^{y}}{2i} \\ &= \frac{\cos x(e^{-y} - e^{y}) + i\sin x(e^{-y} + e^{y})}{2i} = \frac{\sin x(e^{y} + e^{-y})}{2} + i\frac{\cos x(e^{y} - e^{-y})}{2} \\ &= \sin x \text{ch} y + i\cos x \text{sh} y;\end{aligned}$$

b) The above relation $\sin z = \sin x\text{ch}y + i\cos x\text{sh}y$ leads to

$$|\sin z| = \sqrt{\sin^2 x\text{ch}^2 y + \cos^2 x\text{sh}^2 y}.$$

It is well known that $\sin^2 x = 1 - \cos^2 x,\ \text{sh}^2 y = \text{ch}^2 y - 1$ hence

$$|\sin z| = \sqrt{(1 - \cos^2 x)\text{ch}^2 y + \cos^2 x(\text{ch}^2 y - 1)} = \sqrt{\text{ch}^2 y - \cos^2 x}.$$

4.3 Laplace's transform.

Laplace's transform is defined for any function $f : \mathbb{R} \to \mathbb{C}$ by

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt,$$

where $s \in D \subseteq \mathbb{C}$. By D we denote the set of complex numbers s for which the above integral is convergent. If $D \neq \emptyset$ we say that the function f has Laplace's transform on D.

A function $f : \mathbb{R} \to \mathbb{C}$ is called an *original* if the following statements are true

- $f(t) = 0, t < 0$;
- f has a finite number of first septets? discontinuity for any bounded interval;

- f has an *exponential growing order*, that is there are constants $M, \sigma \geq 0$ such that $|f(t)| \leq Me^{\sigma t}, \forall t > 0$

The set of all original functions is denoted by O. The constant σ is called a growing index and the smallest growing index σ is denoted $\sigma_o = \sigma_o(f)$.

Theorem 4.3.1. *If $f \in O$ then $F(s)$ exists and is holomorphic in the semiplane* $\text{Re}s > \sigma_o(f)$.

In the next example we show how to compute the Laplace's transform using its definition.

Example 4.3.2. For any $a \in \mathbb{C}$ we have

$$L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \frac{1}{s-a},$$

for all s with $\text{Re}s > \text{Re}a$.

However in practice we will use the properties of Laplace's transform which we will give in the next theorem and a basic list of Laplace's transform for some elementary functions.

Theorem 4.3.3. *(1) (Linearity). For any $\alpha, \beta \in \mathbb{C}$ and $f, g \in O$ we have*

$$L\{\alpha f(t) + \beta g(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\};$$

(2) If $f \in O$ and $a > 0$ then $L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$;

(3) If $f \in O$ and $a \in CC$ then $L\{e^{at}f(t)\} = F(s-a)$ for $\text{Re } s > \text{Re } a + \sigma_o(f)$*;*

(4) If $f \in O$ and $a > 0$ then $L\{f(t-a)\} = e^{-as}F(s)$ for $\text{Re } s > \sigma_o(f)$*;*

(5) (The differentiation of the original). If f is continuous and $f, f', f'' \in O$ then for any s such that $\text{Re } s > max\{\sigma_o(f), \sigma_o(f')\}$ *we have*

$$L\{f'(t)\} = sF(s) - f(0+0),$$

and for any s such that Re $s > max\{\sigma_o(f), \sigma_o(f'), \sigma_o(f'')\}$ *we have*

$$L\{f''(t)\} = s^2F(s) - sf(0+0) - f'(0+0);$$

(6) *(The differentiation of the transform). If $f \in O$ is a continuous function then for any s such that* Re $s > \sigma_0(f)$ *we have*

$$F'(s) = -L\{tf(t)\}.$$

(7) *(The integration of the transform). If $\frac{f(t)}{t} \in O$ then for any s such that* Re $s > \sigma_o(\frac{f(t)}{t})$ *we have*

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds;$$

In particular

$$\int_0^\infty \frac{f(t)}{t} = \int_0^\infty F(s)ds.$$

It is recommended that the next list which contains some fundamental Laplace's transforms of elementary functions to be memorized.

Table with Laplace's transforms:

(1) $L\{e^{at}\} = \frac{1}{s-a}$ for $a \in \mathbb{C}$, Re$s >$ Re a;

More generally for $n \geq 0$ any integer, we have

$$L\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}};$$

In particular $L\{1\} = \frac{1}{s}$ and $L\{at\} = \frac{a}{s^2}$;

(2) $L\{\sin\ at\} = \frac{a}{s^2+a^2}$ for $a > 0$;

(3) $L\{\cos\ at\} = \frac{s}{s^2+a^2}$ for $a > 0$;

(4) $L\{\text{sh}\ at\} = \frac{a}{s^2-a^2}$ for $a > 0$;

(5) $L\{\text{ch}\ at\} = \frac{s}{s^2-a^2}$ for $a > 0$;

To see how to apply the above formulas we present some solved problems.

Problem 4.3.4. Compute the transform $L\{\cos^2 t\}$.

Solution. The well-known trigonometrical formula $\cos^2 t = \frac{1+\cos\ 2t}{2}$ gives us

$$L\{\cos^2 t\} = \frac{1}{2}(L\{1\} + L\{\cos\ 2t\}) = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+2^2}\right)$$
$$= \frac{1}{2} \cdot \frac{2s^2+4}{s^3+4s}.$$

For the second equality we used formula (1) and (3) from the above table.

Problem 4.3.5. Determine the next improper integral $\int_0^\infty \frac{\cos 2t - \cos t}{t} dt$.

Solution. From Theorem 4.3.3,(7) we have that $\int_0^\infty \frac{\cos 2t-\cos t}{t}dt = \int_0^\infty F(s)ds$, where $F(s) = L\{\cos\ 2t - \cos\ t\}$. Then

$$F(s) = L\{\cos\ 2t\} - L\{\cos\ t\} = \frac{s}{s^2+4} - \frac{s}{s^2+1}.$$

The above integral is now

$$\int_0^\infty \frac{\cos 2t - \cos t}{t} dt = \int_0^\infty \frac{s}{s^2+4} ds - \int_0^\infty \frac{s}{s^2+1} ds$$
$$= \frac{1}{2}\left[\ln \frac{s^2+4}{s^2+1}\right]_0^\infty = \frac{1}{2}(0 - \ln 4) = \ln \frac{1}{2}.$$

Laplace's transform has also an inverse. To compute the inverse of Laplace' s transform we have the next theorem.

Theorem 4.3.6. *(Mellin-Fourier.)*

If $f \in O$ *and* $L\{f(t)\} = F(s)$, $\mathrm{Re}s > \sigma_o(f)$ *then*

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2i\pi}\int_{x-i\infty}^{x+i\infty} e^{st}F(s)ds, \quad x > \sigma_o(f).$$

The properties from Theorem 4.3.3 can be translated to its inverse. If we know the table for the direct transform then we can find easily the inverse. For example:

(I1). $L^{-1}\{\frac{1}{(s-a)^{n+1}}\} = \frac{e^{at}t^n}{n!}$;
(I2). $L^{-1}\{\frac{a}{s^2+a^2}\} = \sin\ at$;
(I3). $L^{-1}\{\frac{s}{s^2+a^2}\} = \cos\ at$.

Using Laplace's transform and its inverses we can easily solve Cauchy's problems for linear equations or linear system with constant coefficients

Problem 4.3.7. Solve Cauchy's problem for the differential equation

$$y'' - 4y = \frac{e^{2t} + e^{-2t}}{2}, \quad y(0) = y'(0) = 0.$$

Solution. We denote by Y the Laplace's transform

$$L\{y(t)\} = Y(s).$$

The differentiation formulas Theorem 4.3.3,(6) and the initial conditions of Cauchy's problem give us

$$L\{y'(t)\} = sY(s) - y(0+0) = sY(s);$$

$$L\{y''(t)\} = s^2Y(s) - sy(0+0) - y'(0+0) = s^2Y(s).$$

We apply Laplace's transform on the given differential equation to obtain the equation in Y:

$$s^2Y(s) - 4Y(s) = \frac{1}{2}L\{e^{2t}\} + \frac{1}{2}L\{e^{-2t}\},$$

hence

$$(s^2-4)Y(s) = \frac{1}{2}\left(\frac{1}{s-2} + \frac{1}{s+2}\right) = \frac{1}{2} \cdot \frac{s+2+s-2}{s^2-4} = \frac{s}{s^2-4}.$$

It follows that $Y(s) = \frac{s}{(s^2-4)^2}$ and we want to find A, B, C, D constants such that

$$Y(s) = \frac{s}{(s-2)^(s+2)^2} = \frac{As+B}{(s-2)^2} + \frac{Cs+D}{(s+2)^2}.$$

We obtain $s = (As+B)(s+2)^2 + (Cs+D)(s-2)^2$, which give us the system

$$\begin{cases} A + C = 0 \\ 4A + B - 4C + D = 0 \\ 4A + 4B + 4C - 4D = 1 \\ 4B + 4D = 0 \end{cases}$$

The solutions are $A = 0, C = 0, B = \frac{1}{8}, D = -\frac{1}{8}$, thus

$$Y(s) = \frac{1}{8}\left[\frac{1}{(s-2)^2} - \frac{1}{(s+2)^2}\right].$$

We apply the inverse L^{-1} to obtain

$$y(t) = \frac{1}{8}L^{-1}\left\{\frac{1}{(s-2)^2}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$
$$= \frac{1}{8}e^{2t}t - \frac{1}{8}e^{-2t}t = \frac{t}{8}(e^{2t} - e^{-2t}).$$

For the second equality we used (I1).

Problem 4.3.8. Integrate the system

$$\begin{cases} x'(t) = x(t) + 2y(t), \quad x(0) = 1 \\ y'(t) = 2x(t) + y(t), \quad y(0) = 1 \end{cases}$$

Solution. Consider the notations $L\{x(t)\} = X(s), L\{Y(t)\} = Y(s)$ and apply L on the equations to obtain

$$\begin{cases} L\{x'(t)\} = L\{x(t)\} + 2L\{y(t)\} \\ L\{y'(t)\} = 2L\{x(t)\} + L\{y(t)\} \end{cases}$$

hence

$$\begin{cases} sX - 1 = X + Y \\ sY + 1 = 2X + Y \end{cases}$$

that is, the linear system

$$\begin{cases} (s-1)X - 2Y = 1 \\ -2X + (s-1)Y = -1 \end{cases}.$$

To solve this system we multiply the first equation by 2, the second equation by $s-1$ and we add to get

$$-4Y + (s-1)^2Y = 2 - s + 1,$$

hence

$$Y = \frac{3-s}{-4+s^2-2s+1} = \frac{3-s}{s^2-2s+3} = \frac{3-s}{(s-3)(s+1)} = \frac{-1}{s+1}.$$

Then

$$y(t) = -L^{-1}\left\{\frac{1}{s+1}\right\} = -e^{-t}.$$

Similarly we obtain $X(t) = \frac{1}{s+1}$ and then

$$x(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

4.4 Problems

Problem 4.4.1. Let $z_1 = -2 + 11i, z_2 = 2 - i$. Compute or find the algebraic form $x + iy, x, y \in \mathbb{R}$ where is the case:

a) $\text{Im}z_1$;

b) $z_1 z_2$;

c) $\frac{z_1}{z_2}$;

d) $\text{Re}z_1^2,\ (\text{Re}z_1)^2$.

e) $|z_1|$;

f) $|z_1 z_2|$;

g) $|z_1||z_2|$;

h) $|z_1 + z_2|$;

i) $|z_1| + |z_2|$; Compare the result with the result of h) and explain geometrically.

Problem 4.4.2. Prove the relations from Proposition 4.1.1 by using that $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2,\ x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Problem 4.4.3. Find the real part and the imaginary part of:

a) $\text{ch}(2 + i)$;

b) 1^i;

c) $\text{Log}(1 + i)$;

d) $\cos z$, where $z = x + iy \in \mathbb{C}$;

e) $z^2\overline{z}$, where $z = x + iy \in \mathbb{C}$.

Problem 4.4.4. Find the holomorphic functions

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = u(x, y) + iv(x, y), \quad x, y \in \mathbb{R}$$

if:

a) $u(x, y) = x^2 - y^2 + xy,\ f(0) = 0$;

b) $u(x, y) = e^x(x \cos y - y \sin y),\ f(0) = 0$;

c) $v(x, y) = \arctan \frac{y}{x}, x > 0, f(1) = 0$;

d) $u(x, y) = x^2 - y^2 + 2x,\ f(i) = 2i - 1$;

Problem 4.4.5. Find $z \in \mathbb{C}$ such that:

a) $\mathrm{sh} z = \frac{i}{2}$;

b) $\mathrm{ch} z = \frac{i}{2}$;

c) $\cos z = 5$;

d) $\sin z = 0$.

Problem 4.4.6. Prove the following relations, for any $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$:

a) $\mathrm{Re} \cos z = \cos x \mathrm{ch} y$;

b) $|\cos z| = \sqrt{\mathrm{ch}^2 y - \sin^2 x}$;

c) $\cos z = \cos x \mathrm{ch} y - i \sin x \mathrm{sh} y$;

d) $\sin z = \sin x \text{ch} y + i \cos x \text{sh} y$;

e) $|\text{Re} z| \le |z|$;

f) $|\text{Im} z| \le |z|$;

g) (**Parallelogram equality**)

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2),$$

where $z_1, z_2 \in \mathbb{C}$. Explain the name!

Problem 4.4.7. Find Laplace's transforms $L\{f(t)\}$ for the next functions:

(1) $f(t) = \sin^2 t$;

(2) $f(t) = \sin^3 t$;

(3) $f(t) = \cos^3 t$;

(4) $f(t) = e^{2t} \sin t$;

(5) $f(t) = t \sin^3 t$;

(6) $f(t) = \frac{1 - \cos t}{t}$;

(7) $f(t) = \frac{e^t - 1}{t}$;

Problem 4.4.8. Compute the integrals:

(1) $\int_0^\infty \frac{\cos 2t - \cos 3t}{t} dt$;

(2) $\int_0^\infty \frac{e^{-t} - e^{-2t}}{t} dt$;

(3) $\int_0^\infty \frac{\sin t \cdot \sin 2t}{t} dt$;

Problem 4.4.9. Integrate the following linear differential equations and systems of linear differential with constant coefficients:

(1) $y'' + 4y = \cos 2t,\ y(0) = y'(0) = 0;$

(2) $y'' + y = \sin t,\ y(0) = y'(0) = 0;$

(3) $y'' - 4y = \mathrm{ch}2t,\ y(0) = y'(0) = 0;$

(4) $y'' - 9y = \mathrm{sh}3t,\ y(0) = y'(0) = 0;$

(5) $\begin{cases} x' = y + 2e^t \\ y' = x + t^2 \end{cases},\ x(0) = y(0) = 1$

Solutions **4.4.1** a) 11; b) $7 + 24i$; c) $-3 + 4i$; d) $-117, 4$; e) $\sqrt{125}$; f) 25; g) 25; h) 10; i) $6\sqrt{5}$; This number is larger than the result from h), since it may be viewed as the sum of two sides of a triangle which is always larger than the third side: the triangle inequality. **4.4.3** a) $\frac{1}{2e}[(e^2 + 1)\cos 1 + i(e^2 - 1)\sin 1]$; b) $e^{-2k\pi}, k \in \mathbb{Z}$; c) $\ln\sqrt{2} + i(\frac{\pi}{2} + 2k\pi), k \in \mathbb{Z}$; d) $\cos x\mathrm{ch}y - i\sin x\mathrm{sh}y$; e) $x^3 + xy^2 + i(y^3 + x^2y)$; **4.4.4** a) $f(z) = 1 - \frac{i}{2}z^2$; b) $f(z) = ze^z$; c) $f(z) = \ln z$; d) $f(z) = z^2 + 2z$; **4.4.5** a) $z \in \{(\frac{\pi}{6} + 2k\pi)i | k \in \mathbb{Z}\} \cup \{(\frac{5\pi}{6} + 2k\pi)i\pi k \in \mathbb{Z}\}$; c)$z \in \{2k\pi + i\ln(5 \pm \sqrt{24}) | k \in \mathbb{Z}\}$; d)$z \in \{k\pi | k \in \mathbb{Z}\}$; **4.4.6** See 4.2.10; **4.4.7** (1) $\frac{2}{(s^2+4)s}$; (2) $\frac{6}{(s^2+1)(s^2+9)}$, use $\sin^3 t = \left(\frac{e^{it}-e^{-it}}{2i}\right)^3$; (3) $\frac{s(s^2+7)}{(s^2+1)(s^2+9)}$; (4) $\frac{1}{(s-2)^2+1}$; (5) $\frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$, use Theorem 4.3.3,(6); (6) $\ln\frac{s+1}{s}$, use Theorem 4.3.3,(7); (7) $\ln\frac{s}{s-1}$, use Theorem 4.3.3,(7). **4.4.8** (1) $\ln\frac{3}{2}$; (2) $\ln 2$; (3) $-\ln\sqrt{3}$; **4.4.9** (1) $y(t) = \frac{t}{4}\sin 2t$; (2) $y(t) = \frac{1}{2}(\sin t - t\cos t)$; (3) $y(t) = \frac{t}{4}\mathrm{sh}2t$; (4) $y(t) = \frac{1}{18}(3t\mathrm{ch}3t - \mathrm{sh}3t)$; (5) $x(t) = \frac{5}{2}e^t + \frac{1}{2}e^{-t} + te^t - t^2 - 2,\ y(t) = \frac{5}{2}e^t - \frac{1}{2}e^{-t} + (t-1)e^t - 2t$.

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